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DIRECT AND INDIRECT JACKSON'S TYPE THEOREMS IN SOME FRÉCHET FUNCTIONS SPACES

Summary. In this paper some direct and indirect Jackson's type theorems are proved, in which functions from a Frèchet space are approximated by trigonometric polynomials.

Designations

A is a set in R^1.

L^{Φ} is a Frèchet space with the F-norm $\|F\| = \int_A \Phi(|f(x)|)dx < \infty$.

$$E_n(f)_{\Phi} = \inf_{T_n} \|F-T_n\|.$$

$$\Delta_h^k f(x) = \sum_{i=0}^{k} (-1)^{k+i} C_k^i f(x+ih).$$

For $\delta > 0$

$$\omega_k(\delta, f)_{\Phi} = \sup_{|h|\leq\delta} \|\Delta_h^k f(x)\|.$$

1. Introduction

Let $\Phi(s)$ be a continuous and non-decreasing function, defined for $s \geqslant 0$, such that $\Phi(0) = 0$, $\Phi(s) > 0$ for $s > 0$ and $\Phi(s_1+s_2) \leqslant \Phi(s_1) + \Phi(s_2)$

Φ space is called a linear space L^{Φ}, measurable and finite almost everywhere functions $f(x)$ such that $\|f\| = \int_A \Phi(|f(x)|)dx < \infty$ with metric $\varrho(f,\varphi) = \|f-\varphi\|$ $(f,\varphi \in L^{\Phi})$.

Every Φ space is a Frèchet space with the F-norm $\|f\|$ ([1] p. 584).

Let $G = \{g_k\}$ be linear independent system functions from L^{Φ} and $f \in L^{\Phi}$.

Let's denote the best approximation of $f \in L^{\Phi}$ by polynomials of degree $\leqslant n$ from G system by

$$E_n(f,G)_{\Phi} = E_n(f)_{\Phi} = \inf_{\{a_k\}} \| f - \sum_{k=0}^{n} a_k g_k \|.$$

$E_n(f)_{\Phi}$ of cours decreases with respect to n.

<u>Lemma 1</u>

For any function $f \in L^{\Phi}$ there exists an element of the best approximation

$$P_n = \sum_{k=0}^{n} c_k g_k \quad \text{i.e.} \quad E_n(f)_{\Phi} = \|f - P_n\|.$$

<u>Proof</u>

The lemma is a consequence of the theorem p. 590 [1].

<u>Lemma 2</u>

If $f_i \in L^{\Phi}$ for $i = 1,2$, then $E_n(f_1+f_2)_{\Phi} \leqslant E_n(f_1)_{\Phi} + E_n(f_2)_{\Phi}$.

<u>Proof</u>

Let $P_i = \sum_{k=0}^{n} c_k^{(i)} g_k$ be polynomial of the best approsimation of the function f_i $(i = 1,2)$, then $E_n(f_1+f_2)_{\Phi} \leqslant \|(f_1+f_2) - (P_1+P_2)\| \leqslant \|f_1-P_1\| + \|f_2-P_2\| = E_n(f_1)_{\Phi} + E_n(f_2)_{\Phi}$.

When $\delta > 0$ and $f \in L^{\Phi}$, then $\omega_k(\delta,f)_{\Phi} = \sup_{|h| \leqslant \delta} \|\Delta_h^k f(x)\|$, where

$$\Delta_h^k f(x) = \sum_{i=0}^{k} (-1)^{k+i} C_k^i f(x+ih).$$

<u>Lemma 3</u>

Let $f \in L^{\Phi}$, then for $\delta \geqslant 0$ and $\eta \geqslant 0$

$$0 = \omega(0,f)_{\Phi} \leqslant \omega(\delta,f)_{\Phi} \leqslant \omega(\delta+\eta,f)_{\Phi} \leqslant \omega(\delta,f)_{\Phi} + \omega(\eta,f)_{\Phi}.$$

Hance in $n \in N$ we get

$$\omega(n\delta, f)_{\Phi} \leqslant n\omega(\delta, f)_{\Phi}.$$

Lemma 4

Let $f, g \in L^{\Phi}$, then $\omega_k(\delta, f+g)_{\Phi} \leqslant \omega_k(\delta, f)_{\Phi} + \omega_k(\delta, g)_{\Phi}$.

Proof

$$\|\Delta_h^k(f+g)\| = \|\sum_{i=0}^{k} (-1)^{k+i} C_k^i [f(x+ih) + g(x+ih)]\| \leqslant$$

$$\leqslant \|\Delta_h^k f\| + \|\Delta_h^k g\|.$$

Lemma 5

Let f be A-periodic function or $A = (-\infty, \infty)$, then $\omega_k(\delta, f)_{\Phi} \leqslant 2^k \|f\|$.

Proof

$$\omega_k(\delta, f)_{\Phi} = \sup_{|h| \leqslant \delta} \|\Delta_h^k f(x)\| = \sup_{|h| \leqslant \delta} \int_A \Phi(|\sum_{i=0}^{k} (-1)^{k+i} C_k^i f(x+ih)|) \, dx =$$

$$= \int_A \Phi(|\sum_{i=0}^{k} (-1)^{k+i} C_k^i f(x)|) dx \leqslant \int_A \Phi(|f(x)| 2^k) dx \leqslant$$

$$\leqslant 2^k \int_A \Phi(|f(x)|) dx = 2^k \|f\|.$$

2. Approximation in the L^{Φ} spaces

Let $f \in L^{\Phi}$ be 2π-periodic function and $T_n(x)$ be a trigonometric polynomial of degree $\leqslant n$ $(n \geqslant 0)$.

If $\lim_{t \to \infty} \Phi(t) = \infty$ and $\psi(\alpha) = \sup_{s \geqslant 0} \frac{\Phi(\alpha s)}{\Phi(s)}$ is denoted for $\alpha \geqslant 0$, then the theorems 1 and 2 are true in L^{Φ} space.

Theorem 1

If $f \in L^{\Phi}[0,2\pi]$ and for any $p \in N$ $\int_1^{\infty} \psi(\frac{1}{x^p})dx < \infty$, then for $n \geqslant 1$ $E_{n-1}(f)_{\Phi} \leqslant C_{\Phi}\,\omega(\frac{\pi}{n},f)_{\Phi}$

(C_{Φ} is a constant depending on Φ).

Proof

It is known from the theorem 3 [2], that there exists function $l_n(x) = c_i$, $x \in \left[\frac{\pi(i-1)}{n}, \frac{\pi i}{n}\right)$, $i = 1,\dots,2n$, such that

$$\int_0^{2\pi} \Phi(|f(x) - l_n(x)|)dx \leqslant 4\,\omega(\tfrac{\pi}{n},f)_{\Phi} \tag{1}$$

When we denote $c_{2n+1} = c_1$, then

$$\frac{\pi}{n}\sum_{i=1}^{2n} \Phi(|c_{i+1} - c_i|) = \int_0^{2\pi} \Phi(|l_n(x+\tfrac{\pi}{n}) - l_n(x)|)dx =$$

$$= \int_0^{2\pi} \Phi(|-f(x) + f(x+\tfrac{\pi}{n}) - f(x+\tfrac{\pi}{n}) + l_n(x+\tfrac{\pi}{n}) + f(x) - l_n(x)|)dx \leqslant$$

$$\leqslant \int_0^{2\pi} \Phi(|\Delta^1_{\frac{\pi}{n}} f(x)|)dx + 2\int_0^{2\pi} \Phi(|f(x) - l_n(x)|)dx$$

Hance applying (1) we obtain

$$\frac{\pi}{n}\sum_{i=1}^{2n}\Phi(|c_{i+1} - c_i|) \leqslant 9\omega(\tfrac{\pi}{n},f)_{\Phi} \tag{2}$$

Let $H_y(x)$ be 2π-periodic Heaviside function for $y \in (0,2\pi)$ i.e.

$$H_y(x) = \begin{cases} 0 & \text{for } 0 \leqslant x < y \\ 1 & \text{for } y \leqslant x < 2\pi \end{cases}$$

Then ([2] p. 647 lemma 3) for every $y \in (0,2\pi)$, determined $m \in N$ and $n \geqslant 1$ there exists a trigonometric polynomial $T_y(x)$ of degree $\leqslant (n-1)n$ such that, for $x \in [0,2\pi]$

$$|H_y(x) - T_y(x)| \leqslant C_m \left\{ \frac{1}{(n|\sin \frac{x}{2}|+1)^{2m-1}} + \frac{1}{(n|\sin \frac{x-y}{2}|+1)^{2m-1}} \right\}$$

Of course $l_n(x) = c_1 + \sum_{i=1}^{2n-1} H_{x_i}(x)(c_{i+1} - c_i)$

for almost all $x \in [0,2\pi]$, $x_i = \frac{\pi i}{n}$ $(i = 1,\dots,2n)$.

Let for every n

$$T_n(x) = c_1 + \sum_{i=1}^{2n-1} T_{x_i}(x)(c_{i+1} - c_i)$$

then

$$\int_0^{2\pi} \Phi(|l_n(x) - T_n(x)|)dx \leqslant$$

$$\leqslant \int_0^{2\pi} \Phi\Big(\sum_{i=1}^{2n-1} |c_{i+1} - c_i|\,|H_{x_i}(x) - T_{x_i}(x)|\Big)dx \leqslant$$

$$\leqslant \sum_{i=1}^{2n-1} \Phi(|c_{i+1} - c_i|) \int_0^{2\pi} \psi(|H_{x_i}(x) - T_{x_i}(x)|)dx$$

If $\psi(\alpha) = \sup\limits_{s>0} \frac{\Phi(\alpha s)}{\Phi(s)}$ then $\psi(\alpha\beta) \leqslant \psi(\alpha)\psi(\beta)$ and

$$\int_0^{2\pi} \Phi(|l_n(x) - T_n(x)|)dx \leqslant$$

$$\leqslant \sum_{i=1}^{2n} \Phi(|c_{i+1} - c_i|) \int_0^{2\pi} \psi(|c_m|)\, \psi\left\{\frac{1}{(n|\sin \frac{x}{2}|+1)^{2m-1}} + \right.$$

$$\left. + \frac{1}{(n|\sin \frac{x-x_i}{2}|+1)^{2m-1}} \right\} dx \leqslant$$

$$\leqslant 2\sum_{i=1}^{2n}\Phi(|c_{i+1}-c_i|)\int_0^{2\pi}\Psi(|c_m|)\psi\left\{\frac{1}{(n\left|\sin\frac{x}{2}\right|+1)^{2m-1}}\right\}dx =$$

$$= 2\psi(|c_m|)\sum_{i=1}^{2n}\Phi(|c_{i+1}-c_i|)\int_0^{2\pi}\psi\left(\frac{1}{(n\sin\frac{x}{2}+1)^{2m-1}}\right)dx \qquad (3)$$

Because $\psi(\alpha)$ is a non-decreasing function we get

$$\int_0^{2\pi}\psi\left(\frac{1}{(n\sin\frac{x}{2}+1)^{2m-1}}\right)dx \leqslant$$

$$\leqslant \frac{\pi}{n}\Big[\psi(1)+\psi\left(\frac{1}{2^{2m-1}}\right)+\psi\left(\frac{1}{3^{2m-1}}\right)+\dots+\psi\left(\frac{1}{n^{2m-1}}\right) +$$

$$+\psi\left(\frac{1}{n^{2m-1}}\right) + \dots + \psi\left(\frac{1}{2^{2m-1}}\right)+\psi(1)\Big] =$$

$$= \frac{2\pi}{n}\left[\psi(1)+\psi\left(\frac{1}{2^{2m-1}}\right)+\dots+\psi\left(\frac{1}{n^{2m-1}}\right)\right] = \frac{2\pi}{n}S_n.$$

We denote by S_n the n-th partial sum of the series $\sum_{i=1}^{\infty}\psi\left(\frac{1}{i^{2m-1}}\right)$ where $\psi\left(\frac{1}{i^{2m-1}}\right)\geqslant 0$ for $i = 1,2,\dots$ and $\psi(1)\geqslant\psi\left(\frac{1}{2^{2m-1}}\right)\geqslant\dots$

Choosing m such that $2m-1\geqslant p$, $\int_1^{\infty}\psi\left(\frac{1}{x^{2m-1}}\right)dx < \infty$ is obtained and consequently this series is convergent.

Hence

$$\sum_{i=1}^{\infty}\psi\left(\frac{1}{i^{2m-1}}\right) = A_\psi \geqslant S_n$$

and

$$\int_0^{2\pi}\psi\left(\frac{1}{(n\sin\frac{x}{2}+1)^{2m-1}}\right)dx \leqslant \frac{2\pi}{n}A_\psi$$

Hence by (3) and (2)

$$\int_0^{2\pi} \Phi(|l_n(x) - T_n(x)|)dx \leqslant$$

$$\leqslant B_\psi \frac{\pi}{n} \sum_{i=1}^{2n} \Phi(|c_{i+1} - c_i|) \leqslant B_\psi\, 9\omega(\tfrac{\pi}{n}, f)_\Phi$$

Hence by (1)

$$\int_0^{2\pi} \Phi(|f(x) - T_n(x)|)dx \leqslant$$

$$\leqslant \int_0^{2\pi} \Phi(|f(x) - l_n(x)|)dx + \int_0^{2\pi} \Phi(|l_n(x) - T_n(x)|)dx \leqslant$$

$$\leqslant 4\omega(\tfrac{\pi}{n}, f)_\Phi + 9\, B_\psi \omega(\tfrac{\pi}{n}, f)_\Phi = C_\Phi \omega(\tfrac{\pi}{n}, f)_\Phi$$

In that way we get the estimation as desired.

Lemma 6

Let ψ be such that for any $p \in N$ $\sum_{i=1}^{\infty} \psi(\frac{1}{i^p}) < \infty$, then if $k \in N$, $h \in R$ and $n \geqslant 1$ we get

$$\|\Delta_h^k T_n\| \leqslant C_{\Phi,k}\, n^k \cdot \psi^k(|h|)\, \|T_n\|$$

where $T_n = T_n(x)$ is a trigonometric polynomial of degree $\leqslant n$

Proof

Let $$S_{nl}(x) = \left(\frac{\sin \frac{(n+1)x}{2}}{(n+1)\sin \frac{x}{2}} \right)^{2l}$$

where $l \in N$

From [2] p. 651 we get

$$|\Delta_h^1 T_n(x)| \leqslant n(l+1)|h| \left\{|T_n(x+h)| + \sum_{i=0}^{2n(l+1)} S_{nl}(x_i)|T_n(x+x_i)|\right\}$$

where $x_i = \frac{2\pi i}{2n(l+1)+1}$ $\quad i = 0,1,\dots,2n(l+1)$

Hence

$$\|\Delta_h^1 T_n(x)\| \leqslant \int_0^{2\pi} \Phi\Big[n(l+1)|h| \Big\{|T_n(x+h)| +$$

$$+ \sum_{i=0}^{2n(l+1)} S_{nl}(x_i)|T_n(x+x_i)|\Big\}\Big] dx \leqslant$$

$$\leqslant n(l+1)\psi(|h|)\left\{\int_0^{2\pi} \Phi(|T_n(x+h)|)dx +\right.$$

$$\left. + \sum_{i=0}^{2n(l+1)} \psi(S_{nl}(x_i)) \int_0^{2\pi} \Phi(|T_n(x+x_i)|)dx\right\} \leqslant$$

$$\leqslant n(l+1)\psi(|h|)\left\{1 + \sum_{i=0}^{2n(l+1)} \psi(S_{nl}(x_i))\right\} \|T_n\|$$

Since for $\frac{x_i}{2} \in [0,\frac{\pi}{2})$

$$(n+1)\sin\frac{x_i}{2} = (n+1)\sin\frac{2\pi i}{2n(l+1)+1} \geqslant \frac{4(n+1)i}{2n(l+1)+1} \geqslant \frac{i}{l+1}$$

then

$$\left(\frac{\sin\frac{(n+1)x_i}{2}}{(n+1)\sin\frac{x_i}{2}}\right)^{2l} \leqslant \frac{1}{\left[(n+1)\sin\frac{x_i}{2}\right]^{2l}} \leqslant \left(\frac{l+1}{i}\right)^{2l}.$$

Therefore

$$\sum_{i=0}^{2n(l+1)} \psi[S_{nl}(x_i)] \leqslant \psi(1) + 2\sum_{i=1}^{n(l+1)} \psi\left[\left(\frac{l+1}{i}\right)^{2l}\right] \leqslant$$

$$< \psi(1) + 2(l+1)^{2l}\sum_{i=0}^{\infty} \psi\left(\frac{1}{i^{2l}}\right).$$

Choosing $l=l_\psi$ such that the last series is convergent $\sum_{i=0}^{2n(l+1)} \psi[S_{nl}(x_i)] < C_\psi$ is obtained.

Hence

$$\|\Delta_h^1 T_n(x)\| \leqslant C_\psi^* \, n\psi(|h|)\,\|T_n\|$$

and

$$\|\Delta_h^k T_n\| = \|\Delta_h^1(\Delta_h^{k-1} T_n)\| \leqslant n\, C_\psi^* \psi(|h|)\,\|\Delta_h^{k-1} T_n\| \leqslant$$

$$\leqslant n^k C_{\Phi,k}\,\psi^k(|h|)\,\|T_n\|.$$

Corollary 1

Let ψ be such that for any $p \in N$ $\sum_{i=0}^{\infty} \psi\left(\frac{1}{i^p}\right) < \infty$, then if $k \in N$, $n \geqslant 1$ we get $\omega_k(\delta, T_n)_\Phi \leqslant C_{\Phi,k}\, n^k \psi^k(\delta)\,\|T_n\|$.

Theorem 2

Let $f \in L^\Phi$ $k, n \in N$ and ψ is such that for any $p \in N$ $\sum_{i=0}^{\infty} \psi\left(\frac{1}{i^p}\right) < \infty$ then

$$\omega_k\left(\frac{1}{n}, f\right)_\Phi \leqslant C^*_{\Phi,k}\,\psi^k\left(\frac{1}{n}\right)\sum_{\nu=0}^{n} (\nu+1)^{k-1} E_\nu(f)_\Phi + 2^k E_n(f).$$

Proof

Let $t_n(x)$ be a polynomial of the best approximation of degree $\leqslant n$, then for integer $m \geqslant 0$, from the lemmas 4 and 5 we obtain

$$\omega_k(\tfrac{1}{n},f)_\Phi \leqslant \omega_k(\tfrac{1}{n},f-t_{2^{m+1}})_\Phi + \omega_k(\tfrac{1}{n},\ t_{2^{m+1}})_\Phi \leqslant$$

$$\leqslant 2^k\ E_{2^{m+1}}(f) + \omega_k(\tfrac{1}{n},\ t_{2^{m+1}})_\Phi \tag{4}$$

From corollary 1 we get

$$\omega_k(\tfrac{1}{n},t_{2^{m+1}})_\Phi \leqslant \omega_k(\tfrac{1}{n},t_{2^0} - t_0)_\Phi + \sum_{\nu=0}^{m} \omega_k(\tfrac{1}{n},t_{2^{\nu+1}} - t_{2^\nu})_\Phi \leqslant$$

$$\leqslant 2\ C_{\Phi,k}\ \psi^k(\tfrac{1}{n}) \left\{ E_0(t)_\Phi + \sum_{\nu=0}^{m} 2^{(\nu+1)k}\ E_{2^\nu}(f)_\Phi \right\} \tag{5}$$

Since for $\nu \geqslant 1$

$$2^{2k} \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{k-1}\ E_\mu(f)_\Phi = 2^{2k}[(2^{\nu-1}+1)^{k-1}\ E_{2^{\nu-1}+1} +$$

$$+ (2^{\nu-1}+2)^{k-1}\ E_{2^{\nu-1}+2} + \ldots + (2^\nu)^{k-1}\ E_{2^\nu}] \geqslant 2^{2k}(2^{\nu-1})^{k-1}\ 2^{\nu-1}\ E_{2^\nu} =$$

$$= 2^{(\nu+1)k}\ E_{2^\nu}(f)_\Phi$$

From the above and (5), it comes out that

$$\omega_k(\tfrac{1}{n},t_{2^{m+1}})_\Phi \leqslant C^*_{\Phi,k}\psi^k(\tfrac{1}{n}) \left\{ E_0(f)_\Phi + E_1(f)_\Phi + \right.$$

$$\left. + \sum_{\nu=1}^{m} \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{k-1} E_\mu(f)_\Phi \right\} \leqslant$$

$$\leqslant C^*_{\Phi,k}\ \psi^k(\tfrac{1}{n}) \sum_{\nu=0}^{2^m} (\nu+1)^{k-1}\ E_\nu(f)_\Phi$$

Choosing m such that $2^m < n \leqslant 2^{m+1}$ from the above and (4)

$$\omega_k(\tfrac{1}{n},f)_\Phi \leqslant 2^k E_{2^{m+1}}(f)_\Phi + C^*_{\Phi,k}\,\psi^k(\tfrac{1}{n}) \sum_{\nu=0}^{2^m} (\nu+1)^{k-1} E_\nu(f)_\Phi \leqslant$$

$$\leqslant C^*_{\Phi,k}\,\psi^k(\tfrac{1}{n}) \sum_{\nu=0}^{n} (\nu+1)^{k-1} E_\nu(f)_\Phi + 2^k E_n(f)_\Phi$$

results.

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ПРЯМЫЕ И ОБРАТНЫЕ ТЕОРЕМЫ ТИПА ЖДЕКСОНА
В НЕКОТОРЫХ ФУНКЦИОНАЛЬНЫХ ПРОСТРАНСТВАХ ФРЕШЕ

Р е з ю м е

Для приближения функций тригонометрическими полиномами в пространстве Фреше доказываются некоторые теоремы типа Джексона.

PROSTE I ODWROTNE TWIERDZENIA TYPU JACKSONA W PEWNYCH FUNKCYJNYCH PRZESTRZENIACH FRÉCHETA

S t r e s z c z e n i e

W pracy wykazano pewne twierdzenia typu Jacksona, aproksymując funkcje w przestrzeni Frécheta wielomianami trygonometrycznymi.

Wpłynęło do Redakcji 27.IV.1980 r.

Recenzent

Prof. dr hab. Julian Musielak