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COUNTABLE SUBMODELS IN ZF AND SEPARABILITY*

Summary. A modification of the notion of elementary submodel is defined in the paper for certain model X of the power of continuum. There is proved in ZF (in section 2) that there exists a sufficiently large class of countable submodels of X, that are elementary in the sense, mentioned above. An example of application is sketched in section 3.

1. COUNTABLE ELEMENTARY SUBMODELS IN ZF

The downward Skolem-Löwenheim theorem on countable elementary submodels depends essentially on the axiom of choice. More exactly, it depends on the countable axiom of choice, asserting the existence of the function of choice for any countable family of non-empty sets. For several concrete structures, however, we can prove the existence of countable elementary submodels in ZF. We quote some well-known and important examples below.

Denote by R the set of all reals, by Q the set of all rationals, by A the set of all real algebraic numbers and use the usual symbols $+, \cdot, <$ to denote algebraic operations and the natural ordering in R,Q,A. Add to the language L of the elementary first order predicate calculus with equality as the only non-logical symbols the sign $<$, the signs $<, +$, and the signs $<, +, \cdot$ respectively for the cases 1.1, 1.2, 1.3 below. Then the following are theorems of ZF.

1.1. The ordered set $\langle Q, < \rangle$ is an elementary submodel of $\langle R, < \rangle$.

1.2. The ordered additive group $\langle Q, <, + \rangle$ is an elementary submodel of $\langle R, <, + \rangle$.

1.3. The ordered field $\langle A, <, +, \cdot \rangle$ is an elementary submodel of $\langle R, <, +, \cdot \rangle$.

All those submodels are countable.

We sketch briefly the uniform method of proof in ZF of 1.1-1.3 Consider, at first, the following property of the at most countable set K of sentences:

(P) for any countable model M of K if the extension E of M is a model of K, then E is an elementary extension of M.

The existential formula $\bigvee x_1 \ldots \bigvee x_n\ X$ is said to be a <u>primitive formula</u> iff X is a conjunction of some atomic and negated atomic formulae. Consider the statement

(L) a non-empty and at most countable set K of sentences has the property (P) iff for every pair M, M' of models of K, where M' is an extension of the countable M and for any $a_1, \dots, a_s \in M$ and primitive formula $Y(y_1, \dots, y_s)$ with free variables $y_1, \dots, y_s$ $M \models Y(a_1, \dots, a_s)$ iff $M' \models Y(a_1, \dots, a_s)$.

(P) is a weakening of model completeness and (L) is clearly the adapted to (P) test of model completeness (2.3.1 in [1]). We may ape the proof of 2.3.1 in [1], using, however, instead of the general completeness theorem for predicate calculus, the theorem of completeness for languages of countable signature. Since the last is a theroem of ZF, so is (L) too.

Choosing K to be the set of axioms of the linear dense ordering without first and last elements for 1.1, of the ordered abelian group for 1.2, of the ordered commutative field for 1.3 and using the same standard algebraic facts as in [1] and properties of dense ordering, we obtain proofs of 1.1–1.3 in ZF.

2. THE MODEL X

We shall not generalize the well-known facts, mentioned above. We construct instead a model $\overline{X}$ which, being of the power of continuum, has a class of countable submodels, that are "weakly elementary" in the sense, described exactly below.

The idea of weakly elementary submodel is, in principle, taken from the exposition of non-standard analysis as presented, e.g. in [2], [3].

Let X_1 be the set of all rationals; for positive integer n let X_{n+1} be the set of those countable sequences s, with terms in $X_1 \cup \dots \cup X_n$, for which $s \notin X_1 \cup \dots \cup X_n$.

Let $X = \bigcup_{n \geq 1} X_n$. Denote the set of all positive integers by N. Let N, X_n, $n \in N$, be also the unary relational symbols, denoting themselves. Let $<$ be the natural ordering of X_1 and a binary relational symbol for itself. Let the ternary relational symbol e denotes the set e of all ordered triples $\langle s,n,x \rangle$ such, that $s \in X \setminus X_1$ and x is the n-th term of s. Denote also by $+, \cdot, -,$, E the functions, defined on the suitable cartesian powers of X accordingly to the usual meaning of symbols (E stands for the entier function), if all the arguments are rational and arbitrarily otherwise. Use also the familiar constants to denote the rationals. Add all the symbols, defined above, to the language of the elementary first order predicate calculus with equality as the only extralogical signs and denote the obtained language by L. The structure with the universe X and with the symbols of L interpreted as above will be our <u>model $\overline{X}$</u>.

Denote by B the set of all formulae of L, which are of the form

$$K_1x_1 \dots K_nx_n \; A(x_1, \dots, x_n, y_1, \dots, y_m) \tag{1}$$

where $A(\ldots)$ contains no quantifier and for any $i = 1,\ldots,n$ $K_i x_i$ is a shortening of either $\bigwedge x_i X_{p_i}(x_i) \rightarrow \ldots$ or $\bigvee x_i X_{p_i}(x_i) \& \ldots$ where p_i is a positive integer. We shall say, that a submodel $\overline{Y}$ of $\overline{X}$ is a <u>weakly elementary</u> submodel of $\overline{X}$ iff for any formula $S(y_1,\ldots,y_m)$ of the form (1) (any formula from B) and any $a_1,\ldots,a_m \in \overline{Y}$ $\overline{X} \models S(a_1,\ldots,a_m)$ iff $\overline{Y} \models S(a_1,\ldots,a_m)$.

Our main theorem is the following theorem of ZF

<u>Theorem 1</u>. If C is any countable subset of X, then $\overline{X}$ has a weakly elementary submodel $\overline{Y}$, with C included in its universe.

Before proving the theorem we make some remarks. At first, "countable" in the formulation of thm. 1 may be clearly replaced by "finite" and we shall use this corollary of the theorem. At the second, adding to L the set of constants C for the elements of C, we can easily prove in ZF that the countable set of all true in $\overline{X}$ substitutions of the constants from $C \cup X_1$ into the formulae of L has a countable model, consisting of the intepretations of constants only, Unfortunately in general case it does not give en elementary submodel of $\overline{X}$. We perform our proof of the theorem, modifying the routine methods mentioned above. The modifications are easy, but we must describe them carefully.

<u>Proof of the theorem 1. in ZF</u>

A. Fix the countable $C \subseteq X$, Let any $c \in C$ be a constant, denoting itself and denote by L_C the language L with added constants form C and by $\overline{X}_C$ the corresponding expansion of $\overline{X}$. Denote by B_C the set of all substitutions of the constants from C into the formulae from B. Write for brevity $\bigwedge_m x_i$ instead of $\bigwedge x_i\, X_m(x_i) \rightarrow \ldots$ and $\bigvee_m x_i \ldots$ for $\bigvee x_i\, X_m(x_i) \& \ldots$.

Call the formula F from B a <u>k, m-formula</u> iff k is a non-negative integer and F is a sentence of the form

$$\bigvee_m y\, W(y) \tag{20}$$

for k = 0, or, for positive k, a formula of the form

$$\bigvee_m y\, W(x_1,\ldots,x_k,y). \tag{2k}$$

Assign to each k,m-formula F the <u>Skolem term</u> $f^{(m)}_{k,F}(x_1,\ldots,x_k)$ if $k > 0$, the <u>Skolem constant</u> $f^{(m)}_{0,F}$ if F is a sentence (20).

Every constant of L_C is said to be <u>constant term</u>; a subsitution of constant terms for all the variables in Skolem term is a <u>constant term</u> again. The constant terms of the form $f^{(1)}_{\ldots}$, will be called <u>rational terms</u>.

Let S_0 be the set of all true in $\bar{X}_C$ sentences from B_C. Define for given S_{2n} the sets S_{2n+1}, S_{2n+2} as follows. S_{2n+1} is S_{2n} with added formulae $X_m(f^{(m)}_{k,F}(c_1,\ldots,c_k))$, $W(c_1,\ldots,c_k,f^{(m)}_{k,F}(c_1,\ldots,c_k))$ for any F of the form (2k) $c_1,\ldots,c_k$ constant terms, $F(c_1,\ldots,c_k)$ in S_{2n} and the formulae $X_m(f^{(m)}_{0,F})$, $W(f^{(m)}_{0,F})$ whenever F of the form (20) is in S_{2n}.
S_{2n+2} is S_{2n+1} with added formula $Y(c)$ for any Y, c such that for certain m $X_m(c)$, $\bigwedge_m y\, Y(y)$ are in S_{2n+1}.

Observe, that for any sentence in $\bigcup_{n\geqslant 0} S_n$ we can choose the values of its Skolem terms in X, with each $f^{(m)}_{\ldots}$ in X_m and so, that the obtained sentence is true in X_C. We perform it below systematically for the sentence without quantifiers.

B. Let S be the set of those sentences from $\bigcup_{n\geqslant 0} S_n$, which do not contain variables. We shall say, that the constant term c_{m+1} is essential on (m+1)-th place in a sentence $F(c_1,\ldots,c_{m+s})$ from S iff there exists in $\bigcup_{n\geqslant 0} S_n$ a setence H of the form

$$\bigvee_{n_1} x_{m+1} K_{n_2}^2 x_{m+2} \ldots K^s_{n_s} x_{m+s}\; F(c_1,\ldots,c_m,\, x_{m+1},\ldots,x_{m+s}) \tag{3}$$

(K for quantifiers) such that $c_{m+1} = f^{(n_1)}_{m,H}\;(c_1,\ldots,c_m)$, $m \geqslant 0$.
Any constant term, which is not a constant of L_C is essential on the exactly one place in exactly one sentence of S.

Let $\langle F_n\rangle_{n\geqslant 1}$ be the 1-1 sequence of all sentences from S, arranged so, that $X_p\;(c_{m+1})$ precedes in it the sentence $F(c_1,\ldots,c_{m+s})$ whenever $X_p\;(c_{m+1})$ is in S and c_{m+1} is essential on the m+1-th place in $F(c_1,\ldots,c_{m+1})$. Fix arbitrary 1-1 sequence $\langle w_n\rangle_{n\geqslant 1}$ of all rationals and let $\langle t_n\rangle_{n\geqslant 1}$ be the 1-1 sequence of all rational terms, occuring in S, arranged so, that for every $m < p$ either the formula in which t_m is essential precedes in $\langle F_n\rangle_{n\geqslant 1}$ that one in which t_p is essential, or t_m, t_p are essential in the same F_n, but t_m precedes t_p in it. For the variables $z_1,z_2,\ldots,z_m,\ldots$ let, at the end $\langle G_n\rangle_{n\geqslant 1}$ be sequence of formulae G_n obtained from F_n by the replacement of t_m by z_m for any m.

Consider the following two conditions, depending on m.

(4,m) $a_1,\ldots,a_m$ are rationals and the set

$$S \cup \{t_1 = a_1,\ldots,t_m = a_m\} \text{ of sentences is consistent.}$$

(5,m) $a_1,\ldots,a_m$ are as above and if t_m is essential on certain place in F_{n_m}, then there exists a valuation a of z_i's with $a(z_i) = a_i$ for $i = 1,\ldots,m$, such, that

$$\bar{X}_C \models (G_1 \,\&\ldots\&\, G_{n_m})\quad [a].$$

We define inductively the sequence $\langle a_n \rangle_{n \geq 1}$ of rationals such, that (4,m), (5,m) are fulfilled for any $m \in N$ by $a_1,\dots,a_m$. Suppose, that (4,m), (5,m) are fulfilled by $a_1,\dots,a_m$. It follows from the observation at the end of section A. of proof, that there exist w_p such, that $a_1,\dots,w_p$ fulfil (5,m+1). We have to prove, that $a_1,\dots,a_{m+1}$ with a_{m+1} equal to the first w_p as above fulfil also (4,m+1). Obviously, from (5,m+1) it follows, that $\{F_1,\dots,F_{n_{m+1}}\} \cup \{t_1 = a_1,\dots,t_{m+1} = a_{m+1}\}$ is a consistent set. If $n > n_{m+1}$, then each occurence of $t_1,\dots,t_{m+1}$ in F_n is not essential. Hence, for F_n of the form $F(c_1,\dots,c_{j+1})$ there exists formula from $\bigcup_{n \geq U} S_n$ of the form

$$K_1 x_{j+1} \dots K_{r_s} x_{j+s} F(c_1,\dots,c_j,x_{j+1},\dots,x_{j+s}) \tag{6}$$

(with K_i for quantifiers), where K_1 is $\wedge$ and $X_1(c_{j+1})$ is in S, whenever $j \geq 0$, c_{j+1} is among the $t_1,\dots,t_{m+1}$. Hence, $a_1,\dots,a_{m+1}$ satisfy also (4,m+1).

For the sequence $\langle a_n \rangle_{n \geq 1}$ so defined also the set $S' = S \cup \{t_1 = a_1,\dots, t_n = a_n,\dots\}$ is consistent. If $\overline{Y}'$ is its model (Y' exists and can be costructed in ZF) then the field of rationals of $\overline{Y}'$ is isomorphic with X, for S includes all the equations and inequalities, true in $\overline{X}$.

C. Suppose, that $\overline{Y}'$ is a model of S' and identify its rationals with the corresponding rationals of $\overline{X}_C$. Denote by Y_n the realization of X_n in $\overline{Y}'$ and define the function J_1 to be identity function on Y_1. For given $J_1,\dots,J_m$ with values of J_i in X_i for $i = 1,\dots,m$ define J_{m+1} as follows.

Let $y \in Y_{m+1}$ Then S' contains for any $n \in N$ just one sentence of the form $e(y,n,y_n)$ and the sentence $X_p(y_n)$ with $p \leq m$ too. For $J' = J_1 \cup \dots \cup J_m$ put $J_{m+1}(y) = \langle J'_m(y_n) \rangle_{n \geq 1}$ with $y_n^{,s}$ defined as above. Then $J_{m+1} : Y_{m+1} \xrightarrow{1-1} X_{m+1}$.

Let $J = \bigcup_{m \geq 1} J_m$. We define now the predicates of L in $J[Y] \subseteq X$ (Y is a universe of $\overline{Y}'$) transferring isomorphically the definitions from $\overline{Y}'$. The obtained model is clearly a submodel of $\overline{X}_C$ and contains C in its universe (we omit the easy proof). Denote it by $\overline{Y}_C$. Obviously it is model of B_C and we deduce from it, that the model $\overline{Y}$, obtained from $\overline{Y}_C$ by interpretation of the symbols of L as in $\overline{Y}_C$ is a weakly elementary submodel of $\overline{X}$, q.e.d.

3. REMARKS ON SEPARABILITY

We end the paper with some applications of the theorem 1. At first, observe, that with the axioms of the dense linear unbounded ordering, of the linearly ordered group, and of the ordered field there correspond the sentences from B, true in $\overline{X}$. This gives us the another method of proving 1.1-1.3 and the similar theorems.

Consider now the formula

$$X_2(x)\& \bigwedge_1 E(E>0 \to \bigvee_1 n(N(n)\& \bigwedge_1 m \bigwedge_1 p \bigwedge_1 y \bigwedge_1 z$$

$$(N(m)\&N(p)\&m>n\&p>n\&e(x,m,y)\&e(x,p,z) \to \tag{7}$$

$$|y-z|<E)).$$

Of course (7) defines in $\overline{X}$ the set of all Cauchy sequences of rationals and is equivalent to certain formula from B. Also the equivalence of Cauchy sequences of rationals is defined in $\overline{X}$ by a formula from B equivalent to

$$X_2(x)\&X_2(y)\& \bigwedge_1 E(E>0 \to \bigvee_1 n \bigwedge_1 m \bigwedge_1 x_1 \bigwedge_1 y_1$$

$$(m>n\&e(x,m,x_1)\&e(y,m,y_1) \to |x_1-y_1|<E)). \tag{8}$$

We say, that the set A of reals is <u>weakly definable</u> in $\overline{X}$ iff there exists a set $A' \subseteq X_2$, which is definable in $\overline{X}$ by a formula from B, consists of Cauchy sequences and A is the set of all classes of equivalence (8) of elements of A'. The following is a theorem of ZF.

<u>Theorem 2</u>. If the set $A \subseteq R$ is weekly definable in $\overline{X}$ then it is separable.

<u>Proof</u>. Let X_A be a formula from B, defining in $\overline{X}$ the set A' of Cauchy sequences of rationals such, that A is a set of all limits of sequences from A'. Let $\overline{Y}$ be countable weakly elementary submodel of $\overline{X}$. Consider $a \in A$ and an arbitrary sequence $\langle a_n \rangle_{n\geq 1}$ of rationals, convergent to a. For any $k \in N$, $m \in N$, such, that $|a_k - a| \leq 1/m$ the formula $K(x,y,z)$ of the form

$$X_A(x)\& \bigvee_1 n \bigwedge_1 p \bigwedge_1 u\ (e(x,p,u)\&p>n \to |y-x| \leq z) \tag{9}$$

holds in $\overline{X}$ for $y = a_k$, $z = 1/m$ and a Cauchy sequence x from A' convergent to a. Since (9) has an equivalent formula in B, there must hold $\overline{Y} \models K(c_m, a_k, 1/m)$ for certain c_m which is also, in A'. Putting $c_m = \langle c_{mn} \rangle_{n \geq 1}$ we obtain $\lim_n |c_{mn} - a| \leq 1/m$. Thus the set of all equivalence classes of those sequences from A', which belong to the universe of $\overline{Y}$ is at most countable and dense subset of A, q.e.d.

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PRZELICZALNE PODMODELE W ZF A OŚRODKOWOŚĆ

S t r e s z c z e n i e

W pracy zdefiniowano dla pewnego modelu X mocy continuum wariant słaby pojęcia podmodelu elementarnego. Wykazano w ZF, że model $\overline{X}$ ma dostatecznie wiele podmodeli przeliczalnych, elementarnych w tym zmodyfikowanym sensie. Naszkicowano przykładowe zastosowanie.

СЧЁТНЫЕ ПОДМОДЕЛИ В ZF И СЕПАРАБЕЛЬНОСТЬ

Р е з ю м е

В работе даётся модифицированное понятие элементарной подмодели для некоторой модели X мощности континуума. Доказывается в ZF существование достаточно широкого класса счётных подмоделей X, элементарных в этом модифицированном смысле. Намечен пример приложения.