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AN INFINITE NIELSEN TRANSFORMATION AS A LIMIT
OF A SEQUENCE OF FINITE ONES

Abstract. We consider here a natural notion of infinite Nielsen transformation in a free group of countable rank. For a sequence of transformations the notion of convergency is defined and it is shown that every infinite Nielsen transformation is a limit of a sequence of Nielsen transformations of finite rank. An example shows that the converse statement is not true. The necessary and sufficient conditions for transformation to be infinite Nielsen are given.

The notion of infinite Nielsen transformation could enable to extend many known results for infinite case if analogue of theorem 3.1.1 for infinite case were proved.

Let F be the free group of countable rank on free generators x_i , $i = 1, 2, \dots$. An endomorphism α of F is determined by its effect on the x_i , $i = 1, 2, \dots$. If $\alpha(x_i) = y_1(x_j)$, and y_1 is an automorphism, then y_1, y_2, \dots freely generate F . Conversely, if (y_1, y_2, \dots) is a set of words, then the mapping $\bar{\alpha}: x_i \rightarrow y_1(x_j)$ determines a unique endomorphism α of F , namely $\alpha: u(x_i) \rightarrow u(y_1(x_j))$. Moreover, if the set (y_1, y_2, \dots) freely generates F , then α is an automorphism. The mapping $\bar{\alpha}$ is called a substitution on the x_i , $i = 1, 2, \dots$, and if y_1, y_2, \dots freely generate F , the substitution is called a free substitution on x_i , $i = 1, 2, \dots$. If we have a second substitution $\bar{\beta}: x_i \rightarrow z_1(x_k)$, then the product $\bar{\beta}\bar{\alpha}$ is the substitution

$$\bar{\beta}\bar{\alpha}: x_i \rightarrow y_1(z_1(x_k))$$

It is clear that $\bar{\beta}\bar{\alpha} = \bar{\alpha}\bar{\beta}$. In particular, the group of automorphisms of F is isomorphic to the group of free substitutions on x_i , $i = 1, 2, \dots$.

Any infinite countable set of freely reduced words in F we call a chain, and denote, for example, by $W = (w_1, w_2, \dots)$. The chain of generators we denote by $V = (x_1, x_2, \dots)$. With each substitution $\bar{\alpha}: x_i \rightarrow y_1(x_j)$ we may associate not only an endomorphism α of F but also a transformation N_α of chains in F , that carries a chain $W = (w_1, w_2, \dots)$ into $N_\alpha W = (N_\alpha w_1, N_\alpha w_2, \dots)$, where $N_\alpha w_i$ is a reduced word obtained from $y_1(x_j)$ by substituting w_j for x_j . Shortly, N_α replaces the i -th entry by the y_1 - combination of the other entries. If $\bar{\alpha}$ is a free substitution. Moreover if $\bar{\beta}$ is then N is called an infinite Nielsen transformation. Moreover if $\bar{\beta}$ is

another substitution then $N_\alpha N_\beta = N_{\beta\alpha}$. It follows that the correspondence $\alpha \mapsto N_\alpha$ is an anti-isomorphism of the group of automorphisms of F onto the group of infinite Nielsen transformations. A transformation N is called invertible if there exists a transformation M such that $NM = MN = E$, where E is the identity transformation. It follows now that a transformation N is an infinite Nielsen transformation if and only if it is invertible. A standard Nielsen transformation N of rank n will be considered as an infinite Nielsen transformation of rank n i.e. for $i > n$, $Nx_i = x_i$, and for $i < n$ $Nx_i = y_j(x_i)$ with $j < n$.

Let N_1, N_2, \dots be a sequence of transformations. We say that this sequence is convergent to a transformation N , if for every m there exists k_m such that for all k satisfying $k \geq k_m$, we have $N_k x_m = Nx_m$.

Theorem

N is an infinite Nielsen transformation if and only if N is a limit of a sequence N_k , $k = 1, 2, \dots$ of Nielsen transformations of finite ranks and sequence N_k^{-1} , $k = 1, 2, \dots$ is convergent.

Proof

Let N be an infinite Nielsen transformation and $Nx_i = y_i$, $i = 1, 2, \dots$. Denote $X_n = gp(x_1, \dots, x_n)$, $Y_n = gp(y_1, \dots, y_n)$. Let l_1, L_1 , ($l_1 = 1$) be successively defined as the minimal numbers satisfying $Y_{l_1} \subset X_{L_1} \subset Y_{l_1+1} \subset \dots$. Every subset (y_1, \dots, y_{l_1}) can be completed by Kurosh's theorem [2] to a set of free generators in X_{L_1} , $k \geq 1$, then $X_{L_1} = gp(x_1, \dots, x_{L_1}) = gp(y_1, \dots, y_{l_1}, a_{l_1+1}, \dots, a_{L_1})$. Denote by N_k , $k \geq 1$ the Nielsen transformation of rank L_k that carries (x_1, \dots, x_{L_k}) into $(y_1, \dots, y_{l_k}, a_{l_k+1}, \dots, a_{L_k})$, $k \geq 1$. The sequence N_k , $k = 1, 2, \dots$ is obviously convergent to N which was required. We shall show now that the sequence N_k^{-1} is also convergent, namely that $N^{-1} = \lim N_k^{-1}$. Let $N^{-1} x_m = w(x_1, x_2, \dots, x_s)$ be a word in x_i , $i < s$. By convergence of the sequence N_k , $k = 1, 2, \dots$ to N there exists k_s such that, for $k \geq k_s$, we have $N_k x_i = Nx_i$ for $i < s$. Then each transformation $T_k = N^{-1} N_k$ ($k \geq k_s$) carries x_m into $T_k x_m = (N^{-1} N_k) x_m = w(N_k x_1, \dots, N_k x_s) = w(Nx_1, \dots, Nx_s) = N^{-1}(Nx_m) = x_m$. Now $N^{-1} x_m = T_k(N^{-1} x_m) = N_k^{-1} x_m$ for $k \geq k_s$, so that $N^{-1} = \lim N_k^{-1}$ which completes the proof of necessity.

Let now $N = \lim N_k$ for a sequence N_k , $k = 1, 2, \dots$ of finite Nielsen transformations and the sequence N_k , $k = 1, 2, \dots$ is convergent to a transformation M . To show that N is an infinite Nielsen transformation it is enough to establish that N is invertible. From our conditions we get that for every m there exist k_m , l_m such that, for $k \geq k_m$, $l \geq l_m$, we have $N_k x_m = N_l x_m = Nx_m$ and N_k^{-1} and N_l^{-1} and $N_l^{-1} x_m = Mx_m$. Choose subsequence T_m , $m = 1, 2, \dots$, where $T_m = N_{\max(k_m, l_m)}$ then $T_m x_i = Nx_i$, $T_m^{-1} x_i = Mx_i$ ($i < m$).

Suppose that m first words Nx_1, Nx_2, \dots, Nx_m (and hence $T_m x_1, \dots, T_m x_m$) are expressed through x_i , $i < s$, for some s . We can always take $s > m$. Then $T_s^{-1} : (x_1, x_2, \dots) \rightarrow (Nx_1, \dots, Nx_m, \dots, Nx_s, y_{s+1}, \dots)$, $T_m T_s^{-1} : (x_1, x_2, \dots) \rightarrow (N(Nx_1), \dots, N(Nx_m), z_{m+1}, \dots)$. Since $s > m$, the first m elements $T_m x_1, \dots, T_m x_m$ and $T_s x_1, \dots, T_s x_m$ coincide, hence the same is true for $T_m w_1, \dots, T_m w_m$ and $T_s w_1, \dots, T_s w_m$ for any $w = (w_1, w_2, \dots)$. Take $w = -T_s^{-1} v$; then $T_m w = T_m (T_s^{-1} v)$ and $T_s w = T_s (T_s^{-1} v) = v$; hence $N(Nx_i) = x_i$ for $i < m$. Since this is true for arbitrary m we get $NM = E$, and in a similar way we get $NM' = E$, which shows that N is an infinite Nielsen transformation with $N^{-1} = M$. The proof is complete.

We give here an example of convergent sequence of Nielsen transformations of finite rank with a limit that is not an infinite Nielsen transformation. Define N_k , $k \geq 1$, by

$$N_k v = (x_1 x_2, x_2 x_3, \dots, x_k x_{k+1}, x_{k+1}, \dots), \quad k \geq 1.$$

Then $\lim N_k = N$ for $Nv = (x_1 x_2, x_2 x_3, \dots)$. The correspondent substitution $\bar{\alpha} : x_i \rightarrow x_i x_{i+1}$, $i = 1, 2, \dots$ is not a free one because in a gp($x_1 x_2, x_2 x_3, \dots$) every element has an even x -length, hence N is not an infinite Nielsen transformation.

REFERENCES

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- [2] Magnus W., Karrass A. and D. Solitar: Combinatorial group theory, Interscience, New York 1966.

NIESKOŃCZONE PRZEKSZTAŁCENIE NIELSENA JAKO GRANICA CIĄGU PRZEKSZTAŁCENI SKOŃCZONEJ RANDZE

S t r e s z c z e n i e

W pracy uogólnia się pojęcie przekształcenia Nielsena na przypadek grupy wolnej o randze nieskończonej i wprowadza się nieskończone przekształcenie Nielsena. Określa się również zbieżność ciągu przekształceń. Pokazane jest, że każde nieskończone przekształcenie Nielsena jest granicą ciągu przekształceń Nielsena o skończonej randze.

Przykład pokazuje, że warunek ten nie jest wystarczający. W twierdzeniu podane są warunki konieczne i wystarczające na to, by przekształcenie było nieskończonym przekształceniem Nielsena.

БЕЗКОНЕЧНОЕ ПРЕОБРАЗОВАНИЕ НИЛЬСЕНА

Резюме

В работе вводится понятие бесконечного преобразования Нильсена, которое является естественным обобщением преобразований конечного ранга. Вводится также понятие сходимости последовательности преобразований. Показано, что каждое бесконечное преобразование Нильсена является пределом последовательности преобразований конечного ранга, однако, как следует из примера, не каждое преобразование, являющееся пределом последовательности преобразований Нильсена конечного ранга является бесконечным преобразованием Нильсена. В теореме даны необходимые и достаточные условия для того чтобы преобразование было преобразованием Нильсена.