ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1978

Seria: MATEMATYKA-FIZYKA z. 30 Nr kol. 560

Grażyna KOZŁOWSKA
Instytut Matematyki - Politechnika Śląska

O RZĘDZIE PRZYBLIŻENIA FUNKCJI $f(x,y)\in L^{P}(-\pi,\pi;-\pi,\pi)$ PEWNYMI LINIOWYMI ŚREDNIMI ICH SZEREGÓW FOURIERA

Streszczenie. Rozpatruje się rząd przybliżenia funkcji $f(x,y)\in L^{P}(-\pi,\pi;-\pi,\pi)$ liniowymi średnimi ich szeregów Fouriera spełniającymi uogólniony warunek S.B. Stieczkina

OZNACZENIA

$P = (p_1,p_2)$, gdzie $1\leq p_1,p_2<\infty$.

$L^{P}(-\pi,\pi;-\pi,\pi)$ przestrzeń funkcji mierzalnych, okresowych o okresie 2π ze względu na każdą zmienną, określonych w kwadracie $[-\pi,\pi;-\pi,\pi]$, dla których

$$\left\{\int_{-\pi}^{\pi}\left[\int_{-\pi}^{\pi}|f(x,y)|^{p_2}\,dy\right]^{p_1/p_2}dx\right\}^{1/p_1}<\infty$$

$$\|f(x,y)\|_P=\left\{\int_{-\pi}^{\pi}\left[\int_{-\pi}^{\pi}|f(x,y)|^{p_2}\,dy\right]^{p_1/p_2}dx\right\}^{1/p_1}$$ norma w $L^{P}(-\pi,\pi;-\pi,\pi)$

$$\omega(f;u,v)_P=\sup_{|h|\leq u,\,|k|\leq v}\|f(x+h,y+k)-f(x,y)\|_P \quad u>0,\ v>0$$

$$\Lambda=\left\|\lambda_k^{(n)}\right\| \quad \lambda_0^{(n)}=1,\quad \lambda_k^{(n)}=0 \quad k>n \ (n=0,1,2,\ldots;\ k=0,1,\ldots)$$

$$\Delta\lambda_k^{(n)}=\lambda_k^{(n)}-\lambda_{k+1}^{(n)} \qquad n=0,1,\ldots;\ k=0,1,\ldots,n$$

$$U_{n_1,n_2}(t_1,t_2)=\left(\frac{1}{2}+\sum_{k_1=1}^{n_1}\lambda_{k_1}^{(n_1)}\cos k_1t_1\right)\left(\frac{1}{2}+\sum_{k_2=1}^{n_2}\lambda_{k_2}^{(n_2)}\cos k_2t_2\right)$$

$$L_{n_1,n_2}(f;x,y;\Lambda)=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(x+t_1,y+t_2)\,U_{n_1,n_2}(t_1,t_2)\,dt_1dt_2$$

Λ^* klasa macierzy spełniających uogólniony warunek Stieczkina

$$\left\{(n_1+1)^{p_1-1}\sum_{k_1=0}^{n_1}\left|\Delta\lambda_{k_1}^{(n_1)}\right|^{p_1}\right\}^{1/p_1}\left\{(n_2+1)^{p_2-1}\sum_{k_2=0}^{n_2}\left|\Delta\lambda_{k_2}^{(n_2)}\right|^{p_2}\right\}^{1/p_2}\leqslant K$$

$H(\omega)_p$ klasa funkcji $f(x,y)\in L^p(-\pi,\pi;-\pi,\pi)$, dla których $\omega(f;u,v)_p\leqslant\omega(u,v)$ $0<u\leqslant\pi$, $0<v\leqslant\pi$, gdzie $\omega(u,v)$ jest funkcją dodatnią.

$$A_{n,m}(\omega)=\sup_{\Lambda\in\Lambda^*}\ \sup_{f\in H(\omega)_p}\left\|L_{n_1,n_2}(f;x,y;\Lambda)-f(x,y)\right\|_p$$

$$F=\left\{F_{n_1,n_2}\right\}_{n_1=0,\ n_2=0}^{\infty}\qquad F_{n_1,n_2}\downarrow 0$$

$C(F)$ klasa funkcji $f(x,y)\in L^p(-\pi,\pi;-\pi,\pi)$ dla których $E_{n_1,n_2}(f)_p\leqslant F_{n_1,n_2}$ $n_1=0,1,2,\ldots,$ $n_2=0,1,2,\ldots$

$$A_{n,m}(F)=\sup_{\Lambda\in\Lambda^*}\ \sup_{f\in C(F)}\left\|L_{n_1,n_2}(f;x,y;\Lambda)-f(x,y)\right\|_p$$

$$T_{m,n}(x,y)=\sum_{k=0}^{m}\sum_{l=0}^{n}(a_k\cos kx+b_k\sin kx)(c_l\cos ly+d_l\sin ly)$$

$$E_{m,n}(f)_p=\inf_{T_{m,n}}\left\|f(x,y)-T_{m,n}(x,y)\right\|_p$$

$$U_{n,m}(f;x,y;\Lambda)=\sum_{k=0}^{n}\sum_{l=0}^{m}\left|\Delta\lambda_k^{(n)}\Delta\lambda_l^{(m)}\right|\left|S_{k,l}(f;x,y)-f(x,y)\right|$$

$$S_{m,n}(f;x,y)=\sum_{k=0}^{n}\sum_{l=0}^{m}(\tilde{a}_{kl}\cos kx\cos ly+\tilde{b}_{kl}\cos kx\sin ly+$$

$+\tilde{c}_{kl}\sin kx\sin ly+\tilde{d}_{kl}\sin kx\cos ly)$ suma częściowa szeregu Fouriera

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(\tilde{a}_{kl}\cos kx\cos ly+\tilde{b}_{kl}\cos kx\sin ly+\tilde{c}_{kl}\sin kx\sin ly+$$

$$+\tilde{d}_{kl}\sin kx\cos ly)=\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(a_{kl}\cos kx\cos ly+b_{kl}\cos kx\sin ly+$$

$$+ c_{kl}\sin kx \sin ly + d_{kl}\sin kx \cos ly) + \frac{1}{2}\sum_{k=1}^{\infty}(a_{k0}\cos kx + d_{k0}\sin kx) +$$

$$+ \frac{1}{2}\sum_{l=1}^{\infty}(a_{0l}\cos ly + b_{0l}\sin ly) + \frac{a_{00}}{4}, \text{ gdzie}$$

$$a_{kl} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(t,\tau)\cos kt \cos l\tau \, dt\, d\tau$$

$$b_{kl} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(t,\tau)\cos kt \sin l\tau \, dt\, d\tau$$

$$c_{kl} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(t,\tau)\sin kt \sin l\tau \, dt\, d\tau$$

$$d_{kl} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(t,\tau)\sin kt \cos l\tau \, dt\, d\tau$$

W niniejszej pracy funkcje $f(x,y) \in L^P(-\pi,\pi;-\pi,\pi)$ aproksymuje się średnimi

$$L_{m,n}(f;x,y;\Lambda) = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+t_1,y+t_2)\; U_{m,n}(t_1,t_2)dt_1dt_2$$

spełniającymi uogólniony warunek Stieczkina

$$\left[(m+1)^{p_1-1}\sum_{k_1=0}^{m}\left|\Delta\lambda_{k_1}^{(m)}\right|^{p_1}\right]^{1/p_1}\left[(n+1)^{p_2-1}\sum_{k_2=0}^{n}\left|\Delta\lambda_{k_2}^{(n)}\right|^{p_2}\right]^{1/p_2} \leqslant K$$

W twierdzeniach 2 i 3 zawarte są uzyskane oszacowania.

Lemat 1

$$L_{m,n}(f;x,y;\Lambda) = \sum_{k=0}^{m}\sum_{l=0}^{n}\lambda_k^{(m)}\lambda_l^{(n)}(\tilde{a}_{kl}\cos kx \cos ly + \tilde{b}_{kl}\cos kx \sin ly +$$

$$+ \tilde{c}_{kl}\sin kx \sin ly + \tilde{d}_{kl}\sin kx \cos ly)$$

Prawdziwość lematu łatwo stwierdzić wykorzystując wzory na współczynniki w rozwinięciu funkcji f(x,y) w szereg Fouriera i wykonując proste przekształcenia strony prawej.

Lemat 2

$$L_{m,n}(f;x,y;\Lambda) = \sum_{k=0}^{m}\sum_{l=0}^{n} \Delta\lambda_k^{(m)}\,\Delta\lambda_l^{(n)}\,S_{k,l}$$

Prawdziwość lematu jest oczywista po rozpisaniu strony prawej i wykorzystaniu lematu 1.

Lemat 3

$$\sum_{k=0}^{m}\sum_{l=0}^{n} \Delta\lambda_k^{(m)}\,\Delta\lambda_l^{(n)} = 1$$

Dowód wynika z definicji $\Delta\lambda_k^{(m)}$ i $\Delta\lambda_l^{(n)}$.

Lemat 4

$$S_{m,n}(f;x,y) = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+t,y+\tau)\left[\frac{1}{2}+\sum_{k=1}^{m}\cos kt\right]\left[\frac{1}{2}+\sum_{l=1}^{n}\cos l\tau\right]dt\,d\tau =$$

$$= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+t,y+\tau)\,D_{m,n}(t,\tau)\,dt\,d\tau$$

Prawdziwość lematu uzyskuje się po rozpisaniu strony lewej zgodnie z definicją, wykorzystaniu wzorów na współczynniki w rozwinięciu funkcji $f(x,y)$ w szereg Fouriera i wykonaniu prostych przekształceń.

Lemat 5

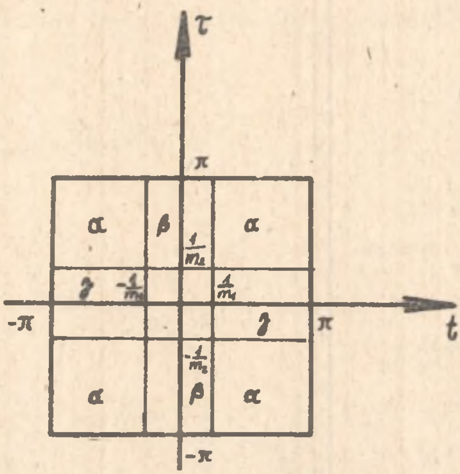
$$\left[\sum_{k_1=l_1}^{m_1}\left(\sum_{k_2=l_2}^{m_2}\left\|S_{k_1,k_2}(g;x,y)\right\|_p^{q_2}\right)^{q_1/q_2}\right]^{1/q_1} \leqslant C(q_1,q_2)\,m_1^{1/q_1}\,m_2^{1/q_2}\,\|g\|_p$$

Dowód

Z lematu 4 i [6] wiadomo, że

$$S_{k_1,k_2}(g;x,y) = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(x+t,y+\tau)\,D_{k_1,k_2}(t,\tau)\,dt\,d\tau, \quad \text{gdzie}$$

$$D_{k_1,k_2}(t,\tau) = \left[\frac{1}{2}+\sum_{k=1}^{k_1}\cos kt\right]\left[\frac{1}{2}+\sum_{l=1}^{k_2}\cos l\tau\right] = \frac{\sin\frac{2k_1+1}{2}t}{2\sin\frac{t}{2}}\cdot\frac{\sin\frac{2k_2+1}{2}\tau}{2\sin\frac{\tau}{2}}$$



Prawdą jest również wobec nierówności

$$|a+b|^p \leqslant 2^p(|a|^p + |b|^p) \tag{*}$$

[3], że

$$\left\{\sum_{k_1=l_1}^{m_1}\left(\sum_{k_2=l_2}^{m_2}\|S_{k_1,k_2}(g;x,y)\|_p^{q_2}\right)^{q_1/q_2}\right\}^{1/q_1} \leqslant$$

$$\leqslant C_1(q_1,q_2)\left\{\sum_{k_1=l_1}^{m_1}\left(\sum_{k_2=l_2}^{m_2}\left\|\int_{-\frac{1}{m_1}}^{\frac{1}{m_1}}\int_{-\frac{1}{m_2}}^{\frac{1}{m_2}} g(x+t,y+\tau)D_{k_1,k_2}(t,\tau)\,dt\,d\tau\right\|_p^{q_2}+\right.\right.$$

$$\left.\left.+\left\|\iint_{J_\alpha+J_\beta+J_\gamma} g(x+t,y+\tau)\,D_{k_1,k_2}(t,\tau)\,dt\,d\tau\right\|_p^{q_2}\right)^{q_1/q_2}\right\}^{1/q_1} \leqslant$$

$$\leqslant C_2(q_1,q_2)\left\{\left[\sum_{k_1=l_1}^{m_1}\left(\sum_{k_2=l_2}^{m_2}\left\|\int_{-\frac{1}{m_1}}^{\frac{1}{m_1}}\int_{-\frac{1}{m_2}}^{\frac{1}{m_2}} g(x+t,y+\tau)D_{k_1,k_2}(t,\tau)\,dt\,d\tau\right\|_p^{q_2}\right)^{q_1/q_2}\right]^{1/q_1}+\right.$$

$$\left.+\left[\sum_{k_1=l_1}^{m_1}\left(\sum_{k_2=l_2}^{m_2}\left\|\iint_{J_\alpha+J_\beta+J_\gamma} g(x+t,y+\tau)\,D_{k_1,k_2}(t,\tau)dt\,d\tau\right\|_p^{q_2}\right)^{q_1/q_2}\right]^{1/q_1}\right\}.$$

Z [4] s. 26 wł. 15 i stąd, że dla małych kątów sint ≅ t

$$\left[\sum_{k_1=l_1}^{m_1}\left(\sum_{k_2=l_2}^{m_2}\left\|\int_{-\frac{1}{m_1}}^{\frac{1}{m_1}}\int_{-\frac{1}{m_2}}^{\frac{1}{m_2}} g(x+t,y+\tau)\, D_{k_1,k_2}(t,\tau)\,dt\,d\tau\right\|_p^{q_2}\right)^{q_1/q_2}\right]^{1/q_1} \leqslant$$

$$\leqslant \left\{\sum_{k_1=l_1}^{m_1}\left[\sum_{k_2=l_2}^{m_2}\left(\int_{-\frac{1}{m_1}}^{\frac{1}{m_1}}\int_{-\frac{1}{m_2}}^{\frac{1}{m_2}}\|g(x+t,y+\tau)\|_p\left|D_{k_1,k_2}(t,\tau)\right|dt\,d\tau\right)^{q_2}\right]^{q_1/q_2}\right\}^{1/q_1} \leqslant$$

$$\leqslant \sup_{(t,\tau)}\|g(x+t,y+\tau)\|_p\left\{\sum_{k_1=l_1}^{m_1}\left[\sum_{k_2=l_2}^{m_2}\left(\int_{-\frac{1}{m_1}}^{\frac{1}{m_1}}\int_{-\frac{1}{m_2}}^{\frac{1}{m_2}}\left|D_{k_1,k_2}(t,\tau)\right|dt\,d\tau\right)^{q_2}\right]^{q_1/q_2}\right\}^{1/q_1} \leqslant$$

$$\leqslant \sup_{(t,\tau)}\|g(x+t,y+\tau)\|_p\left\{\sum_{k_1=l_1}^{m_1}\left[\sum_{k_2=l_2}^{m_2}\left(\tfrac{1}{2}(2k_1+1)\,\tfrac{1}{2}(2k_2+1)\,\frac{2}{m_1}\,\frac{2}{m_2}\right)^{q_2}\right]^{q_1/q_2}\right\}^{1/q_1} \leqslant$$

$$\leqslant C_3(q_1,q_2)\, m_1^{1/q_1}\, m_2^{1/q_2}\sup_{(t,\tau)}\|g(x+t,y+\tau)\|_p$$

Z własności całek i nierówności (*) otrzymujemy również

$$\left[\sum_{k_1=l_1}^{m_1}\left(\sum_{k_2=l_2}^{m_2}\left\|\iint_{J_\alpha+J_\beta+J_\gamma} g(x+t,y+\tau)\, D_{k_1,k_2}(t,\tau)dt\,d\tau\right\|_p^{q_2}\right)^{q_1/q_2}\right]^{1/q_1} \leqslant$$

$$\leqslant C_4(q_1,q_2)\left\{\left[\sum_{k_1=l_1}^{m_1}\left(\sum_{k_2=l_2}^{m_2}\left\|\iint_{J_\alpha} g(x+t,y+\tau)\, D_{k_1,k_2}(t,\tau)dt\,d\tau\right\|_p^{q_2}\right)^{q_1/q_2}\right]^{1/q_1} +\right.$$

$$+\left[\sum_{k_1=l_1}^{m_1}\left(\sum_{k_2=l_2}^{m_2}\left\|\iint_{J_\beta} g(x+t;y+\tau)\, D_{k_1,k_2}(t,\tau)dt\,d\tau\right\|_p^{q_2}\right)^{q_1/q_2}\right]^{1/q_1} +$$

$$+\left[\sum_{k_1=l_1}^{m_1}\left(\sum_{k_2=l_2}^{m_2}\left\|\iint_{J_\gamma} g(x+t,y+\tau)\, D_{k_1,k_2}(t,\tau)dt\, d\tau\right\|_P^{q_2}\right)^{q_1/q_2}\right]^{1/q_1}\Bigg] =$$

$$= C_4(q_1,q_2)\left\{\sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3\right\}$$

$$\sum\nolimits_1 = \left[\sum_{k_1=l_1}^{m_1}\left(\sum_{k_2=l_2}^{m_2}\left\|\iint_{J_\alpha} g(x+t,y+\tau)\, D_{k_1,k_2}(t,\tau)dt\, d\tau\right\|_P^{q_2}\right)^{q_1/q_2}\right]^{1/q_1}$$

Ponieważ

$$D_{k_1,k_2}(t,\tau) = \tfrac{1}{4}(\sin k_1 t \operatorname{ctg}\tfrac{t}{2}\sin k_2\tau \operatorname{ctg}\tfrac{\tau}{2} + \sin k_1 t \operatorname{ctg}\tfrac{t}{2}\cos k_2\tau +$$

$$+ \cos k_1 t \sin k_2\tau \operatorname{ctg}\tfrac{\tau}{2} + \cos k_1 t \cos k_2\tau)$$

więc

$$\sum\nolimits_1 \le C_5(q_1,q_2)\left\{\left[\sum_{k_1=l_1}^{m_1}\left(\sum_{k_2=l_2}^{m_2}\left(\iint_{J_\alpha}\|g(x+t,y+\tau)\|_P \times\right.\right.\right.\right.$$

$$\left.\left.\left.\times \operatorname{ctg}\tfrac{t}{2}\operatorname{ctg}\tfrac{\tau}{2}\sin k_1 t \sin k_2\tau\, dt\, d\tau\right)^{q_2}\right)^{q_1/q_2}\right]^{1/q_1} +$$

$$+\left[\sum_{k_1=l_1}^{m_1}\left(\sum_{k_2=l_2}^{m_2}\left(\iint_{J_\alpha}\|g(x+t,y+\tau)\|_P \operatorname{ctg}\tfrac{t}{2}\sin k_1 t \cos k_2\tau\, dt\, d\tau\right)^{q_2}\right)^{q_1/q_2}\right]^{1/q_1} +$$

$$+\left[\sum_{k_1=l_1}^{m_1}\left(\sum_{k_2=l_2}^{m_2}\left(\iint_{J_\alpha}\|g(x+t,y+\tau)\|_P \operatorname{ctg}\tfrac{\tau}{2}\cos k_1 t \sin k_2\tau\, dt\, d\tau\right)^{q_2}\right)^{q_1/q_2}\right]^{1/q_1} +$$

$$+\left[\sum_{k_1=l_1}^{m_1}\left(\sum_{k_2=l_2}^{m_2}\left(\iint_{J_\alpha}\|g(x+t,y+\tau)\|_P \cos k_1 t \cos k_2\tau\, dt\, d\tau\right)^{q_2}\right)^{q_1/q_2}\right]^{1/q_1} =$$

$$= C_5(q_1,q_2)\left\{\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4\right\}.$$

Korzystając z nierówności (*) i Hausdorffa-Younga [7] otrzymujemy

$$\sigma_1 \leqslant C_6(q_1,q_2)\left[\left[\sum_{k_1=l_1}^{m_1}\left[\sum_{k_2=l_2}^{m_2}\left(\int_{\frac{1}{m_1}}^{\pi}\int_{\frac{1}{m_2}}^{\pi}\|g(x+t,y+\tau)\|_p \times\right.\right.\right.\right.$$

$$\left.\left.\left.\times \operatorname{ctg}\frac{t}{2}\operatorname{ctg}\frac{\tau}{2}\sin k_1t\,\sin k_2\tau\,dt\,d\tau\right)^{q_2}\right]^{q_1/q_2}\right]^{1/q_1} +$$

$$+\left[\sum_{k_1=l_1}^{m_1}\left[\sum_{k_2=l_2}^{m_2}\left(\int_{-\pi}^{-\frac{1}{m_1}}\int_{\frac{1}{m_2}}^{\pi}\|g(x+t,y+\tau)\|_p\operatorname{ctg}\frac{t}{2}\operatorname{ctg}\frac{\tau}{2}\sin k_1t\,\sin k_2\tau\,dt\,d\tau\right)^{q_2}\right]^{q_1/q_2}\right]^{1/q_1} +$$

$$+\left[\sum_{k_1=l_1}^{m_1}\left[\sum_{k_2=l_2}^{m_2}\left(\int_{-\pi}^{-\frac{1}{m_1}}\int_{-\pi}^{-\frac{1}{m_2}}\|g(x+t,y+\tau)\|_p\operatorname{ctg}\frac{t}{2}\operatorname{ctg}\frac{\tau}{2}\sin k_1t\,\sin k_2\tau\,dt\,d\tau\right)^{q_2}\right]^{q_1/q_2}\right]^{1/q_1} +$$

$$+\left[\sum_{k_1=l_1}^{m_1}\left[\sum_{k_2=l_2}^{m_2}\left(\int_{\frac{1}{m_1}}^{\pi}\int_{-\pi}^{-\frac{1}{m_2}}\|g(x+t,y+\tau)\|_p\operatorname{ctg}\frac{t}{2}\operatorname{ctg}\frac{\tau}{2}\sin k_1t\,\sin k_2\tau\,dt\,d\tau\right)^{q_2}\right]^{q_1/q_2}\right]^{1/q_1}\right] \leqslant$$

$$\leqslant C_6(q_1,q_2)\left[\left[\int_{\frac{1}{m_1}}^{\pi}\left[\int_{\frac{1}{m_2}}^{\pi}\left|\|g(x+t,y+\tau)\|_p\operatorname{ctg}\frac{t}{2}\operatorname{ctg}\frac{\tau}{2}\right|^{p_2}d\tau\right]^{p_1/p_2}dt\right]^{1/p_1} +\right.$$

$$+\left[\int_{-\pi}^{-\frac{1}{m_1}}\left[\int_{\frac{1}{m_2}}^{\pi}\left|\|g(x+t,y+\tau)\|_p\operatorname{ctg}\frac{t}{2}\operatorname{ctg}\frac{\tau}{2}\right|^{p_2}d\tau\right]^{p_1/p_2}dt\right]^{1/p_1} +$$

$$+\left[\int_{-\pi}^{-\frac{1}{m_1}}\left[\int_{-\pi}^{-\frac{1}{m_2}}\left|\|g(x+t,y+\tau)\|_p\operatorname{ctg}\frac{t}{2}\operatorname{ctg}\frac{\tau}{2}\right|^{p_2}d\tau\right]^{p_1/p_2}dt\right]^{1/p_1} +$$

$$+\left\{\int_{\frac{1}{m_1}}^{\pi}\left[\int_{-\pi}^{-\frac{1}{m_2}}\left|\|g(x+t,y+\tau)\|_p \operatorname{ctg}\frac{t}{2}\operatorname{ctg}\frac{\tau}{2}\right|^{p_2} d\tau\right]^{p_1/p_2} dt\right\}^{1/p_1} \leqslant$$

Ponieważ $\left|\operatorname{ctg}\frac{t}{2}\right| \leqslant \frac{\pi}{|t|}$ [6], więc

$$\sigma_1 = C_7(q_1,q_2) \sup_{(t,\tau)}\|g(x+t,y+\tau)\|_p \left[\int_{\frac{1}{m_1}}^{\pi}\left(\int_{\frac{1}{m_2}}^{\pi}\frac{1}{(t\tau)^{p_2}}\,d\tau\right)^{p_1/p_2} dt\right]^{1/p_1}$$

Stąd, że $\int_{\frac{1}{m_1}}^{\pi}\frac{dt}{t^{p_1}} < \frac{m_1^{p_1-1}}{p_1-1}$ jest

$$\sigma_1 \leqslant C_8(q_1,q_2) \sup_{(t,\tau)}\|g(x+t,y+\tau)\|_p \left[m_1^{p_1-1}\left(m_2^{p_2-1}\right)^{p_1/p_2}\right]^{1/p_1} =$$

$$= C_8(q_1,q_2) \sup_{(t,\tau)}\|g(x+t,y+\tau)\|_p\, m_1^{1/q_1}\, m_2^{1/q_2}$$

Analogicznie

$$\sigma_2 \leqslant C_9(q_1,q_2) \sup_{(t,\tau)}\|g(x+t,y+\tau)\|_p \left[\int_{\frac{1}{m_1}}^{\pi}\left(\int_{\frac{1}{m_2}}^{\pi}\frac{1}{|t|^{p_2}}\,d\tau\right)^{p_1/p_2} dt\right]^{1/p_1} \leqslant$$

$$\leqslant C_{11}(q_1,q_2)m_1^{1/q_1} \sup_{(t,\tau)}\|g(x+t,y+\tau)\|_p$$

$$\sigma_3 \leqslant C_{12}(q_1,q_2)m_2^{1/q_2} \sup_{(t,\tau)}\|g(x+t,y+\tau)\|_p$$

$$\sigma_4 \leqslant C_{13}(q_1,q_2) \sup_{(t,\tau)}\|g(x+t,y+\tau)\|_p$$

Stąd

$$\sum\nolimits_1 \leqslant C_5(q_1,q_2) \sup_{(t,\tau)}\|g(x+t,y+\tau)\|_p\Big\{C_8(q_1,q_2)m_1^{1/q_1}\, m_2^{1/q_2} +$$

$$+ C_{11}(q_1,q_2)m_1^{1/q_1} + C_{12}(q_1,q_2)m_2^{1/q_2} + C_{13}(q_1,q_2)\Big\} \leqslant$$

$$\leqslant C_{14}(q_1,q_2)m_1^{1/q_1} m_2^{1/q_2} \sup_{(t,\tau)} \|g(x+t,y+\tau)\|_p$$

$$\sum\nolimits_2 = \left[\sum_{k_1=l_1}^{m_1}\left(\sum_{k_2=l_2}^{m_2}\Big\|\iint_{J_\beta} g(x+t,y+\tau)\, D_{k_1,k_2}(t,\tau)\, dt\, d\tau\Big\|_p^{q_2}\right)^{q_1/q_2}\right]^{1/q_1} \leqslant$$

$$\leqslant \sup_{(t,\tau)}\|g(x+t,y+\tau)\|_p \left\{\sum_{k_1=l_1}^{m_1}\left[\sum_{k_2=l_2}^{m_2}\left(\iint_{J_\beta}|D_{k_1,k_2}(t,\tau)|\,dt\, d\tau\right)^{q_2}\right]^{q_1/q_2}\right\}^{1/q_1} =$$

$$= \sup_{(t,\tau)}\|g(x+t,y+\tau)\|_p \left\{\sum_{k_1=l_1}^{m_1}\left[\sum_{k_2=l_2}^{m_2}\left(\int_{-\frac{1}{m_1}}^{\frac{1}{m_1}}\int_{\frac{1}{m_2}}^{\pi}\left|\frac{\sin\frac{2k_1+1}{2}t}{2\sin\frac{t}{2}}\left(\frac{1}{2}\sin k_2\tau \operatorname{ctg}\frac{\tau}{2} + \right.\right.\right.\right.\right.$$

$$\left. + \frac{1}{2}\cos k_2\tau)\right|dt\,d\tau + \int_{-\frac{1}{m_1}}^{\frac{1}{m_2}}\int_{-\pi}^{-\frac{1}{m_2}}\left|\frac{\sin\frac{2k_1+1}{2}t}{2\sin\frac{t}{2}}\left(\frac{1}{2}\sin k_2\tau\operatorname{ctg}\frac{\tau}{2} + \frac{1}{2}\cos k_2\tau\right)\right|dt\,d\tau\Bigg)^{q_2}\Bigg]^{q_1/q_2}\Bigg\}^{1/q_1} \leqslant$$

$$\leqslant C_{15}(q_1,q_2)\sup_{(t,\tau)}\|g(x+t,y+\tau)\|_p\left(\left\{\sum_{k_1=l_1}^{m_1}\left[\sum_{k_2=l_2}^{m_2}\left[\frac{1}{2}(2k_1+1)\frac{2}{m_1}\left(\int_{\frac{1}{m_2}}^{\pi}\frac{1}{2}\operatorname{ctg}\frac{\tau}{2}\sin k_2\tau\, d\tau + \right.\right.\right.\right.\right.$$

$$\left. + \int_{\frac{1}{m_2}}^{\pi}\left|\frac{1}{2}\cos k_2\tau\right|d\tau\right)\Bigg]^{q_2}\Bigg]^{q_1/q_2}\Bigg\}^{1/q_1} +$$

$$+ \left\{\sum_{k_1=l_1}^{m_1}\left[\sum_{k_2=l_1}^{m_2}\left[\frac{1}{2}(2k_1+1)\,\frac{2}{m_1}\left(\int_{-\pi}^{-\frac{1}{m_2}}\frac{1}{2}\operatorname{ctg}\frac{\tau}{2}\sin k_2\tau\, d\tau + \int_{-\pi}^{-\frac{1}{m_2}}\left|\frac{1}{2}\cos k_2\tau\right|d\tau\right)\right]^{q_2}\right]^{q_1/q_2}\right\}^{1/q_1} \leqslant$$

$$\leqslant C_{16}(q_1,q_2)\sup_{(t,\tau)}\|g(x+t,y+\tau)\|_p\, m_1^{1/q_1}\left(\left\{\sum_{k_2=l_2}^{m_2}\left[\int_{\frac{1}{m_2}}^{\pi}\frac{1}{2}\operatorname{ctg}\frac{t}{2}\sin k_2\tau\, d\tau + \right.\right.\right.$$

$$+ \int_{\frac{1}{m_2}}^{\pi} \left|\frac{1}{2} \cos k_2\tau\right| d\tau\Bigg]^{q_2}\Bigg\}^{1/q_2} + \left\{\sum_{k_2=l_2}^{m_2}\left[\int_{-\pi}^{-\frac{1}{m_2}} \frac{1}{2} \operatorname{ctg}\frac{\tau}{2} \sin k_2\tau \, d\tau + \right.\right.$$

$$\left.\left. + \int_{-\pi}^{-\frac{1}{m_2}} \left|\frac{1}{2} \cos k_2\tau\right| d\tau\right]^{q_2}\right\}^{1/q_2} \leqslant$$

$$\leqslant C_{18}(q_1,q_2)\, m_1^{1/q_1} m_2^{1/q_2} \sup_{(t,\tau)} \|g(x+t,y+\tau)\|_p$$

Analogicznie

$$\sum\nolimits_3 \leqslant C_{19}(q_1,q_2)\, m_1^{1/q_1} m_2^{1/q_2} \sup_{(t,\tau)} \|g(x+t,y+\tau)\|_p$$

Zatem

$$\left\{\sum_{k_1=l_1}^{m_1}\left(\sum_{k_2=l_2}^{m_2} \|S_{k_1,k_2}(g;x,y)\|_p^{q_2}\right)^{q_1/q_2}\right\}^{1/q_1} \leqslant$$

$$\leqslant C(q_1,q_2)\, m_1^{1/q_1} m_2^{1/q_2} \sup_{(t,\tau)\in(-\pi;\pi;-\pi,\pi)} \|g(x+t,y+\tau)\|_p = C(q_1,q_2)\, m_1^{1/q_1} m_2^{1/q_2} \|g\|_p$$

<u>Lemat 6</u> (uogólniony lemat Leindlera)

Dla dowolnych n_1, n_2, m_1, m_2 i dowolnych $p_1 > 1$, $p_2 > 1$

$$\sum_{k_1=m_1}^{m_2} \sum_{k_2=n_1}^{n_2} \left|\Delta\lambda_{k_1}^{(m)} \Delta\lambda_{k_2}^{(n)}\right| \|S_{k_1,k_2}(f;x;y) - f(x,y\|_p \leqslant$$

$$\leqslant CE_{m_1,n_1}(f)_p \left\{m_2^{p_1-1} \sum_{k_1=m_1}^{m_2} \left|\Delta\lambda_{k_1}^{(m)}\right|^{p_1}\right\}^{1/p_1} \left\{n_2^{p_2-1} \sum_{k_2=n_1}^{n_2} \left|\Delta\lambda_{k_2}^{(n)}\right|^{p_2}\right\}^{1/p_2}$$

gdzie C zależy tylko od p_1, p_2.

Dowód

Dla $p_1 > 1$, $p_2 > 1$ i $\frac{1}{p_1} + \frac{1}{q_1} = 1$, $\frac{1}{p_2} + \frac{1}{q_2} = 1$ z nierówności Höldera

$$\sum_{k_1=m_1}^{m_2} \sum_{k_2=n_2}^{n_2} \left|\Delta\lambda_{k_1}^{(m)} \Delta\lambda_{k_2}^{(n)}\right| \left\|S_{k_1,k_2}(f;x,y) - f(x,y)\right\|_p =$$

$$= \sum_{k_1=m_1}^{m_2} \left(\left|\Delta\lambda_{k_1}^{(m)}\right| \sum_{k_2=n_1}^{n_2} \left|\Delta\lambda_{k_2}^{(n)}\right| \left\|S_{k_1,k_2}(f;x,y) - f(x,y)\right\|_p \right) \leqslant$$

$$\leqslant \left(\sum_{k_1=m_1}^{m_2} \left|\Delta\lambda_{k_1}^{(m)}\right|^{p_1} \right)^{1/p_1} \left(\sum_{k_1=m_1}^{m_2} \left| \sum_{k_2=n_1}^{n_2} \left|\Delta\lambda_{k_2}^{(n)}\right| \left\|S_{k_1,k_2}(f;x,y) - f(x,y)\right\|_p \right|^{q_1} \right)^{1/q_1} \leqslant$$

$$\leqslant \left(\sum_{k_1=m_1}^{m_2} \left|\Delta\lambda_{k_1}^{(m)}\right|^{p_1} \right)^{1/p_1} \left\{ \sum_{k_1=m_1}^{m_2} \left[\left(\sum_{k_2=n_1}^{n_2} \left|\Delta\lambda_{k_2}^{(n)}\right|^{p_2} \right)^{1/p_2} \left(\sum_{k_2=n_1}^{n_2} \left\|S_{k_1,k_2}(f;x,y) - \right.\right.\right.\right.$$

$$\left.\left.\left.\left. - f(x,y)\right\|_p^{q_2} \right)^{1/q_2} \right]^{q_1} \right\}^{1/q_1} =$$

$$= \left(\sum_{k_1=m_1}^{m_2} \left|\Delta\lambda_{k_1}^{(m)}\right|^{p_1} \right)^{1/p_1} \left(\sum_{k_2=n_1}^{n_2} \left|\Delta\lambda_{k_2}^{(n)}\right|^{p_2} \right)^{1/p_2} \left\{ \sum_{k_1=m_1}^{m_2} \left[\sum_{k_2=n_1}^{n_2} \left\|S_{k_1,k_2}(f;x,y) - \right.\right.\right.$$

$$\left.\left.\left. - f(x,y)\right\|_p^{q_2} \right]^{q_1/q_2} \right\}^{1/q_1} \leqslant$$

Jeśli $T^*_{m_1,n_1}(x,y)$ jest wielomianem trygonometrycznym najlepszego przybliżenia, to z [4] s. 104 wł. 5.1 i s. 105 wł. 5.2

$$\leqslant \left(\sum_{k_1=m_1}^{m_2} \left|\Delta\lambda_{k_1}^{(m)}\right|^{p_1} \right)^{1/p_1} \left(\sum_{k_2=n_1}^{n_2} \left|\Delta\lambda_{k_2}^{(n)}\right|^{p_2} \right)^{1/p_2} \left\{ \sum_{k_1=m_1}^{m_2} \left[\sum_{k_2=n_1}^{n_2} \left(\left\|S_{k_1,k_2}(T^*_{m_1,n_1} - f\, x,y)\right\|_p + \right.\right.\right.$$

$$\left.\left.\left. + \left\|T^*_{m_1,n_1}(x,y) - f(x,y)\right\|_p^{q_2} \right) \right]^{q_1/q_2} \right\}^{1/q_1} \leqslant$$

Wobec nierówności (*)

$$\leq C(q_1,q_2)\left(\sum_{k_1=m_1}^{m_2}\left|\Delta\lambda_{k_1}^{(m)}\right|^{p_1}\right)^{1/p_1}\left(\sum_{k_2=n_1}^{n_2}\left|\Delta\lambda_{k_2}^{(n)}\right|^{p_2}\right)^{1/p_2}\left\{\sum_{k_1=m_1}^{m_2}\left[\sum_{k_2=n_1}^{n_2}\left(\left\|S_{k_1,k_2}(T^*_{m_1,n_1}-\right.\right.\right.\right.$$

$$\left.\left.\left.\left.-f;x,y)\right\|_p^{q_2}+\left\|T^*_{m_1,n_1}(x,y)-f(x,y)\right\|_p^{q_2}\right)\right]^{q_1/q_2}\right|^{1/q_1}\leq$$

$$\leq C_1(q_1,q_2)\left(\sum_{k_1=m_1}^{m_2}\left|\Delta\lambda_{k_1}^{(m)}\right|^{p_1}\right)^{1/p_1}\left(\sum_{k_2=n_1}^{n_2}\left|\Delta\lambda_{k_2}^{(n)}\right|^{p_2}\right)^{1/p_2}\left\{\sum_{k_1=m_1}^{m_2}\left[\left(\sum_{k_2=n_1}^{n_2}\left\|S_{k_1,k_2}(T^*_{m_1,n_1}-\right.\right.\right.\right.$$

$$\left.\left.\left.\left.-f;x,y)\right\|_p^{q_2}\right)^{q_1/q_2}+\left(\sum_{k_2=n_1}^{n_2}\left\|T^*_{m_1,n_1}(x,y)-f(x,y)\right\|_p^{q_2}\right)^{q_1/q_2}\right]\right]^{1/q_1}\leq$$

$$\leq C_2(q_1,q_2)\left(\sum_{k_1=m_1}^{m_2}\left|\Delta\lambda_{k_1}^{(m)}\right|^{p_1}\right)^{1/p_1}\left(\sum_{k_2=n_1}^{n_2}\left|\Delta\lambda_{k_2}^{(n)}\right|^{p_2}\right)^{1/p_2}\left\{\left[\sum_{k_1=m_1}^{m_2}\left(\sum_{k_2=n_1}^{n_2}\left\|S_{k_1,k_2}(T^*_{m_1,n_1}-\right.\right.\right.\right.$$

$$\left.\left.\left.\left.-f;x,y)\right\|_p^{q_2}\right)^{q_1/q_2}\right]^{1/q_1}+\left[\sum_{k_1=m_1}^{m_2}\left(\sum_{k_2=n_1}^{n_2}\left\|T^*_{m_1,n_1}-f\right\|_p^{q_2}\right)^{q_1/q_2}\right]^{1/q_1}\right\}=$$

$$=C_3(p_1,p_2)\left(\sum_{k_1=m_1}^{m_2}\left|\Delta\lambda_{k_1}^{(m)}\right|^{p_1}\right)^{1/p_1}\left(\sum_{k_2=n_1}^{n_2}\left|\Delta\lambda_{k_2}^{(n)}\right|^{p_2}\right)^{1/p_2}\left\{\left[\sum_{k_1=m_1}^{m_2}\left(\sum_{k_2=n_1}^{n_2}\left\|S_{k_1,k_2}(T^*_{m_1,n_1}-\right.\right.\right.\right.$$

$$\left.\left.\left.\left.-f;x,y)\right\|_p^{q_2}\right)^{q_1/q_2}\right]^{1/q_1}+\left[\sum_{k_1=m_1}^{m_2}\left(\sum_{k_2=n_1}^{n_2}\left(E_{m_1,n_1}(f)_p\right)^{q_2}\right)^{q_1/q_2}\right]^{1/q_1}\right\}\leq$$

$$\leq C_4(p_1,p_2)\left(\sum_{k_1=m_1}^{m_2}\left|\Delta\lambda_{k_1}^{(m)}\right|^{p_1}\right)^{1/p_1}\left(\sum_{k_2=n_1}^{n_2}\left|\Delta\lambda_{k_2}^{(n)}\right|^{p_2}\right)^{1/p_2}\left[m_2^{1/q_1}\,n_2^{1/q_2}\left\|T^*_{m_1,n_1}-f\right\|_p+\right.$$

$$\left.+E_{m_1,n_1}(f)_p(n_2-n_1+1)^{1/q_2}(m_2-m_1+1)^{1/q_1}\right]\leq$$

$$\leqslant C(p_1,p_2)E_{m_1,n_1}(f)_p\left(m_2^{p_1-1}\sum_{k_1=m_1}^{m_2}\left|\Delta\Lambda_{k_1}^{(m)}\right|^{p_1}\right)^{1/p_1}\left[n_2^{p_2-1}\sum_{k_2=n_1}^{n_2}\left|\Delta\Lambda_{k_2}^{(n)}\right|^{p_2}\right]^{1/p_2}$$

Lemat 7

$$E_{m,n}(f)_p\leqslant C\,\omega\left(f;\frac{1}{m+1},\frac{1}{n+1}\right)_p$$

Prawdziwość lematu wynika natychmiast z wł. 5 s. 20 [4].

Twierdzenie 1

Jeśli spełniony jest dla pewnych $p_1>1$, $p_2>1$ warunek

$$\left[(n_1+1)^{p_1-1}\sum_{k_1=0}^{n_1}\left|\Delta\lambda_{k_1}^{(n_1)}\right|^{p_1}\right]^{1/p_1}\left[(n_2+1)^{p_2-1}\sum_{k_2=0}^{n_2}\left|\Delta\lambda_{k_2}^{(n_2)}\right|^{p_2}\right]^{1/p_2}\leqslant K,$$

gdzie K jest stałą, to dla

$$U_{n_1,n_2}(f;x,y;\Lambda)=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\left|\Delta\lambda_{k_1}^{(n_1)}\,\Delta\lambda_{k_2}^{(n_2)}\right|\left|S_{k_1,k_2}(f;x,y)-f(x,y)\right|$$

prawdziwa jest nierówność

$$\|U_{n_1,n_2}(f;x,y;\Lambda)\|_p\leqslant C\left(\frac{1}{n_1+1}\right)^{1/q_1}\left(\frac{1}{n_2+1}\right)^{1/q_2}\left[\sum_{k_1=0}^{n_1}\left[\sum_{k_2=0}^{n_2}E_{k_1,k_2}^{q_2}(f)_p\right]^{q_1/q_2}\right]^{1/q_1}$$

$$q_1=\frac{p_1}{p_1-1},\quad q_2=\frac{p_2}{p_2-1},$$

C jest stałą zależną od K, p_1 i p_2.

Dowód

Niech r_1 i r_2 będą liczbami całkowitymi takimi, że $2^{r_1-1}\leqslant n_1<2^{r_1}$ $2^{r_2-1}\leqslant n_2<2^{r_2}$, wtedy

$\|U_{n_1,n_2}(f;x,y;\Lambda)\|_p\leqslant$

$$\leqslant\sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\left|\Delta\lambda_{i_1}^{(n_1)}\,\Delta\lambda_{i_2}^{(n_2)}\right|\left\|S_{i_1,i_2}(f;x,y)-f(x,y)\right\|_p+\qquad\text{(a)}$$

$$+\sum_{s_1=0}^{r_1-2}\ \sum_{i_1=2^{s_1}+1}^{2^{s_1+1}}\ \sum_{s_2=0}^{r_2-2}\ \sum_{i_2=2^{s_2}+1}^{2^{s_2+1}} \left|\Delta\lambda_{i_1}^{(n_1)}\ \Delta\lambda_{i_2}^{(n_2)}\right| \left\|S_{i_1,i_2}(f;x,y)-f(x,y)\right\|_p + \tag{b}$$

$$+\sum_{i_1=2^{r_1-1}+1}^{n_1}\ \sum_{i_2=2^{r_2-1}+1}^{n_2} \left|\Delta\lambda_{i_1}^{(n_1)}\ \Delta\lambda_{i_2}^{(n_2)}\right| \left\|S_{i_1,i_2}(f;x,y)-f(x,y)\right\|_p + \tag{c}$$

$$+\sum_{i_1=0}^{1}\ \sum_{s_2=0}^{r_2-2}\ \sum_{i_2=2^{s_2}+1}^{2^{s_2+1}} \left|\Delta\lambda_{i_1}^{(n_1)}\ \Delta\lambda_{i_2}^{(n_2)}\right| \left\|S_{i_1,i_2}(f;x,y)-f(x,y)\right\|_p + \tag{d}$$

$$+\sum_{i_1=0}^{1}\ \sum_{i_2=2^{r_2-1}+1}^{n_2} \left|\Delta\lambda_{i_1}^{(n_1)}\ \Delta\lambda_{i_2}^{(n_2)}\right| \left\|S_{i_1,i_2}(f;x,y)-f(x,y)\right\|_p + \tag{e}$$

$$+\sum_{s_1=0}^{r_1-2}\ \sum_{i_1=2^{s_1}+1}^{2^{s_1+1}}\ \sum_{i_2=2^{r_2-1}+1}^{n_2} \left|\Delta\lambda_{i_1}^{(n_1)}\ \Delta\lambda_{i_2}^{(n_2)}\right| \left\|S_{i_1,i_2}(f;x,y)-f(x,y)\right\|_p + \tag{f}$$

$$+\sum_{s_1=0}^{r_1-2}\ \sum_{i_1=2^{s_1}+1}^{2^{s_1+1}}\ \sum_{i_2=0}^{1} \left|\Delta\lambda_{i_1}^{(n_1)}\ \Delta\lambda_{i_2}^{(n_2)}\right| \left\|S_{i_1,i_2}(f;x,y)-f(x,y)\right\|_p + \tag{g}$$

$$+\sum_{i_1=2^{r_1-1}+1}^{n_1}\ \sum_{i_2=0}^{1} \left|\Delta\lambda_{i_1}^{(n_1)}\Delta\lambda_{i_2}^{(n_2)}\right| \left\|S_{i_1,i_2}(f;x,y)-f(x,y)\right\|_p + \tag{h}$$

$$+\sum_{i_1=2^{r_1-1}+1}^{n_1}\ \sum_{s_2=0}^{r_2-2}\ \sum_{i_2=2^{s_2}+1}^{2^{s_2+1}} \left|\Delta\lambda_{i_1}^{(n_1)}\ \Delta\lambda_{i_2}^{(n_2)}\right| \left\|S_{i_1,i_2}(f;x,y)-f(x,y)\right\|_p + \tag{i}$$

W oparciu o lemat 6 można oszacować kolejne sumy

$$\text{(a)}\quad \sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\left|\Delta\lambda_{i_1}^{(n_1)}\,\Delta\lambda_{i_2}^{(n_2)}\right|\left\|S_{i_1,i_2}(f;x,y)-f(x,y)\right\|_p \leqslant$$

$$\leqslant C_1(p_1,p_2)\,E_{o,o}(f)_p\left[\sum_{i_1=0}^{1}\left|\Delta\lambda_{i_1}^{(n_1)}\right|^{p_1}\right]^{1/p_1}\left[\sum_{i_2=0}^{1}\left|\Delta\lambda_{i_2}^{(n_2)}\right|^{p_2}\right]^{1/p_2}\leqslant$$

$$\leqslant C_1(p_1,p_2)\,E_{o,o}(f)_p\,K\left(\frac{1}{n_1+1}\right)^{\frac{p_1-1}{p_1}}\left(\frac{1}{n_2+1}\right)^{\frac{p_2-1}{p_2}}\leqslant$$

$$\leqslant C_1\,K\left(\frac{1}{n_1+1}\right)^{1/q_1}\left(\frac{1}{n_2+1}\right)^{1/q_2}\left\{\sum_{k_1=0}^{n_1}\left[\sum_{k_2=0}^{n_2}E_{k_1,k_2}^{q_2}(f)_p\right]^{q_1/q_2}\right\}^{1/q_1},$$

bo $E_{o,o}(f)_p=\left[E_{o,o}^{q_2}\right]^{1/q_2}\leqslant\left[\sum_{k_2=0}^{n_2}E_{o,k_2}^{q_2}\right]^{1/q_2}$, stąd

$$\left[E_{o,o}^{q_2}\right]^{q_1/q_2}\leqslant\left[\sum_{k_2=0}^{n_2}E_{o,k_2}^{q_2}\right]^{q_1/q_2}\leqslant\sum_{k_1=0}^{n_1}\left[\sum_{k_2=0}^{n_2}E_{k_1,k_2}^{q_2}\right]^{q_1/q_2}$$

$$\text{(b)}\quad \sum_{s_1=0}^{r_1-2}\ \sum_{i_1=2^{s_1}+1}^{2^{s_1+1}}\ \sum_{s_2=0}^{r_2-2}\ \sum_{i_2=2^{s_2}+1}^{2^{s_2+1}}\left|\Delta\lambda_{i_1}^{(n_1)}\,\Delta\lambda_{i_2}^{(n_2)}\right|\left\|S_{i_1,i_2}(f;x,y)-f(x,y)\right\|_p\leqslant$$

$$\leqslant\sum_{s_1=0}^{r_1-2}\sum_{s_2=0}^{r_2-2}C_2(p_1,p_2)\,E_{2^{s_1}+1,2^{s_2}+1}(f)_p\left(2^{s_1+1}\right)^{\frac{p_1-1}{p_1}}\left(2^{s_2+1}\right)^{\frac{p_2-1}{p_2}}\times$$

$$\times\left(\sum_{i_1=2^{s_1}+1}^{2^{s_1+1}}\left|\Delta\lambda_{i_1}^{(n_1)}\right|^{p_1}\right)^{1/p_1}\left(\sum_{i_2=2^{s_2}+1}^{2^{s_2+1}}\left|\Delta\lambda_{i_2}^{(n_2)}\right|^{p_2}\right)^{1/p_2}\leqslant$$

$$\leqslant C_2 \sum_{s_1=0}^{r_1-2} \left(s^{s_1+1}\right)^{\frac{p_1-1}{p_1}} \left(\sum_{i_1=2^{s_1}+1}^{2^{s_1+1}} \left|\Delta\lambda_{i_1}^{(n_1)}\right|^{p_1} \right)^{1/p_1} \times$$

$$\times \left(\sum_{s_2=0}^{r_2-2} E^{q_2}_{2^{s_1}+1,2^{s_2}+1}(f)_p \, 2^{s_2+1} \right)^{1/q_2} \left(\sum_{s_2=0}^{r_2-2} \sum_{i_2=2^{s_2}+1}^{2^{s_2+1}} \left|\Delta\lambda_{i_2}^{(n_2)}\right|^{p_2} \right)^{1/p_2} \leqslant$$

$$\leqslant C_2 \left(\sum_{k_2=0}^{n_2} \left|\Delta\lambda_{k_2}^{(n_2)}\right|^{p_2} \right)^{1/p_2} \left(\sum_{s_1=0}^{r_1-2} \sum_{i_1=2^{s_1}+1}^{2^{s_1+1}} \left|\Delta\lambda_{i_1}^{(n_1)}\right|^{p_1} \right)^{1/p_1} \times$$

$$\times \left\{ \sum_{s_1=0}^{r_1-2} 2^{s_1+1} \left(\sum_{s_2=0}^{r_2-2} E^{q_2}_{2^{s_1}+1,2^{s_2}+1}(f)_p \, 2^{s_2+1} \right)^{q_1/q_2} \right\}^{1/q_1} \leqslant$$

$$\leqslant C_2 \left[\sum_{k_2=0}^{n_2} \left|\Delta\lambda_{k_2}^{(n)}\right|^{p_2} \right]^{1/p_2} \left[\sum_{k_1=0}^{n_1} \left|\Delta\lambda_{k_1}^{(n_1)}\right|^{p_1} \right]^{1/p_1} \times$$

$$\times \left\{ 4 \sum_{s_1=1}^{r_1-2} \sum_{i_1=2^{s_1-1}+1}^{2^{s_1}} \left[4 \sum_{s_2=1}^{r_2-2} \sum_{i_2=2^{s_2-1}+1}^{2^{s_2}} E^{q_2}_{i_1,i_2}(f)_p \right]^{q_1/q_2} \right\}^{1/q_1} \leqslant$$

$$\leqslant C_3 \, K\left(\frac{1}{n_1+1}\right)^{\frac{p_1-1}{p_1}} \left(\frac{1}{n_2+1}\right)^{\frac{p_2-1}{p_2}} \left\{ \sum_{k_1=0}^{n_1} \left[\sum_{k_2=0}^{n_2} E^{q_2}_{k_1,k_2}(f)_p \right]^{q_1/q_2} \right\}^{1/q_1}$$

(c) $$\sum_{i_1=2^{r_1-1}+1}^{n_1} \sum_{i_2=2^{r_2-1}+1}^{n_2} \left|\Delta\lambda_{i_1}^{(n_1)} \Delta\lambda_{i_2}^{(n_2)}\right| \left\| S_{i_1,i_2}(f;x,y) - f(x,y) \right\|_p \leqslant$$

$$\leq C_4 E_{2^{r_1-1}+1,\,2^{r_2-1}+1}(f)_p \left\{ n_1^{p_1-1} \sum_{i_1=2^{r_1-1}+1}^{n_1} \left|\Delta\lambda_{i_1}^{(n_1)}\right|^{p_1} \right\}^{1/p_1} \left\{ n_2^{p_2-1} \sum_{i_2=2^{r_2-1}+1}^{n_2} \left|\Delta\lambda_{i_2}^{(n_2)}\right|^{p_2} \right\}^{1/p_2} \leq$$

$$\leq C_4\, E_{2^{r_1}+1,\,2^{r_2-1}+1}(f)_p\, n_1^{1/q_1}\, n_2^{1/q_2}\, K\left(\frac{1}{n_1+1}\right)^{1/q_1} \left(\frac{1}{n_2+1}\right)^{1/q_2} \leq$$

$$\leq C_4\, K\left(\frac{1}{n_1+1}\right)^{1/q_1} \left(\frac{1}{n_2+1}\right)^{1/q_2} \left\{ \sum_{k_1=0}^{n_1} \left[\sum_{k_2=0}^{n_2} E_{k_1,k_2}^{q_2}(f)_p \right]^{q_1/q_2} \right\}^{1/q_1}$$

(d) $$\sum_{i_1=0}^{1} \sum_{s_2=0}^{r_2-2} \sum_{i_2=2^{s_2}+1}^{2^{s_2+1}} \left|\Delta\lambda_{i_1}^{(n_1)}\, \Delta\lambda_{i_2}^{(n_2)}\right| \left\| S_{i_1,i_2}(f;x,y) - f(x,y) \right\|_p \leq$$

$$\leq \sum_{s_2=0}^{r_2-2} C_5\, E_{o,2^{s_2}+1}(f)_p \left\{ \sum_{i_1=0}^{1} \left|\Delta\lambda_{i_1}^{(n_1)}\right|^{p_1} \right\}^{1/p_1} \left\{ \left(2^{s_2+1}\right)^{p_2-1} \sum_{i_2=2^{s_2}+1}^{2^{s_2+1}} \left|\Delta\lambda_{i_2}^{(n_2)}\right|^{p_2} \right\}^{1/p_2} \leq$$

$$\leq C_5 \left\{ \sum_{s_2=0}^{r_2-2} E_{o,2^{s_2}+1}(f)_p\, 2^{s_2+1} \right\}^{1/q_2} \left\{ \sum_{s_2=0}^{r_2-2} \sum_{i_2=2^{s_2}+1}^{2^{s_2+1}} \left|\Delta\lambda_{i_2}^{(n_2)}\right|^{p_2} \right\}^{1/p_2} \times$$

$$\times \left\{ \sum_{i_1=0}^{1} \left|\Delta\lambda_{i_1}^{(n_1)}\right|^{p_1} \right\}^{1/p_1} \leq C_6 \left\{ \sum_{k_1=0}^{n_1} \left|\Delta\lambda_{i_1}^{(n_1)}\right|^{p_1} \right\}^{1/p_1} \left\{ \sum_{k_2=0}^{n_2} \left|\Delta\lambda_{i_2}^{(n_2)}\right|^{p_2} \right\}^{1/p_2} \times$$

$$\times \left\{ \sum_{k_1=0}^{n_1} \left[\sum_{k_2=0}^{n_2} E_{k_1,k_2}^{q_2}(f)_p \right]^{q_1/q_2} \right\}^{1/q_1} \leq$$

$$\leq C_6\, K\left(\frac{1}{n_1+1}\right)^{1/q_1} \left(\frac{1}{n_2+1}\right)^{1/q_2} \left\{ \sum_{k_1=0}^{n_1} \left[\sum_{k_2=0}^{n_2} E_{k_1,k_2}^{q_2}(f)_p \right]^{q_1/q_2} \right\}^{1/q_1}$$

$$(e)\quad \sum_{i_1=0}^{1}\ \sum_{i_2=2^{r_2-1}+1}^{n_2}\left|\Delta\lambda_{i_1}^{(n_1)}\,\Delta\lambda_{i_2}^{(n_2)}\right|\left\|S_{i_1,i_2}(f;x,y)-f(x,y)\right\|_p\leqslant$$

$$\leqslant C_7\,E_{o,2^{r_2}+1}(f)_p\left[\sum_{i_1=0}^{1}\left|\Delta\lambda_{i_1}^{(n_1)}\right|^{p_1}\right]^{1/p_1}\left\{n_2^{p_2-1}\sum_{i_2=2^{r_2-1}+1}^{n_2}\left|\Delta\lambda_{i_2}^{(n_2)}\right|^{p_2}\right\}^{1/p_2}\leqslant$$

$$\leqslant C_7\,K\left(\frac{1}{n_1+1}\right)^{1/q_1}\left(\frac{1}{n_2+1}\right)^{1/q_2}\left\{\sum_{k_1=0}^{n_1}\left[\sum_{k_2=0}^{n_2}E^{q_2}_{k_1,k_2}(f)_p\right]^{q_1/q_2}\right\}^{1/q_1}$$

$$(f)\quad \sum_{s_1=0}^{r_1-2}\ \sum_{i_1=2^{s_1}+1}^{2^{s_1+1}}\ \sum_{i_2=2^{r_2-1}+1}^{n_2}\left|\Delta\lambda_{i_1}^{(n_1)}\,\Delta\lambda_{i_2}^{(n_2)}\right|\left\|S_{i_1,i_2}(f;x,y)-f(x,y)\right\|_p\leqslant$$

$$\leqslant C_8\sum_{s_1=0}^{r_1-2}E_{2^{s_1}+1,2^{r_2}+1}(f)_p\left[\left(2^{s_1+1}\right)^{p_1-1}\sum_{i_1=2^{s_1}+1}^{2^{s_1+1}}\left|\Delta\lambda_{i_1}^{(n_1)}\right|^{p_1}\right]^{1/p_1}\times$$

$$\times\left\{n_2^{p_2-1}\sum_{i_2=2^{r_2-1}+1}^{n_2}\left|\Delta\lambda_{i_2}^{(n_2)}\right|^{p_2}\right\}^{1/p_2}\leqslant$$

$$\leqslant C_8\,n^{1/q_2}\left[\sum_{i_2=2^{r-1}+1}^{n_2}\left|\Delta\lambda_{i_2}^{(n_2)}\right|^{p_2}\right]^{1/p_2}\left[\sum_{s_1=0}^{r_1-2}E^{q_2}_{2^{s_1}+1,2^{r_2-1}+1}(f)_p\,2^{s_1+1}\right]^{1/q_1}\times$$

$$\times\left[\sum_{s_1=0}^{r_1-2}\ \sum_{i_1=2^{s_1}+1}^{2^{s_1+1}}\left|\Delta\lambda_{i_1}^{(n_1)}\right|^{p_1}\right]^{1/p_1}\leqslant C_9\left[\sum_{k_2=0}^{n_2}\left|\Delta\lambda_{k_2}^{(n_2)}\right|^{p_2}\right]^{1/p_2}\times$$

$$\times\left\{\sum_{k_1=0}^{n_1}\left|\Delta\lambda_{k_1}^{(n_1)}\right|^{p_1}\right\}^{1/p_1}\left\{\sum_{k_1=0}^{n_1}\left[\sum_{k_2=0}^{n_2}E_{k_1,k_2}^{q_2}(f)_p\right]^{q_1/q_2}\right\}^{1/q_1}\leqslant$$

$$\leqslant C_9\, K\left(\frac{1}{n_1+1}\right)^{1/q_1}\left(\frac{1}{n_2+1}\right)^{1/q_2}\left\{\sum_{k_1=0}^{n_1}\left[\sum_{k_2=0}^{n_2}E_{k_1,k_2}^{q_2}(f)_p\right]^{q_1/q_2}\right\}^{1/q_1}$$

(g) analogicznie jak (d)
(h) analogicznie jak (e)
(i) analogicznie jak (f)
Z oszacowań (a), (b), (c), (d), (e), (f), (g) wynika teza

Jeśli spełniony jest warunek uogólniony Stieczkina, to prawdziwe są twierdzenia 2 i 3.

Twierdzenie 2

„Istnieje stała dodatnia K^*, zależna tylko od K i p_1, p_2, taka, że dla wszystkich naturalnych n_1 i n_2

$$A_{n_1,n_2}(\omega)\leqslant K^*\left(\frac{1}{n_1+1}\right)^{1/q_1}\left(\frac{1}{n_2+1}\right)^{1/q_2}\left\{\sum_{k_1=1}^{n_1+1}\left[\sum_{k_2=1}^{n_2+1}\omega^{q_2}\left(\frac{1}{k_1},\frac{1}{k_2}\right)_p\right]^{q_1/q_2}\right\}^{1/q_1},$$

$$q_1=\frac{p_1}{p_1-1},\quad q_2=\frac{p_2}{p_2-1}.$$

Dowód

Na podstawie lematu 2, 3, twierdzenia 1 i lematu 7

$$\left\|L_{n_1,n_2}(f;x,y;\Lambda)-f(x,y)\right\|_p=\left\|\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\Delta\lambda_{k_1}^{(n_1)}\Delta\lambda_{k_2}^{(n_2)}S_{k_1,k_2}-f(x,y)\right\|_p=$$

$$=\left\|\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\Delta\lambda_{k_1}^{(n_1)}\Delta\lambda_{k_2}^{(n_2)}\left[S_{k_1,k_2}-f(x,y)\right]\right\|_p=$$

$$=\left\{\int_{-\pi}^{\pi}\left[\int_{-\pi}^{\pi}\left|\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\Delta\lambda_{k_1}^{(n_1)}\Delta\lambda_{k_2}^{(n_2)}\left[S_{k_1,k_2}-f(x,y)\right]\right|^{p_2}dy\right]^{p_1/p_2}dx\right\}^{1/p_1}\leqslant$$

$$\leqslant \left\{ \int_{-\pi}^{\pi} \left[\int_{-\pi}^{\pi} \left(\sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \left| \Delta\lambda_{k_1}^{(n_1)} \Delta\lambda_{k_2}^{(n_2)} \right| \left| S_{k_1,k_2} - f(x,y) \right| \right)^{p_2} dy \right]^{p_1/p_2} dx \right\}^{1/p_1} =$$

$$= \left\| U_{n_1,n_2}(f;x,y;\Lambda) \right\|_p \leqslant$$

$$\leqslant C \left(\frac{1}{n_1+1}\right)^{1/q_1} \left(\frac{1}{n_2+1}\right)^{1/q_2} \left\{ \sum_{k_1=0}^{n_1} \left[\sum_{k_2=0}^{n_2} E_{k_1,k_2}^{q_2}(f)_p \right]^{q_1/q_2} \right\}^{1/q_1} \leqslant$$

$$\leqslant C^* \left(\frac{1}{n_2+1}\right)^{1/q_1} \left(\frac{1}{n_2+1}\right)^{1/q_2} \left\{ \sum_{k_1=0}^{n_1} \left[\sum_{k_2=0}^{n_2} \omega^{q_2}\left(f; \frac{1}{k_1+1}, \frac{1}{k_2+1}\right)_p \right]^{q_1/q_2} \right\}^{1/q_1}$$

a stąd teza.

TWIERDZENIE 3

Istnieje stała dodatnia K_1, zależna tylko od K i p_1, p_2, taka, że dla wszystkich naturalnych n_1 i n_2

$$A_{n_1,n_2}(F) \leqslant K_1 \left(\frac{1}{n_1+1}\right)^{1/q_1} \left(\frac{1}{n_2+1}\right)^{1/q_2} \left\{ \sum_{k_1=0}^{n_1} \left[\sum_{k_2=0}^{n_2} F_{k_1,k_2}^{q_2} \right]^{q_1/q_2} \right\}^{1/q_1} \leqslant$$

$$q_1 = \frac{p_1}{p_1-1}, \quad q_2 = \frac{p_2}{p_2-1}.$$

Dowód

Analogicznie jak w twierdzeniu 2 wykazuje się, że

$$\left\| L_{n_1,n_2}(f;x,y;\Lambda) - f(x,y) \right\|_p \leqslant$$

$$\leqslant K_1 \left(\frac{1}{n_1+1}\right)^{1/q_1} \left(\frac{1}{n_2+1}\right)^{1/q_2} \left\{ \sum_{k_1=0}^{n_1} \left[\sum_{k_2=0}^{n_2} E_{k_1,k_2}^{q_2}(f)_p \right]^{q_1/q_2} \right\}^{1/q_1} \leqslant$$

$$\leqslant K_1 \left(\frac{1}{n_1+1}\right)^{1/q_1} \left(\frac{1}{n_2+1}\right)^{1/q_2} \left\{ \sum_{k_1=0}^{n_1} \left[\sum_{k_2=0}^{n_2} F_{k_1,k_2}^{q_2} \right]^{q_1/q_2} \right\}^{1/q_1}.$$

Panu Profesorowi dr hab. Julianowi Musielakowi dziękuję za wielce pomocne sugestie przy opracowywaniu tematu.

LITERATURA

[1] Баскаков В.А.: Порядок приближения непрерывных функций линейными методами суммирования рядов Фурье. Труды международной конференции. Варна, 19-25 мая 1970. 7-18.

[2] Баскаков В.А.: О порядке приближения непрерывных функций некоторыми линейными средними их рядов Фурье. Математические заметки 9 (1971), 617--627.

[3] Качмаж С., Штейнгауз Г.: Теория ортогональных рядов. Москва 1958, 18.

[4] Kozłowska G.: O aproksymacji wielomianami trygonometrycznymi w przestrzeniach funkcji całkowalnych z mieszanymi potęgami. Praca doktorska. UAM, Poznań 1974.

[5] Von L. Leindler: Uber die Approximation instarken Sinne. Acta. Math. Acad. Scient. Hung. 16 1965, 255-262.

[6] Натансон И.П.: Конструктивная теория функций. Москва 1949, 190.

[7] Szelemeczka J.: On absolute convergence of multiple Fourier series of functions p-integrable with mixed powers. Roczniki PTM, Prace Matematyczne XVIII. 2, Warszawa 1975, 298.

О ПОРЯДКЕ ПРИБЛИЖЕНИЯ ФУНКЦИЙ $f(x,y) \in L^P(-\pi,\pi;-\pi,\pi)$ НЕКОТОРЫМИ ЛИНЕЙНЫМИ СРЕДНИМИ ИХ РЯДОВ ФУРЬЕ

Р е з ю м е

Рассматривается порядок приближения функций $f(x,y) \in L^P(-\pi,\pi;-\pi,\pi)$ линейными средними удовлетворяющими обобщенное условие С.Б. Стечкина.

APPROXIMATION OF FUNCTIONS $f(x,y) \in L^P(-\pi;\pi;-\pi,\pi)$ BY THEIR LINEAR MEANS OF FOURIER SERIES

S u m m a r y

In this paper there are given estimations of the approximation of the functions $f(x,y) \in L^P(-\pi,\pi;-\pi,\pi)$ by their linear means of Fourier series satisfying the generalized condition of Stieczkin.