ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1986

Seria: MATEMATYKA-FIZYKA z. 48 Nr kol. 853

by Jacek Marek JĘDRZEJEWSKI

SPACES IN WHICH EVERY CONTINUOUS FUNCTION IS UNIFORMLY CONTINUOUS

Summary. Characterizations of metric (uniform) spaces such that every continous function in another metric (uniform) space is uniformly continous are presented. In particular. An answer to the problem of S.B. Seidman and J.A. Childress is given.

It is well known that every continuous function defined on a compact metric space with values in another metric space is uniformly continuous. But there are not only compact spaces which have the same property.

Some results concerning that problem have been proved by N. Levine and W.G. Saunders in the article [4] and N. Levine in [2]. The present article will concern to uniform or metric spaces in which that condition holds. It gives an answer to a problem of S.B. Seidman and J.A. Childress [5].

We shall say that a uniform (metric) space X has the property J if every continuous function from X to another uniform (metric) space is uniform (metric) space is uniformly continuous.

A uniform space X with the uniformity $\mathfrak{X}$ has the property J (it is uniformly isolated in N. Levine's [3] terminology) if there exists an entourage $U \in \mathfrak{X}$ such that $x,y \notin U$ for every distinct x,y from the set X.

A uniform space X has the property K if $X = X_1 \cup X_2$, where X_1 is a compact subspace of X and X_2 has the property J.

A metric space X with the metrix ϱ has the property $\mathcal{X}$ if for every pair of sequences x_n and y_n such that $0 < \varrho(x_n,y_n) < \frac{1}{n}$ there exist accumulation points of those sequences.

The following lemma can be proved in the analogous manner as lemma 8.2.5 in [1].

Lemma. Let X be a uniform space with a uniformity $\mathfrak{X}$, X_0 its compact subspace, $\{U_i\}_{i \in I}$ an open covering of X_0. Then there exists $U \in \mathfrak{X}$ such that for every $x \in X_0$ $U[x] = \{y \in X : (x,y) \in U\}$ is contained in some U_i.

Theorem 1. Let X and Y be two uniform spaces and X have the property K.

Then every continuous fucntion $f : X \longrightarrow Y$ is uniformly continuous.

Proof. Let $\mathfrak{X}$ be a uniformity for the space X and $\mathcal{Y}$ - a uniformity for Y. In view of the definition of the property K the space X can be represented as $X_1 \cup X_2$, where X_1 is a compact space, and X_2 is uniformly isolated. Thus there exists an entourage U_0 of the diagonal Δ_X such that $(x_1', x_2'') \in U_0$ for every two different elements x_1', x_2'' from X_2. Now let V be an arbitrary entourage of the diagonal Δ_Y (i.e. the set $\{(y,y) \in Y \times Y : y \in Y\}$). There exists a $V_1 \in \mathcal{Y}$ such that $V_1 \circ V_1 \subset V$. Now for every $x \in X$ there exists $U_x \in \mathfrak{X}$ such that if $\xi \in U_x[x]$ then $(f(\xi), f(x)) \in V_1$. According to lemma there exists $U_1 \in \mathfrak{X}$ such that for every $x_0 \in X_0$ the set $U_1[x_0]$ is contained in some set $U_x[x]$.

Let $U = U_0 \cap U_1$, then $U \in \mathfrak{X}$ and let x', x'' be two different points of the set X for which $(x', x'') \in U$.

Since $U \subset U_0$, then both of those points must not belong simultaneously to the set $X \setminus X_0$. Thus one of those points must belong to X_0; let it be x'. Then

$$x'' \in U[x'] \subset U_1[x']$$

and there exists $x_1 \in X_0$ such that $U_1[x'] \subset U_{x_1}[x_1]$.

It follows from this that

$$(f(x'), f(x_1)) \in V_1$$

and

$$(f(x''), f(x_1)) \in V_1$$

what implies that

$$(f(x'), f(x'')) \in V_1 \circ V_1 \subset V$$

This proves that f is uniformly continuous.

The next theorems give additional properties of investigated spaces.

Theorem 2. Let X be a metric space and $X_1 \subset X$ its subspace. If every continuous fuctnion from X_1 to another metric space Y is uniformly continuous, then X_1 is a closed subset of X.

Proof. Suppose that X_1 is not closed in X, i.e. there exists a sequence (x_n) such that $x_n \in X_1$, $x_n \longrightarrow x$, $x \notin X_1$. We can assume that $x_n \neq x_m$ for $n \neq m$. Then the sequences (x_{2n}), (x_{2n+1}) are convergent to x, so the sets

$$\{x_1, x_3, x_5, \ldots, x_{2n+1}, \ldots\}$$
$$\{x_2, x_4, \ldots, x_{2n}, \ldots\}$$

are disjoint and closed in X_1. There exists a continuous function $f : X_1 \to R$ such that

$$f(x_{2n}) = 0, \quad f(x_{2n+1}) = 1.$$

This function is not, of course, uniformly continuous, what contradicts to our supposition.

The next theorem has been proved by N. Levine and W.G. Saunders in [4].

Theorem 3. If in some metric space X every closed disc is compact, then X has the property $\mathcal{J}$ if and only if X has the property K.

Remark. The above theorem can be generalized a little by the assumption that $X = \bigcup_{n=1} X_n$, where sets X_n are compact and $X_n \subset \operatorname{Int} X_{n+1}$ for $n = 1,2,\ldots$ instead of that every closed disc is compact.

Theorem 4. A metric space has the property $\mathcal{J}$ if and only if it has the property L.

Proof. Assune first that a metric space X has the property L and suppose that there is a continuous function f from X to a metric space Y (with the metric denoted also by ρ) which is not uniformly continuous. Then there exists a positive ε_0 such that for every positive integer n there exists points x_n, $y_n \in X$ such that $\rho(x_n, y_n) < \frac{1}{n}$ and $\rho(f(x_n), f(y_n)) \geq \varepsilon_0$. Since X has the property L then there exists and accumulation point x_0 of the sequence (x_n); it is also an accumulation point of the sequence y_n.

Now there axists a positive δ such that if $x \in K(x_0, \delta) = \{x \in X : \rho(x, x_0) < \delta\}$ then

$$\rho(f(x), f(x_0)) < \frac{1}{2} \varepsilon_0.$$

Then there exists n such that $n > \frac{2}{\delta}$ and $x_n \in K(x_0, \frac{1}{2}\delta)$.
Now $y_n \in K(x_0, \delta)$ and $\rho(f(x_n), f(y_n)) \leq \rho(f(x_n), f(x_0)) + \rho(f(x_0), f(y_n)) < \varepsilon_0$, what contradicts to our supposition.

Let us assume now that X has not the property L, i.e. there are two sequences $(x_n), (y_n)$ such that $\rho(x_n, y_n) < \frac{1}{n}$, $x_n \neq y_m$ for $n,m = 1,2,\ldots$ which have no accumulation point.
Thus the sets

$$\{x_1, \ldots, x_n, \ldots\}, \quad \{y_1, \ldots, y_m, \ldots\}$$

are disjoint and closed. Then there exists a continuous function $f : X \to R$ such that $f(x_n) = 0$, $f(y_n) = 1$. This function is not, of course, uniformly continuous, what completes the proof.

REFERENCES

[1] Engelking R.: General Topology. Warszawa 1977.

[2] Levine N.: Remarks on uniform continuity in metric spaces. Amer.Math. Monthly 67 (1979), p. 562-563.

[3] Levine N.: Uniformly continuous linear sets. Amer. Math. Monthly 62 (1955), p. 579-580.

[4] Levine N., Saundeis W.G.: Uniformly continuous sets in metric spaces. Amer. Math. Monthly 67 (1960), p. 153-156.

[6] Seidman S.B.: Childress J.A., A continuous modulus of continuity. Amer. Math. Monthly 82 (1975), p. 253-254.

Recenzent: Dr hab. Karol Baron

Wpłynęło do redakcji: 20.XII.1983 r.

PRZESTRZENIE W KTÓRYCH KAŻDA FUNKCJA CIĄGŁA JEST JEDNOSTAJNIE CIĄGŁA

S t r e s z c z e n i e

Znaleziono charakteryzacje takich przestrzeni metrycznych (jednostajnych), że każda funkcja ciągła w inną przestrzeń metryczną (jednostajną) jest jednostajnie ciągła. W szczególności podano rozwiązanie pewnego problemu S.B. Seidmana i J.A. Childressa.

ПРОСТРАНСТВА В КОТОРЫХ КАЖДАЯ НЕПРЕРЫВНАЯ ФУНКЦИЯ ЯВЛЯЕТСЯ РАВНОМЕРНО НЕПРЕРЫВНОЙ

Р е з ю м е

В работе даётся характеризация таких метрических (равномерных) пространств, что всякая непрерывная функция в другое метрическое пространство является равномерно непрерывной. Даётся решение проблемы Сейдмана и Чилдреса.