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SOME INDIRECT CONSEQUENCES OF A THEOREM OF ZAHORSKI

Summary. The author gives some consequences of a Zahorski theorem which asserts that for a given set there exists a function whose derivative vanishes on this set. In the third part a new proof of a theorem of Zalcwasser is presented.

1. INTRODUCTION

The impact and influence of Zahorski's work in differentation theory is well known to all researchers in the field. Indeed, his monumetal work [10] is the most widely quoted paper on the subject because it opened so many directions of further research for scholars of the next generations. Those engaged in research on monotonicity, or in the behavior of various forms of generalized differentiation have been very much influenced by both the results and the techniques of Zahorski's work [10].

There may exist hundreds of mathematical articles, written during the last thirty years, in which Zahorski's influence is immediately apparant. An author answers a question posed by Zahorski; or he proves a theorem which makes direct use of a theorem of Zahorski; or he extends a Zahorski theorem to some generalized derivative. In each such case, the reader will quickly understand how the new results were influenced by Zahorski's work. One might call such results "direct" consequences of Zahorski's work. But there are also many examples of "indirect" consequences of Zahorski's work. An author bases a proof on a "direct" consequence of Zahorski's work - the reader may then not be aware of Zahorski's influence.

In section 2 we mention briefly some recent examples of this situation. In each case, the direct consequences are theorems based on special cases of Zahorski's theorem 8 in [10]. The indirect consequences are very simple proofs of what might have very difficult theorems. Then, in section 3, we present two new instances of indirect consequences.

2. RECENT DIRECT AND INDIRECT CONSEQUENCES OF A THEOREM OF ZAHORSKI

Theorem 8 in [10] concerns associated sets of derivatives. We state two important special cases of this theorem.

Theorem a) Let $E \subset [0,1]$ be a set of type F_σ such that every point of E is a point of density of E. Then there exists a differentiable function F such that $F' = 0$ on $[0,1]\setminus E$ and $0 < F' < 1$ on E.

b) Let $E \subset [0,1]$ be of type G_δ and of measure 0. Then there exists a function F such that $F' = \infty$ on E and $0 < F' < \infty$ on $[0,1]\setminus E$.

Various authors have used this theorem to obtain relatively simple proofs of what would otherwise be difficult theorems. For example, Boboc and Marcus [1] characterized the stationary and determining sets of derivatives using part a), Bruckner and Leonard [5] used part a) to obtain differentiable Cantor-like functions, and Fleissner and Foran [6] also used part a) to characterize those functions F for which there exists a homeomorphism H of the real line R onto itself such that the function $H\circ F$ is differentiable. In particular this class includes all absolutely continuous fucntions. Similarly Bruckner and Goffman [4] used part b) to characterize those functions F defined on $[0,1]$ for which there exists a homeomorphism H of $[0,1]$ onto itself such that $F\circ H$ is a differentiable Lipschitz function. A necessary and sufficient condition is that F be continuous and of bounded variation. Kaplan and Slobodnick [7] proved similar change of scale theorems and applied them to obtain a number of relatively simple proofs of difficult theorems. We mention two of their applications, which we view as indirect consequences of Zahorski's theorem. The first application is the construction of a differentiable, nowhere monotonic function. Let E be a set which together with its complement has positive measure in every subinterval of $[0, 1]$. Let $F(x) = \int_0^x x_E\, d\lambda - \int_0^x x_{\tilde{E}}\, d\lambda$. Then $F' = 1$ on a dense set, and $F' = -1$ on a dense set. Thus F is nowhere monotonic. Since F is absolutely continuous, there exists a homeomorphism H of $[0,1]$ onto itself such that $F\circ h$ is differentiable. It is clear that $F\circ H$ is nowhere monotonic.

Kaplan and Slobodnick also used their theorems to obtain a simple construction of a Whitney type function, that is a nonconstant continuously differentiable function of two variables which maps the set on which its gradient vanishes onto the entire range. Their proof used a direct consequence of Zahorski's theorem, part b).

Kelar [8] recently used the Fleissner-Foran theorem to outline a proof of a difficult theorem of Zalcwasser [11] :

<u>Theorem</u>: Let A and B be arbitrary denumerable sets in $[0,1]$ with $A\cap B = \emptyset$. There exists a differentiable function F such that F has strict local maxima on A, strict local minima on B, and no other extrema.

Kelar accomplished this by obtaining a Lipschitz function with the desired extrema properties and then applying the Fleissner-Foran theorem.

In section 3) below we provide another proof of Zalcwasser's theorem. We then show that a much stronger result is possible if we weaken the differentiability requirement to differentiability a.e.

3. FUNCTIONS WITH DENSE SETS OF STRICT EXTREMA

We begin with a proof of Zalcwasser's theorem for the case that A and B are dense. (Zalcwasser had shown that the general case reduces to this one). The proof depends on two lemmas, the first of which is a bit technical.

Lemma 1. Let A, B, A^* and B^* be denumerable, dense subsets of $(0,1)$ such that $A \cap B = \emptyset = A^* \cap B^*$. There exists a homeomorphism h of $[0,1]$ onto itself such that $h(A) = A^*$, $h(B) = B^*$ and $\frac{1}{2} \leq \frac{h(y) - h(x)}{y - x} \leq 2$ for every $x \neq y \in [0,1]$

Proof: Since each of the sets A, B, A^* and B^* are denumerable, each may be enumerated. We shall refer to these enumerations as "preliminary orderings". Let a_1 and b_1 be the first points in the preliminary orderings of A and B, respectively. These points partition $[0,1]$ into three subintervals. Since A^* and B^* are dense, there are elements $a_1^* \in A^*$ and $b_1^* \in B^*$ such that $\frac{1}{2} < \frac{|I|}{|I^*|} < 2$ whenever I is an interval in the partition determined by a_1 and b_1 and I^* is the corresponding interval in the partition determined by a_1^* and b_1^*. Here $|I|$ denotes the length of the interval I.

We proceed by induction.

Suppose we have chosen sets

$$A_n = \{a_1,\dots,a_n\}$$
$$B_n = \{b_1,\dots,b_n\}$$
$$A_n^* = \{a_1^*,\dots,a_n^*\}$$

and $B_n^* = \{b_1^*,\dots,b_n^*\}$ such that

(i) $a_i \in A$, $b_i \in B$, $a_i^* \in A^*$ and $b_i^* \in B^*$ for all $i = 1,\dots,n$

and

(ii) $\frac{1}{2} < \frac{|I|}{|I^*|} < 2$ whenever I is an interval in the partition of $[0,1]$ determined by $A_n \cup B_n$ and I^* is the corresponding interval in the partition determined by $A_n^* \cup B_n^*$.

We shall define a_{n+1}, b_{n+1}, a_{n+1}^* and b_{n+1}^* so that conditions (i) and (ii) are met for the resulting sets A_{n+1}, B_{n+1}, A_{n+1}^* and B_{n+1}^*. If n is even, let a_{n+1} be the first element of the preliminary ordering for A which is not in A_n. Let c and d be those points of

$A_n \cup B_n \cup \{0,1\}$ which satisfy $(c,d) \cap (A_n \cup U_n) = \emptyset$ and $c < a_{n+1} < d$. Let c^* and d^* be the corresponding points in $A_n^* \cup B_n^* \cup \{0,1\}$. Since A^* is dense in $[0,1]$, there exists a point $a_{n+1}^* \in A^*$ such that

$$\frac{1}{2} < \frac{a_{n+1}^* - c^*}{a_{n+1} - c} < 2 \quad \text{and} \quad \frac{1}{2} < \frac{a_{n+1}^* - d^*}{a_{n+1} - d} < 2.$$

Now obtain $b_{n+1} \in B$ and $b_{n+1}^* \in B^*$ in the same manner, using the set $A_{n+1} \cup B_n$ to determine the points c and d. It is clear that conditions (i) and (ii) are satisfied by the sets A_{n+1}, B_{n+1}, A_{n+1}^* and B_{n+1}^*. If n is odd we proceed similarly. We choose a_{n+1}^* to be the first element of the preliminary ordering of A^* which is not in A_n^*. We then obtain a_{n+1} as before. We then obtain b_{n+1}^* and b_{n+1} by the same method.

Since we alternated the process for n even and for n odd, each point in $A \cup B$ and each point in $A^* \cup B^*$ is chosen at some stage of this process. Thus the corre_pondence obtained gives rise to a function h_1 defined on $A \cup B$ such that $h_1(a_n) = a_n^*$ adn $h_1(b_n) = b_n^*$. It is easy to verify that h_1 is a strictly increasing function mapping A onto A^* and B onto B^*. Now let $h(x) = \sup\{h_1(t) : t \in A \cup B,\ 0 \leqslant t \leqslant x\}$ with $h(0) = 0$. Then h is an increasing homeomorphism of $[0,1]$ onto itself. We show h has the desired properties. First let $c, d \in A \cup B$ $(c < d)$ and let $c^* = h(c)$ and $d^* = h(d)$. Choose n such that $c \in A_n \cup B_n$ and $d \in A_n \cup B_n$. There are a finite number of points of $A_n \cup B_n$ in the interval $[c,d]$. The inequalities (ii) are valid for each neighboring pair of such points and their corresponding points in $[c^*, d^*]$. It follows readily that

$$\frac{1}{2} < \frac{d^* - c^*}{d - c} = \frac{h(d) - h(c)}{d - c} < 2. \tag{1}$$

Now let x and y be arbitrary points of $[0,1]$, $(x \neq y)$. Since $A \cup B$ and $A^* \cup B^*$ are dense, and h is continuous, the inequalities $\frac{1}{2} \leqq \frac{h(y) - h(x)}{y - x} \leqslant 2$ follow from (1). This completes the proof of Lemma 1.

Lemma 2, below, provides a quick proof that there exist differentiable functions with <u>strict</u> local extrema on dense sets, and no other local extrema. Observe that the local extrema of a differentiable nowhere monotonic function must be dense but need not be strict.

Let Δ_0' be the set of bounded derivatives on $[0,1]$ which vanish on a dense set. For $f \in \Delta_0'$, let $\|f\| = \sup\{|f(t)| : 0 \leqslant t \leqslant 1\}$. Then $(\Delta_0', \|\ \|)$ is a Banach space. Weil [9] showed that the set $\{f \in \Delta_0' : f$ takes both signs in every interval$\}$ is a residuial (co-meager) subset of Δ_0'. We now show that the set of $f \in \Delta_0'$ whose primitives assume only strict extrema, is also residual in Δ_0'. This, together with Weil's result, will establish the existence of differentiable functions with strict extrema on dense sets and no other extrema. For convenience, we norm the space of differentiable functions using the norm on their derivatives.

Lemma 2. Let $\Delta_0 = \{F = F' \in \Delta_0' \text{ and } F(0) = 0\}$. Define $\|F\| = \|F'\|$, as defined above. Then $E \equiv \{F : F$ achives only strict extrema$\}$ is residual in the Banach space $(\Delta_0, \| \ \|)$.

Proof. Let $M = \{f \in \Delta_0 : F$ achieves only strict local maxima$\}$. For disjoint closed intervals I and J contained in $[0,1]$, let $M_{I,J} = \{F \in \Delta_0 : F$ achieves different maxima on I and $J\}$. It is easy to verify that $M_{I,J}$ is dense and open. If F achieves a non-strict local maximum at a point $x_0 \in [0,1]$ then there exist intervals I and J with rational endpoints such that $F \in \tilde{M}_{I,J} \equiv \Delta_0 - M_{I,J}$. Thus $\Delta_0 - M \subset \cup \tilde{M}_{I,J}$ the union being taken over all pairs of disjoint rational subintervals of $[0,1]$. Since we have shown each $\tilde{M}_{I,J}$ is nowhere dense in Δ_0 and since the union is a denumerable one, $\tilde{M}$ is a first category subset of Δ_0 and M is residual.

A similar proof shows that $N \equiv \{F \cup \Delta_0 : F$ achieves only strict local minima$\}$ is residual in Δ_0. Thus $E = M \cap N$ is also residual, as was to be proved.

Theorem 1. (Zalcwasser). Let A and B be disjoint, denumerable and dense in $[0,1]$. There exists a differentiable function F that has strict local maxima at every point of A, strict local minima at every point of B, and no other local extrema.

Proof. By Weil's Theorem and Lemma 2, there exists a differentiable function F, (with bounded derivative), whose sets of local extrema are dense and all local extrema are strict. Let A^* and B^* be the sets of maxima and minima, respectively of F. Let h be the homeomorphism of Lemma 1 and let $G = F_o h^{-1}$. Since h satisfies a bilipschitz condition, G is a Lipschitz function. It is clear that G has the desired extrema behavior. The Fleissner-Foran theorem guarantees the existence of a homeomorphism H of R into itself such that $H \circ G$ is differentiable. The function $H \circ G$ has all the desired properties.

If we weaken the requirement that F be differentiable everywhere to the requirement that F be differentiable almost everywhere, an interesting phenomenom occurs. "Most" continuous functions can be transformed into ones which are defferentiable almost everywhere ana have prescribed denumerable, disjoint, dense sets of maxima and minima, all strict. Theorem 2 makes this statement precise.

Theorem 2. Let C denote the space of continuous functions on $[0,1]$ furnished with the sup norm : $\|F\| = \sup\{|F(x)| \ : 0 \leqslant x \leqslant 1\}$. There exists a residual subset T of C such that if A and B are disjoint, denumerable, dense subsets of $[0,1]$, then for every $F \in T$ there exists a homeomorphism H of $[0,1]$ onto itself such that

(i) $F \circ H$ is differentiable almost everywhere.

(ii) $(F \circ H)' = 0$ wherever the derivative exists.

(iii) $A = \{x : F \circ H$ achieves a strict local maximum at $x\}$

$B = \{x : F \circ H$ achieves a strict local minimum at $x\}$

and

(iv) $F \circ H$ achieves no other extrema.

Proof. Let T consist of those functions F in C which satisfy the following conditions:

a) F achieves local extrema on dense sets.
b) All local extrema of F are strict.
c) Every isolated point of a level set of F corresponds to a local extrema of F.

It follows immediately from Lemma 2.3 and theorem 3.3 of [3] that T is residual in C. Let $F \in T$. Since F is continuous, there exists a homeomorphism g of $[0,1]$ onto itself such that $F \circ g$ is differentiable a.e. [2]. Let A^* and B^* be the sets of strict maxima and strict minima of F, respectively. It is clear that both A^* and B^* are denumerable and dense. Let h be as in Lemma 1. Since h^{-1} satisifes a Lipschitz condition on $[0,1]$, the function $(F \circ g) \circ h^{-1}$ is also differentiable a.e. This function clearly has the desired extrema properties. Let $H = g \circ h^{-1}$.

Then $F \circ H$ satisifes conditions (i), (III) and (iv). To verify condition (ii) observe first that $F \circ H$ is in T. If $F \circ H$ is differentiable at x_0 then $(F \circ H)'$ must vanish at x_0. This is clear if x_0 is a limit point of a level set of $F \circ H$. If x_0 is an isolated point of a level set of $F \circ H$, then $(F \circ H)'(x_0) = 0$ because of condition c). Thus $(F \circ H)'$ vanishes wherever it exists.

Remark 1. It is interesting to compare the behavior of a singular function (that is, a nonconstant continuous function of bounded variation whose derivative vanishes a.e.) with the a.e. differentiable functions in T. In the former case, the function must have an infinite derivative on a set having cardinality of the continium. A function in T cannot have an infinite derivative (or a finite nonzero derivative) at any point, even if it is differentiable a.e.

Remark 2. The proof of Theorem 2 relies on the fact that every continuous function can be transformed into one that is differentiable a.e. by a suitable change of variable. The same is true of any function which is continuous on a dense set [2]. The bounded functions of this class of functions form a Banach space when furnished with the sup norm. We have been unable to determine what sorts of analogues of Theorem 2 are available for this class of for its important subclasses (e.g., the class of bounded Baire 1 functions). The difficulty lies in the fact that the extrema of functions in there classes is not fully understood.

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NIEKTÓRE KONSEKWENCJE TWIERDZENIA ZAHORSKIEGO

S t r e s z c z e n i e

Praca poświęcona wnioskom z twierdzenia Zahorskiego o istnieniu funkcji, której pochodna zeruje się na z góry zadanym zbiorze. W części trzeciej autor podaje nowy dowód twierdzenia Zalcwasera: Dla dowolnych rozłącznych przeliczalnych podzbiorów A i B odcinka [0,1] istnieje funkcja różniczkowalna, która ma minima w A, maxima w B i żadnych innych ekstremów nie posiada.

НЕКОТОРЫЕ ПОСРЕДСТВЕННЫЕ ПОСЛЕДСТВИЯ ТЕОРЕМЫ ЗАХОРСКОГО

Р е з ю м е

Автором указываются некоторые следствия из теоремы Захорского о существовании функции, производная которой обращается в нуль на наперёд заданном множестве. В работе даётся новое доказательство теоремы Зальцвассера.