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FUNCTIONALLY CONNCECTED FUNCTIONS

Summary. In this paper the notion of functional connectedness at a point and global property of functionally connected real functions are investigated. The relations between these properties and the Darboux property is studied.

In several articles were considered Darboux points and connectivity points of an arbitrary function $f : R \to R$. The local properties of Darboux and of connectedness are close connected to global property of Darboux and global connectedness of a function (it means that the graph of f is a connected set in R^2). Between the two classes of functions: the class of connected functions and the class of Darboux functions, is contained the class of functionally connected functions. It has very similar properties and can be shortly characterized with the aid of the class of Darboux functions.

We shall use the following denotations.

For a function $f : R \to R$ and $x_0 \in R$ $L^+(f,x_0)$ ($L^-(f,x_0)$ or $L(f,x_0)$, respectively) will denote the set of all right-sided limit numbers (left-sided limit numbers or all limit numbers, respectively) of the function f at the point x_0. Moreover, $L^*(f,x_0) = \{x_0\} \cup L(f,x_0)$.

For a set $L(x) \subset R$ for $x \in R$ let

$$\alpha(x) = \{x\} \times L(x)$$

The symbol $\operatorname{ls}_{t \in T} E_t$ will denote the upper topological limit of a class of sets $\{E_t\}_{t \in T}$ (see |4|, |5|).

DEFINITION 1. The function $f : R \to R$ is called functionally connected (shortly $f \in F$) if for every nondegenerated interval $[a,b]$ and every continuous function $g : [a,b] \to R$ such that $f(a) < g(a)$ and $f(b) > g(b)$ or $f(a) > g(a)$ and $f(b) < g(b)$ there exists a point $x \in (a,b)$ such that $f(x) = g(x)$.

DEFINITION 2. The function $f : R \to R$ is called functionally connected at a point $x_0 \in R$ from the right-side (shortly $x_0 \in F^+(f)$) if

(1) $f(x_0) \in L^+(f,x_0)$,

(2) If $g : [x_o, x_o + \varepsilon] \to R$, where ε is any positive number, is a continuous function such that $\liminf_{x \to x_o^+} f(x) < g(x) < \limsup_{x \to x_o^+} f(x)$, then there exists a point $x \in (x_o, x_o + \varepsilon)$ such that $f(x) = g(x)$.

In the analogous way we define functional connectedness of f at x_o from the left-sde (shortly $(x_o \in F^-(f))$.

If $x_o \in F^+(f) \cap F^-(f)$, then we say that f is functionally connected at the point x_o.

By $F(f)$, $F^+(f)$, $F^-(f)$, respectively, we shall denote the set of all points at which the function f is functionally connected, functionally connected from the right side or functionally connected from the left side, respectively. Analogously, $D(f)$ will denote the set of all Darboux points of f.

As we can see, the definition of functionally connected function at a point is analogous to the definition of Darboux function or connected function at a point (see [1], [2]).

It is easy to see that the class F of all functionally connected functions and the class D of Darboux functions are defferent but $F \subset D$. Moreover, the existence of a continuum in a plane which contains no graph of a continuous fuction on any interval implies that the class $\mathcal{C}$ of all connected functions is different to F but $\mathcal{C} \subset F$.

The first theorem gives a characterization of the class F in terms of local property of functional connectedness.

<u>Theorem I</u>. The function $f : R \to R$ belongs to F if and only if for every $x \in R$ $f \in F_x$.

PROOF. I. Assume first that $f \in F$ and suppose that there exists a point $x_o \in R$ such that for example $x_o \notin F^+_{x_o}(f)$. Then there are two possibilities

a) $f(x_o) \notin L^+(f, x_o)$

b) there is a continuous function $g : [x_o, x_o + \varepsilon] \to R$, $\varepsilon > 0$, such that

$$\liminf_{x \to x_o^+} f(x) < g(x) < \limsup_{x \to x_o^+} f(x)$$

and $f(x) \neq g(x)$ for $x \in [x_o, x_o + \varepsilon]$.

The first case is impossible in view of that f is Darboux function. In the case b) the function g is this one which fulfills all conditions in definition 1 except the last one. This is a contradiction to our assumption.

II. Assume now that for every $x_o \in R$ $x_o \in F(f)$ but $f \notin F$. Then there exists a continuous function $g : [a,b] \to R$ such that for example $f(a) < g(a)$, $f(b) > g(b)$ and $f(x) \neq g(x)$ for $x \in (a,b)$. Consider now two sets

$$A = \{x \in [a,b] \mid f(x) < g(x)\},$$

$$B = \{x \in [a,b] \mid f(x) > g(x)\}.$$

Let $K = \mathrm{Fr}\, A$. Since $x \in F(f) \subset D(f)$ for every $x \in R$, then f is a Darboux function and in view of lemma from the article [1].
$K = \mathrm{Fr}\, B$ and K is a perfect set in which the sets $K \cap A$ and $K \cap B$ are dense.

For $x \in K$ we have the following relations

$$L(f|A,x) \subset [-\infty, g(x)],$$

$$L(f|B,x) \subset [g(x), +\infty]$$

and $g(x) \in L(f|A,x) \cap L(f|B,x)$. Since $f \in D$, then the set

$$L(f,x) = L(f|A,x) \cup L(f|B,x)$$

is a connected set and

$$L(f|A,x) \cap L(f|B,x) = \{g(x)\}.$$

In this way both of sets $L(f|A,x)$, $L(f|B,x)$ are intervals and at least one of them is nondegenerated. Let

$$A_n = \{x \in K \mid g(x) - \tfrac{1}{n} \in L(f|A,x)\},$$

$$B_n = \{x \in K \mid g(x) + \tfrac{1}{n} \in L(f|B,x)\}.$$

Since $f(x) \neq g(x)$ and $f(x) \in L(f,x)$ for $x \in [a,b]$, then

$$K = \bigcup_{n=1}^{\infty} A_n \cup \bigcup_{n=1}^{\infty} B_n.$$

The sets A_n, B_n are closed ([4]).

If A_n and B_n for every $n \in N$ are nowhere dense in K, then the perfect set K would be of the I category (in itself), what is impossible.

Thus for example A_n, for some $n \in N$, is dense in K on some partition of K. Then there exists a nondegenerated interval $[c,d] \subset [a,b]$ such that

$$\overline{A_n \cap [c,d]} = K \cap [c,d],$$

then also

$$A_n \cap [c,d] = K \cap [c,d].$$

From the definition of sets A, B and K we know that there is a point $x_o \in K \cap B \cap (c,d)$ and $x_o \in A_n$. In this way

$$g(x_o) + \frac{1}{n} \in L(f,x_o), \quad f(x_o) \in L^+(f,x_o), \quad f(x_o) < g(x_o)$$

and for example $g(x_o) + \frac{1}{n} \in L^+(f,x_o)$. It follows then that the function $g \mid [x_o,d]$ does not fulfill all conditions of definition 2. The contradiction finishes the proof.

Theorem 2. For every function $f : R \to R$ the set $F^+(f) \triangle F^-(f)$ is at most cuntable ($\triangle$ means the symmetrical difference of sets).

PROOF. It is enough to prove that the set $A = F^-(f) \setminus F^+(f)$ is at most countable. Let A_o be the set of all points $x \in A$ for which $f(x) \notin L^+(f,x)$. This set is countable. Now, let, for n = 1,2,..., A_n be the set of points $x_o \in A$ for which $f(x_o) \in L^+(f,x_o)$ and there exists a function $g_{x_o} : [x_o, x_o + \frac{1}{n}] \to R$ such that $f(x) \neq g_{x_o}(x)$ for $x \in [x_o, x_o + \frac{1}{n}]$, and moreover

$$\liminf_{x \to x_o^+} f(x) + \frac{1}{n} < g_{x_o}(x) < \limsup_{x \to x_o^+} f(x) - \frac{1}{n}$$

for $x \in [x_o, x_o + \frac{1}{n}]$.

Then

$$A = \bigcup_{n=1}^{\infty} A_n.$$

We shall show that each set A_n for n = 1,2,... contains no one of its points of left-sided accumulation. Assume that it is not true. Then there exists a sequence (x_k) and $x_o \in R$ such that

$$x_k \to x_o, \quad x_k \in A_n, \quad x_o \in A_n, \quad x_k < x_o.$$

Then $f(x_o) \in L^-(f,x_o)$ and according to [4] the following holds

$$P(x_o) \cap L^-(f,x_o) = P(x_o) \cap \operatorname*{ls}_{x < x_o} L^*(f,x) \supset P(x_o) \cap \operatorname*{ls}_{x < x_o} L(f,x) \supset P(x_o) \cap \operatorname*{ls}_{k \in N} L(f,x_k),$$

where $P(x_o) = \{(x_o,y) \mid y \in R\}$.

Since each of sets $L(f,x_k)$ has a diameter not less than $\frac{2}{n}$, then the diameter of $L^-(f,x_o)$ is so. Moreover, $x_o - x_k < \frac{1}{n}$ for sufficiently large k and $k > n$, and then

$$\liminf_{x\to x_o^-} f(x) + \frac{1}{n} < g_{x_k}(x_o) < \limsup_{x\to x_o^-} f(x) - \frac{1}{n}.$$

Since $f(x) \neq g_{x_k}(x)$ for $x \in [x_k, x_k + \frac{1}{n}]$, then $f(x) \neq g_{x_k}(x)$ for $x \in [x_k, x_o]$, what proves that f is not functionally connected at the point x_o from the left-side. Thus $x_o \notin A$, which completes the proof.

Without any essential change of proof of Rosen's theorem [6] one can prove the following theorem.

<u>Theorem 3</u>. For every function $f : R \to R$ the set $F(f)$ is of type G_δ.

<u>Theorem 4</u>. Let $f : R \to R$ be an arbitrary function. Then $f \in F$ if and only if for every continuous function $g : R \to R$ the function $f + g$ has the Darboux property.

PROOF. I. Assume that $f \in F$ and there is a function $g : R \to R$ such that $f + g \notin D$. Then there exist points $c,d \in R$ and $m \in R$ such that

$$f(c) + g(c) < m < f(d) + g(d)$$

and

$$f(x) + g(x) \neq m \quad \text{for} \quad x \in [c,d].$$

Then the graph of the function $g - m$ has common points with both of sets $\{(x,y) \in R^2 \mid f(x) > y\}$ and $\{(x,y) \in R^2 \mid f(x) < y\}$ but $f(x) \neq g(x) - m$ for $x \in [c,d]$. This contradicts our assumptions.

II. Assume now that $f \notin F$. Then there exists a continuous function $g : [a,b] \to R$ for some nondegenerated interval $[a,b]$, such that there are two points $x_1, x_2 \in [a,b]$ fulfilling the inequalities

$$f(x_1) < g(x_1) \quad \text{and} \quad f(x_2) > g(x_2)$$

but $f(x) \neq g(x)$ for $x \in [a,b]$. Let

$$\bar{g}(x) = \begin{cases} g(x) & \text{for} \quad x \in [a,b], \\ g(a) & \text{for} \quad x < a, \\ g(b) & \text{for} \quad x > b. \end{cases}$$

Then the function $f - \bar{g}$ has not the Darboux property. This completes the proof.

Theorem 5. The class of all continuous functions on R is a maximal additive class for the class F.

PROOF. We shall prove first that for every function $f \in F$ and every continuous function g, $f + g \in F$. Suppose that it is not true. Then there exists a function h defined on some interval $[a,b]$ which is continuous on $[a,b]$ and such that there exist points $x_1, x_2 \in [a,b]$ such that

$$h(x_1) < f(x_1) + g(x_1), \quad h(x_2) > f(x_2) + g(x_2)$$

and $f(x) + g(x) \neq h(x)$ for $x \in [a,b]$. Then $f + g - h \notin D$ what contradicts to the Theorem 4.

Now let $g : R \to R$ be any discontinuous function. Consider two cases:

1^o $g \notin F$,

2^o $g \in F$.

In the first case, let $f = 0$. Of course, $f \in F$ and $f + g \notin F$.

In the second case, let x_o be any point of discontinuity of g. The function $g \in F$ then also $g \in D$ and at least one of the sets $L^+(g,x_o)$, $L^-(g,x_o)$ is a nondegenerated interval and

$$g(x_o) \in L^+(g,x_o) \cap L^-(g,x_o).$$

Let, for example, $L^+(g,x_o)$ be nondegenerated interval, and let $1 \neq g(x_o)$ be any of numbers from $L^+(g,x_o)$. Let $f : R \to R$ be defined as follows:

$$f(x) = \begin{cases} -g(x) & \text{for } x > x_o, \\ -\frac{1}{2}(1 + g(x_o)) & \text{for } x \leqslant x_o. \end{cases}$$

Then $f \in F$ but $f + g \notin D$. This completes the proof.

Theorem 6. For every function $f : R \to R$ fulfilling the condition $f(x) \in L(f,x)$ for $x \in R$ the set $R \setminus F(f)$ is dense in itself.

PROOF. Let us suppose that there are points x_o, $a,b \in R$ such that

$$\{x_o\} = (a,b) \cap [R \setminus F(f)].$$

Then $x_o \notin F^+(f)$ or $x_o \notin F^-(f)$. Let, for example, $x_o \notin F^+(f)$. There exist a number $\delta > 0$ and a continuous function $g : [x_o, x_o + \delta] \to R$ such that

$$\liminf_{x \to x_o^+} f(x) < g(x_o) < \limsup_{x \to x_o^+} f(x)$$

and

$$f(x) \neq g(x) \quad \text{for} \quad x \in [x_o, x_o + \delta] \,. \tag{3}$$

Let $[c,d] \subset (x_o, x_o + \delta)$ be an interval such that there are points $t_1, t_2 \in [c,d]$ fulfilling the inequalities

$$f(t_1) < g(t_1) \quad \text{and} \quad f(t_2) > g(t_2). \tag{4}$$

The function $f | [c,d]$ is functionally connectd, what is impossible in view of (3) and (4).

It is worth to say that the construction of a function f in the main theorem of the article [3] is a good construction in our case. Then we have the following theorem:

Theorem 7. For every set A of type G_δ such that $R \setminus A$ is dense in itself there exists a function $f : R \to R$ such that $A = F(f)$ and f is a Darboux function.

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FUNKCJONALNIE SPÓJNE FUNKCJE

S t r e s z c z e n i e

Praca poświęcona jest badaniom pojęć funkcyjnej spójności w punkcie funkcji rzeczywistych. Główne wyniki, to porównanie tych pojęć oraz związki z własnością Darboux.

ФУНКЦИОНАЛЬНО СВЯЗНЫЕ ФУНКЦИИ

Р е з ю м е

Работа посвящена исследованию понятия функциональной связности в точке и глобальной. Рассматривается связь этого понятия со свойством Дарбу.