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ON CONVERGENT SEQUENCES IN AUT(F)

Abstract. For a natural definition of convergency and an infinite product in Aut(F) the necessary and sufficient conditions for a sequence of automorphisms to have the limit and the product in Aut(F) are given here.

Let F be the free group of countable rank on free generators x_i, $i \in I$, where I is the set of natural numbers. Let $Aut_f(F)$ be the subgroup of Aut(F) generated by the elementary Nielsen transformations. By theorem 4.1 [1] the subgroup $Aut_f(F)$ is dense in Aut(F) in the sense that if $U_1, \dots, U_n$ are elements of F and $\alpha \in Aut(F)$ there exists $\beta \in Aut_f(F)$ such that $\alpha U_i = \beta U_i$, $i=1,2,\dots,n$.

Definition 1

A sequence of automorphisms α_K, $K \in I$ is called convergent to an endomorphism α if $\forall m \in I$, $\exists K_m \in I$ such that $\alpha_K x_i = \alpha x_i$ for $i \leq m$ whenever $K \geq K_m$. We shall write then $\alpha = \lim \alpha_K$. It does not depend obviously on choise of free generators.

From the above theorem it follows that $\forall \alpha \in Aut(F)$ there exists a sequence α_K, $K \in I$, $\alpha_K \in Aut_f F$ such that $\alpha = \lim \alpha_K$. It is enough to define α_K to coincide with α on x_i, $i \leq K$. Let us note however that a limit of a convergent sequence of automorphisms is not necessary an automorphism. Indeed, let α_K, $K \in I$ be defined by $\alpha_K x_i = x_i x_{i+1}$ for $i \leq K$; $\alpha_K x_i = x_i$ for $i > K$. Then $\alpha_K \in Aut_f(F)$, $K \in I$, $\lim \alpha_K = \alpha$, where $\alpha x_i = x_i x_{i+1}$ for all $i \in I$. Obviously α is not an epimorphism as in $gp\{\alpha x_i, i \in I\}$ every element has an even x-length.

Definition 2

Sequence π_K, $K \in I$, $\pi_K \in Aut(F)$ is called a sequence of partial products for some sequence α_K, $K \in I$, $\alpha_K \in Aut(F)$, if $\pi_1 = \alpha_1$, $\pi_K = \alpha_K \pi_{K-1}$, $K > 1$. (here $(\beta\alpha)x = \beta(\alpha x)$). If the sequence π_K, $K \in I$ is convergent then $\pi = \lim \pi_K$ is called a product of the sequence α_K, $K \in I$.

We will give now an example of a (convergent to identity) sequence α_K, $K \in I$, $\alpha_K \in Aut(F)$ that has no product. Let α_K be defined by $\alpha_K x_K = x_K x_{K+1}$, $\alpha_K x_i = x_i$, $i \neq K$, $i, K \in I$. Then $\pi_1 x_1 = \alpha_1 x_1 = x_1 x_2$. Suppose $\pi_{K-1} x_1 = x_1 x_2 \cdot \ldots \cdot x_K$ then $\pi_K x_1 = \alpha_K \pi_{K-1} x_1 = \alpha_K x_1 x_2 \cdot \ldots \cdot x_K = \alpha_K x_1 \alpha_K x_2 \cdot \ldots \cdot \alpha_K x_K = x_1 x_2 \cdot \ldots \cdot x_K x_{K+1}$.

It means that for $m = 1$ in def 1 K_1 does not exist, hence the sequence π_K, $K \in I$ is not convergent.

For automorphisms $\alpha_K, \beta_K, \pi_K, \pi_K^{-1}$ we denote by $A_K(m)$, $B_K(m)$, $P_K(m)$, $P_K^{-1}(m)$ respectively the sets of indexes $i \in I$ of generators x_i in a reduced form of words $\alpha_K x_m, \beta_K x_m, \pi_K x_m, \pi_K^{-1} x_m$. For any $S \subseteq I$ we denote $A_K(S) = \bigcup_m A_K(m)$, $m \in S$. Notice here that $\pi_K = \alpha_K \alpha_{K-1} \cdot \ldots \cdot \alpha_1$ follows

$$P_K(m) \subseteq A_K(A_{K-1}(\ldots(A_1(m))\ldots)) \tag{1}$$

In the set $I \times I$ define a partial order by $(K,i) < (l,j)$ if and only if $(K < l)$ $(i < j)$. Let α_K, $K \in I$ and β_K, $K \in I$ be two convergent sequences of automorphisms, $\lim \alpha_K = \alpha$, $\lim \beta_K = \beta$. Denote $\gamma_{(K,i)} = \alpha_K \beta_i$.

<u>Lemma 1</u>

For every increasing sequence $J \subseteq I \times I$ the sequence $\gamma_{(K,i)}$, $(K,i) \in J$ is convergent and $\lim \gamma_{(K,i)} = \lim \alpha_K \lim \beta_i = \alpha.\beta$.

For any $m \in I$ $\lim \beta_i = \beta$ implies the existence of i_m such that $\beta_i x_m = \beta x_m$ and hence $B_i(m) = B(m)$ whenever $i \geq i_m$. The set $B(m)$ is obviously finite then $\lim \alpha_K = \alpha$ implies the existence of $K(m)$ such that $\alpha_K x_j = \alpha x_j$ for all $j \in B(m)$ whenever $K \geq K(m)$. We will show now that $\gamma_{(K,i)} x_m = (\alpha\beta) x_m$ whenever $(K,i) \geq (K(m), i_m)$ which implies by def 1 our statement. Suppose $\beta x_m = W(x_j)$, $j \in B(m)$, then for $K \geq K(m)$, $i \geq i_m$ $\gamma_{(K,i)} x_m = (\alpha_K \beta_i) x_m = \alpha_K(\beta_i x_m) = \alpha_K(\beta x_m) = \alpha_K W(x_j) = W(\alpha_K x_j) = W(\alpha x_j) = \alpha W(x_j) = \alpha(\beta x_m) = (\alpha\beta) x_m$ which was required.

<u>Lemma 2</u>

The limit of a convergent sequence of automorphisms is a monomorphism.

Suppose $\lim \alpha_K = \alpha$, $\alpha_K \in \mathrm{Aut}(F)$ and for some $v \in F$, $\alpha v = 1$. Denote by S the set of indexes of generators in the reduced form of v. Then $\lim \alpha_K = \alpha$ implies the existence of $\bar{K}$ such that whenever $K \geq \bar{K}$ $\alpha_K x_i = \alpha x_i$ holds for all $i \in S$, which means $\alpha_K v = \alpha v = 1$ and hence $v = 1$.

<u>Lemma 3</u>

The limit of a convergent sequence α_K, $K \in I$ $\alpha_K \in \mathrm{Aut}(F)$ is an automorphism if and only if the sequence α_K^{-1}, $K \in I$ is convergent. Moreover, it follows that $\lim \alpha_K^{-1} = \alpha^{-1}$.

Suppose that $\lim \alpha_K = \alpha$, $\lim \alpha_K^{-1} = \beta$, then by lemma 1 for $\beta_K = \alpha_K^{-1}$, $K \in I$ we obtain $\lim \gamma_{(K,K)} = \lim(\alpha_K \beta_K) = e = \lim \alpha_K \lim \beta_K = \alpha, \beta$ where $\alpha_K \beta_K = \alpha_K \alpha_K^{-1} = e$ is an identical automorphism. In the same way we can state that $e = \beta\alpha$, hence $\alpha^{-1} = \beta = \lim \alpha_K^{-1}$ and $\alpha \in \mathrm{Aut}(F)$. Conversely if $\lim \alpha_K = \alpha \in \mathrm{Aut}(F)$ we will show that $\alpha^{-1} = \lim \alpha_K^{-1}$. Using lemma 1 for

$\beta_i = \alpha^{-1}$, $i \in I$, $\lim(\alpha_K\alpha^{-1}) = \lim \alpha_K \cdot \lim \alpha^{-1} = \alpha\alpha^{-1} = e$. By def 1 it implies that $\forall_m$, $\exists K_m$ such that whenever $K > K_m$, $(\alpha_K\alpha^{-1})x_m = x_m$. Now $\alpha^{-1}x_m = \alpha_K^{-1}(\alpha_K\alpha^{-1})x_m = \alpha_K^{-1} x_m$ whenever $K \geqslant K_m$ which means $\lim \alpha_K^{-1} = \alpha^{-1}$.

Corollary $\lim \alpha_K = e$ if and only if $\lim \alpha_K^{-1} = e$.

Lemma 4

The product π of a sequence α_K, $K \in I$, $\alpha_K \in \mathrm{Aut}(F)$ if existent is an automorphism if and only if $\lim \alpha_K = e$.

Let $\lim \pi_K = \pi$ be an automorphism, then by lemma 3 $\lim \pi_K^{-1} = \pi^{-1}$. Since $\alpha_K = \pi_K \cdot \pi_{K-1}^{-1}$ it follows by lemma 1 that $\lim \alpha_K = \lim \pi_K \cdot \lim \pi_{K-1}^{-1} = \pi \cdot \pi^{-1} = e$ which was required. Conversely since $\lim \alpha_K = e$ it follows by previous corollary that $\lim \alpha_K^{-1} = e$ and by def 1 $\forall m$ $\exists K_m$ that for $K \geqslant K_m$ $\alpha_K^{-1} x_m = x_m$ holds. Now for $K \geqslant K_m$ $\pi_K^{-1}x_m = (\pi_{K-1}^{-1}\alpha_K^{-1})x_m = \pi_{K-1}^{-1}(\alpha_K^{-1}x_m) = \pi_{K-1}^{-1}x_m$. It means that sequence π_K^{-1}, $K \in I$ is convergent, hence by lemma 3 π is an automorphism.

Notice here that the condition of $\lim \alpha_K = e$ is not sufficient for the existence of the product as it is shown in example following def 2.

Lemma 5

If $\lim \alpha_K = e$, $\alpha_K \in \mathrm{Aut}(F)$ and there exists convergent subsequence $\pi_{K(i)}$, $i \in I$ in a sequence of partial products, then there exists $\pi = \lim \pi_{K(i)} = \lim \pi_K$.

Every π_K can be written in the form of $\pi_K = \alpha_K \cdot \ldots \cdot \alpha_{K(i)+1}\pi_{K(i)}$ where $K(i+1) > K \geqslant K(i)$.

If we denote $\lim \pi_{K(i)} = \pi$ then by def 1 it implies that $\forall m$ $\exists K_m(i)$ such that $\pi_{K(i)}x_m = \pi x_m$ whenever $K(i) \geqslant K_m(i)$ and hence $P_{K(i)}(m) = P(m)$ for $K(i) \geqslant K_m(i)$. Since $\lim \alpha_K = e$ and $P(m)$ is finite there exists by def 1 a $\bar{K}$ such that $\alpha_K x_i = x_i$ for all $i \in P(m)$ whenever $K > \bar{K}$. If we denote that $K_m = \max(K_m(i), \bar{K})$ then for all $K \geqslant K_m$ it follws that $\pi_K x_m = \alpha_K \ldots \alpha_{K(i)+j}\pi_{K(i)}x_m = \alpha_K \cdot \ldots \cdot \alpha_{K(i)+1}\pi x_m = \pi x_m$ which means that $\pi = \lim \pi_K$ is the product of the sequence α_K, $K \in I$.

To find the necessary and sufficient conditions for sequence α_K, $K \in I$ having product we need two more definitons.

For a given sequence α_K, $K \in I$ and $m \in I$ we denote $T_1(m) = A_1(m)$, $T_K(m) = A_K(T_{K-1}(m))$ and $T(m) = \bigcup_{K \in I} T_K(m)$.

Definition 3

Sequence α_K, $K \in I$ is called regular if $\forall m \in J$ $T(m)$ is finite.

Lemma 6

Let sequence α_K, $K \in I$ be regular and $\lim \alpha_K = e$, it warrants existence of a product π for this sequence.

We have to show that the sequence π_K, $K \in I$ of partial products is convergent. Since $\pi_K = \alpha_K \alpha_{K-1} \cdot \ldots \cdot \alpha_1$ we obtain from (1)

$$P_K(m) \subseteq T_K(m) \subseteq T(m) \tag{2}$$

We see now that $\pi_{K+1} x_m = (\alpha_{K+1} \pi_K) x_m = \alpha_{K+1}(\pi_K x_m)$ where by (2) $\pi_K x_m$ is a word in x_i, $i \in T(m)$ Since $T(m)$ is finite and $\lim \alpha_K = e$ it follws that there exists a $\bar{K}$ such that whenever $K > \bar{K}$ $\alpha_{K+1} x_i = x_i$ holds for all $i \in T(m)$. Hence $\pi_{K+1} x_m = \pi_K x_m$ whenever $K > \bar{K}$ which means convergency of the sequence π_K, $K \in I$.

Notice that we did not need α_K, $K \in I$ to be automorphism in lemmas 5, 6. Notice as well that condition $\lim \alpha_K = e$ in lemma 6 cannot be weakened for the convergency of α_K, $K \in I$ only. Indeed consider automorphism α, where $\alpha x_1 = x_1 x_2$, $\alpha x_i = x_i$, $i \neq 1$. We will define suquence α_K, $K \in I$ by $\alpha_K = \alpha$, $K \in I$. This sequence is obviously convergent to α and is regular since $T(1) = \langle 1,2 \rangle$. $T(m) = \langle m \rangle$ for $m \neq 1$. However $\pi_K x_1 = \alpha^K x_1 = x_1 x_2^K$ implies that sequence π_K $K \in I$ is not convergent.

We will now state that the condition for sequence α_K, $K \in I$ to be regular is not a necessary one for existence of a product of this sequence. Let us define automorphism β_l, $l \in I$ by $\beta_l x_l = x_l x_{l+1}$, $\beta_l x_i = x_i$, $i \neq l$

Let now $\alpha_1 = e$, $\alpha_K = \beta_l$ for $K = 2l$ and $\alpha_K = \beta_l^{-1}$ for $K = 2l + 1$.

Obviously $\lim \alpha_K = e$ Consider the sequence of partial products for α_K, $K \in I$. $\pi_K = \alpha_K = \beta_l$ for $K = 2l$ and $\pi_K = \alpha_K \pi_{K-1} = \beta_l^{-1} \cdot \beta_l = e$ for $K = 2l+1$. Naturally this sequence is convergent to e and hence α_K, $K \in I$ has a product e. Show however that the sequence α_K, $K \in I$ is not regular since e.g. $T(1)$ is infinite. Using induction shows that $T_{2l}(1) = \langle 1,2,\ldots,$ $,\ldots,l+1 \rangle$. Indeed, for $l = 1$ $T_2(1) = A_2(A_1(1)) = B_1(1) = \langle 1,2 \rangle$. Let $T_{2(l-1)}(1) = \langle 1,2,\ldots,l \rangle$ then $T_{2l}(1) = A_{2l}(A_{2l-1}(T_{2(l+1)}(1))) =$
$= A_{2l}(A_{2l-1}(\langle 1,2,\ldots,l \rangle)) = B_l(B_{l-1}^{-1} \langle 1,2,\ldots,l \rangle)) = B_e(\langle 1,2,\ldots,l \rangle) =$
$= \langle 1,2,\ldots, l+1 \rangle$. hence our sequence α_K, $K \in I$ is not regular.

Definition 4

We say that sequence β_K, $K \in I$ is obtained from sequence α_K, $K \in I$ by blocking if for some increasing sequence of naturals $r(1) < r(2) < r(3) < \ldots$ $\beta_1 = \alpha_{r(1)} \ldots \alpha_2 \alpha_1$, $\beta_K = \alpha_{r(K)} \ldots \alpha_{r(K-1)+1}$, $K > 1$, $K \in I$.

The sequence of partial products π_K, $K \in I$ for β_K, $K \in I$ is obviously a subsequence of that for α_K, $K \in I$ as $\pi_K = \pi_{r(K)}$, $K \in I$ and $\lim \pi_K = \pi$ follows $\lim \pi'_K = \pi$.

Lemma 7

Suppose that sequence α_K, $K \in I$, $\alpha_K \in \mathrm{Aut}(F)$ has a product $\pi \in \mathrm{Aut}(F)$, then π is a product of a regular sequence β_K, $K \in I$, $\beta_K \in \mathrm{Aut}(F)$ obtained by blocking α_K, $K \in I$.

To prove the statement we need to define sequence $r(1) < r(2) < \ldots$ and to check the regularity of obtained β_K, $K \in I$. By def 1, $\lim \pi_K = \pi$ implies the existence of $K(1)$ such that for $K \geq K(1)$ $\pi_K x_1 = \pi x_1$ and hence $P_K(1) = P(1)$. According to lemma 4 we have $\lim \alpha_K = e$ then by def 1 such a $\bar{K}(1)$ exists that for $K > \bar{K}(1)$ $\alpha_K x_i = x_i$ for all $i \in P(1)$ and hence $A_K(i) = \langle i \rangle$, $i \in P(1)$. Let us denote $r(1) = \max(K(1), \bar{K}(1))$ then $\beta = \pi_{r(1)}$, $T_1(1) = B_1(1) = P_{r(1)}(1) = P(1)$. Moreover $A_K(i) = \langle i \rangle$ for. $i \in T_1(1)$, $K > r(1)$. Let $r(1) < r(2) < \ldots < r(m)$ be defined so that for $s \leq m$ two conditions hold:

1° $T_m(s) = T_s(s)$

2° $A_K(i) = \langle i \rangle$ for $i \in T_s(s)$, $K > r(s)$

We can see that independently from the choice of $r(m+1)$ 1° holds for s $m+1$. Indeed for $s = m+1$ it is trivial. For $s \leq m$ using 1°, (1) and 2° $T_{m+1}(s) = B_{m+1}(T_m(s)) = B_{m+1}(T_s(s)) \leq A_{r(m+1)}(\ldots(A_{r(m)+1}(T_s(s))\ldots)) = T_s(s)$.

We have to define $r(m+1)$ to have 2° for $s = m+1$. If $r(m+1)$ is defined, then $\beta_{m+1} = \pi_{r(m+1)}\pi^{-1}_{r(m)}$ and by (1) $T_{m+1}(m+1) = B_{m+1}(T_m(m+1)) \subseteq P_{r(m+1)}(P^{-1}_{r(m)}(T_m(m+1)))$. The set $P^{-1}_{r(m)}(T_m(m+1)) = s$ is known and is finite. By def 1 $\lim \pi_K = \pi$ implies the existence of $K(m+1)$ such that if $r(m+1) \geq K(m+1)$ then $\pi_{r(m+1)} x_s = \pi x_s$ for all $s \in S$ hence $T_{m+1}(m+1) \subseteq P_{r(m+1)}(S) = P(S)$ for any $r(m+1) \geq K(m+1)$ Now by def 1 $\lim \alpha_K = e$ implies the existence of some $\bar{K}(m+1)$ such that for $K > \bar{K}(m+1)$ $\alpha_K x_i = x_i$ for $i \in P(S)$. Let us define $r(m+1) = \max(K(m+1), \bar{K}(m+1), r(m)+1)$ the for $K \geq r(m+1)$ $A_K(i) = \langle i \rangle$, $i \in P(S) \geq T_{m+1}(m+1)$ We can conclude now that 1° holds for every $m \in I$ which imply that $T(s) = T_s(s)$, $s \in I$ is finite, i. e. sequence β_K, $K \in I$ is regular.

Theorem

An automorphism π is a product of a sequence α_K, $K \in I$, $\alpha_K \in \mathrm{Aut}(F)$ if and only if $\lim \alpha_K = e$ and sequence α_K, $K \in I$ can be blocked to a regular on with the product π.

If $\pi \in \mathrm{Aut}(F)$ is a product of α_K, $K \in I$, $\alpha_K \in \mathrm{Aut}(F)$ then by lemma 4 $\lim \alpha_K = e$ and by lemma 7 sequence α_K, $K \in I$ can be blocked to a regular sequence β_K, $K \in I$ with the product π.

Conversely, let $\lim \alpha_K = e$ and a regular sequence β_K $K \in I$ obtained by blocking α_K, $K \in I$ have the product π then by lemma 5 and 4 it follows that π is the product for α_K, $K \in I$ and $\pi \in \mathrm{Aut}(F)$.

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O ZBIEŻNYCH CIĄGACH AUTOMORFIZMÓW

S t r e s z c z e n i e

W grupie automorfizmów grupy wolnej F przeliczalnej rangi wprowadza się pojęcie zbieżności ciągu automorfizmów i pojęcie iloczynu nieskończonego. Dane są warunki konieczne i wystarczające dla istnienia granicy i iloczynu dowolnego ciągu automorfizmów.

О СХОДИМОСТИ ПОСЛЕДОВАТЕЛЬНОСТИ АВТОМОРФИЗМОВ

Р е з ю м е

В группе автоморфизмов свободной группы счетного ранга определяется сходимость последовательности автоморфизмов и их бесконечное произведение. Даются необходимые и достаточные условия для существования предела и произведения для последовательности автоморфизмов.