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ON THE BIEBERBACH CONJECTURE

Summary. In the class S of all univalent functions f of the form $z + a_2(f)z^2 + a_3(f)z^3 + \ldots$ holomorphic in $|z| < 1$ we consider the functional $H_k(f) := \operatorname{re}[a_{k+1}(f) - a_2(f) \cdot k - 1]$. It is shown that

1^0 $\operatorname{re}[a_{k+1}(f) - H_k(f)]$ reaches its maximum and minimum value for Koebe function $z + 2z^2 + 3z^3 + \ldots$ and $z - 2z^2 + 3z^3 - 4z^4 + \ldots$ respectively.

2^0 the maximum and minimum value of $H_k(f)$ in S is equal 4 and 0 resp. (the first result is obtained under the assumption that for all $f \in S$ with $\operatorname{re} a_2(f) < -1{,}8$ the Bieberbach conjecture holds).

3^0 if there exists a function $f_o \in S$ and a natural $k_o > 1$ such that $H_{k_o}(f) = 2\alpha - \operatorname{re} a_2(f_o)$, $\operatorname{re} a_{k_o+1}(f) \leqslant \alpha(1+k_o)$ then $\alpha = 1$ and the Bieberbach conjecture holds.

The coefficient $a_k(f)$ of any function $f \in S$ is given by the integral

$$a_k(f) = (2\pi r^k)^{-1} \int_0^{2\pi} e^{-iks} f(re^{is})ds, \quad 0 < r < 1$$

$a_1(f) = 1$ for all $f \in S$. Put

$$a_k(t;r,f) = (2\pi r^k)^{-1} \int_0^t e^{-iks} f(re^{is})ds$$

then

$$a'_{k+1}(t;r,f) = r^{-k} e^{-ikt} a'_1(t;r,f)$$

Integrating by parts we obtain

$$a_{k+1}(f)r^k = 1 + ik \int_0^{2\pi} e^{-iks} a_1(s;r,f)ds$$

Put

$$I_k(t;r,f) = \int_0^t e^{-iks} a_1(s;r,f)ds$$

then

$$I_k'(t;r,f) = e^{-i(k-1)t} I_1'(t;r,f)$$

Integrating by parts

$$I_k(2\pi;r,f) = I_1(2\pi;r,f) + i(k-1) \int_0^{2\pi} e^{-i(k-1)s} I_1(s;r,f)ds$$

Hence[1)]

$$a_{k+1}(f) + k(k-1) \lim_{r\nearrow 1} \left[\int_0^{2\pi} e^{-i(k-1)s} I_1(s;r,f)ds\right] = 1 + k\left[a_2(f) - 1\right] \tag{1}$$

As $-2 \leqslant \mathrm{re}\, a_2(f) \leqslant 2$ for all $f \in S$, $\mathrm{re}\, a_2 = 2$ for Koebe function only, we obtain the following statement:

The greatest value of the functional

$$\mathrm{re}\, a_{k+1}(f) + k(k-1) \lim_{r\nearrow 1}\left[\mathrm{re}\int_0^{2\pi} e^{-i(k-1)s} I_1(s;r,f)ds\right] \tag{2}$$

is realized in S by Koebe function $K(z) = z + 2z^2 + 3z^3 + \ldots$ only and is equal $1+k$. The minimum value of (2) is realized by the function $z - 2z^2 + 3z^3 - 4z^4 + \ldots$ only and is equal $1 - 3k$.

Put[2)]

$$\sup_k\left[\lim_{r\nearrow 1}\left\{-(k-1)\,\mathrm{re}\int_0^{2\pi} e^{-i(k-1)s} I_1(s;r,f)ds\right\}\right] = c(f)$$

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1) Formula (1) remains true for normalized not univalent functions.

2) We use the fact that $\mathrm{re}\, a_{k+1}(f)/k$ is uniformly bounded in S.

then

$$- k(k-1)\lim_{r \nearrow 1}\left\{ \mathrm{re} \int_0^{2\pi} e^{-i(k-1)s} I_1(s;r,f)\,ds \right\} \leqslant k\, c(f)$$

This inequality is sharp for each function $f \in S$. On the other hand using (1) and the very well known inequality

$$\mathrm{re}\ a_{k+1}(f) \leqslant \alpha(1+k) \text{ for all } f \in S \text{ and } k \in N,$$

$1{,}1 > \alpha \geqslant 1$, we obtain

$$- k(k-1)\lim_{r \nearrow 1}\left\{ \mathrm{re} \int_0^{2\pi} e^{-i(k-1)s} I_1(s;r,f)\,ds \right\} \leqslant \alpha - 1 + k\left[\alpha + 1 - \mathrm{re}\ a_2(f)\right] \quad (3)$$

The greatest value of $\alpha + 1 - \mathrm{re}\, a_2(f)$ in S is equal $\alpha + 3$ and its smallest value is $\alpha - 1$. For Koebe function $K(z) = z + 2z^2 + 3z^3 + \dots$, $c(K) = 0$, hence the inequality (3) is sharp for K only when $\alpha = 1$. For the (Koebe) function $K_1(z) = z - 2z^2 + 3z^3 - \dots$ $c(K_1) = 4$ and the inequality (3) is sharp only when $\alpha = 1$. When $\alpha = 1$ then the Bieberbach conjecture holds and $c(f) \leqslant 2 - \mathrm{re}\ a_2(f)$ for all $f \in S$.

The right hand side of (3) can be rewritten as follows

$$k(\left[\alpha - 1\right]/k + \alpha + 1 - \mathrm{re}\ a_2(f))$$

hence

$$c(f) \leqslant \sup_k(\left[\alpha - 1\right]/k + \alpha + 1 - \mathrm{re}\ a_2(f)) = 2\alpha - \mathrm{re}\ a_2(f)$$

Suppose that for all $f \in S$ with $\mathrm{re}\ a_2(f) < -1{,}8$ the Bieberbach conjecture holds. Now if there exists a function $f_0 \in S$ such that

$$2\alpha - \mathrm{re}\ a_2(f_0) > 4, \quad \alpha > 1$$

then[1)] $\mathrm{re}\ a_2(f_0) < -1{,}8$; hence $\alpha = 1$ for f_0. Therefore

$$2\alpha - \mathrm{re}\ a_2(f) \leqslant 4$$

for all $f \in S$ which do not satisfy the Bieberbach conjecture ($\alpha > 1$).

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1) According to the last results α is smaller then 1,1.

For all $f \in S$ which satisfy the Bieberbach conjecture ($\alpha = 1$)

$$2\alpha - \mathrm{re}\, a_2(f) = 2 - \mathrm{re}\, a_2(f)$$

is not greater then 4. Hence we have the following statement:

If the Bieberbach conjecture holds for all $f \in S$ with $\mathrm{re}\, a_2(f) < -1{,}8$, then $c(f) \leqslant 4$ for all $f \in S$. The maximum value of $c(f)$ is assumend by the Koebe function K_1, the minimum value is equal 0 and is assumed by the function K.

Suppose there exists a function $f_o \in S$ and $k_o \in N$, $k_o > 1$ such that

$$c(f_o) = 2\alpha - \mathrm{re}\, a_2(f_o) \text{ and } \mathrm{re}\, a_{k_o+1}(f_o) \leqslant \alpha(1 + k_o)$$

Then from (1) follows

$$\alpha + \alpha k_o \geqslant \mathrm{re}\, a_{k_o+1}(f_o) = 1 + k_o(2\alpha - 1)$$

As $k_o > 1$ we obtain $\alpha = 1$.

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O HIPOTEZIE BIEBERBACHA

S t r e s z c z e n i e

W klasie S funkcji jednolistnych i holomorficznych w kole $|z| < 1$ postaci $z + a_2(f)z^2 + a_3(f)z^3 + \ldots$ rozważamy funkcjonał $H_k(f) := \mathrm{re}\left[a_{k+1}(f) - a_2(f)k - 1\right]$. Wykazuje się, że:

1^o $\mathrm{re}\left[a_{k+1}(f) - H_k(f)\right]$ przyjmuje największą i najmniejszą wartość odpowiednio dla funkcji Koebego $z + 2z^2 + 3z^3 + \ldots$, względnie $z - 2z^2 + 3z^3 - 4z^4 + \ldots$,

2^o największa względnie najmniejsza wartość funkcjonału $H_k(f)$ w klasie S wynosi odpowiednio 4 i 0 (pierwsza własność została wykazana pod założeniem, że dla wszystkich funkcji $f \in S$, dla których $\mathrm{re}\, a_2(f) < -1{,}8$, zachodzi hipoteza Bieberbacha),

3^o jeśli istnieje funkcja $f_o \in S$ i liczba naturalna $k_o > 1$ taka, że $H_{k_o}(f) = 2\alpha - \mathrm{re}\, a_2(f)$, $\mathrm{re}\, a_{k_o+1}(f) \leqslant \alpha(1 + k_o)$, to $\alpha = 1$ i zachodzi hipoteza Bieberbacha.

О ГИПОТЕЗЕ БИБЕРБАХА

Р е з ю м е

В классе S функций однолистных и голоморфных в круге $|z| < 1$ вида $z + a_2(f)z^2 + a_3(f)z^3 + \ldots$ рассматривается функционал $H_k(f) := \mathrm{re}\left[a_{k+1}(f) - a_k(f)k - 1\right]$. Показано, что:

1^0 $\mathrm{re}\left[a_{k+1}(f) - H_k(f)\right]$ принимает наибольшее и наименьшее значения соответственно для функции Кэбе $z + 2z^2 + 3z^3 + \ldots$, и функции $z - 2z^2 + 3z^3 - 4z^4 + \ldots$,

2^0 наибольшие и наименьшие значения $H_k(f)$ в классе S равно соответственно 4 и 0 (первое показано при условии, что для всех функций $f \in S$, для которых $\mathrm{re}\, a_2(f) < -1{,}8$, выполнена гипотеза Бибербаха),

3^0 если существует функция $f_o \in S$ и натуральное число $k_o > 1$ такое, что $H_{k_o}(f) = 2\alpha - \mathrm{re}\, a_2(f)$, $\mathrm{re}\, a_{k_o+1}(f) \leqslant \alpha(1+k_o)$, то $\alpha = 1$ и выполняется гипотеза Бибербаха.