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ON ALMOST COMPATIBLE SUPPORTS OF LOCALLY INTEGRABLE FUNCTIONS

Dedicated to Professor
Zygmunt Zahorski on His
70th birthday

Summary. In this paper, the notion of almost compatible sets in R^q are introduced in three equivalent ways. The notion plays an analogous role for the convolution of locally integrable functions as the concept of compatible sets for the convolution of Schwartz distributions. It is proved that if A,B are measurable sets in R^q, then the convolution $f * g$ exists for all locally integrable functions with the supports in A,B, respectively, if and only if A,B are almost compatible.

1. IT IS WELL KNOWN THAT THE CONVOLUTION

$$(f * g)(x) = \int_{R^q} f(x-t)g(t)dt \qquad (x \in R^q)$$

of two continuous or two locally integrable functions not always exists. It does if some restrictions are adopted connected with the behaviour of the functions f and g at infinity or under assumptions concerning their supports.

By the support of a given continuous function f in R^q we mean, as usually, the smallest closed set in R^q outside which $f = 0$. The same definition can be adopted for Schwartz distributions in R^q.

For locally integrable functions the above definition of the support is less adequate and it seems to be more natural to define it as the smallest closed set in R^q outside which a locally integrable function vanishes almost everywhere (see [9], p. 196).

We shall adopt the common notation $s(f)$ for the support of a locally integrable function f as well as of a distribution f.

The convolution $f * g$ of locally integrable functions (distributions) exists, for instance, if

1^o $s(f)$ is bounded or $s(g)$ is bounded,

2^o $s(f) \subset R^q_+$ and $s(g) \subset R^q_+$, where R^q_+ denots the set of all points in R^q whose coordinates are non-negative.

Both examples are particular cases a more general situation, which is described in literature in terms of various equivalent conditions on supports of functions and distributions.

Let us formulate some of these conditions.

Given sets $A,B \subset R^q$ and $x \in R^q$, we use the notation:

$$\sigma_{AB}(x) = \left\{y \in R^q: x-y \in A,\ y \in B\right\}$$

and

$$A^\Delta = \left\{(x,y) \in R^{2q}: x+y \in A\right\}.$$

Theorem I (cf. [4], p. 383; [1], p. 125). Let $A,B \subset R^q$.

The following conditions are equivalent:

(i) for each interval $I \subset R^q$ there exists an interval $J \subset R^q$ such that $x \in I$ implies $\sigma_{AB}(x) \subset J$;

(ii) for each interval $I \subset R^q$ there exists an interval $J \subset R^q$ such that $x \in I$ implies $\sigma_{BA}(x) \subset J$;

(iii) for each bounded set $K \subset R^q$ the set $(A \times B) \cap K^\Delta$ is bounded;

(iv) if $x_n \in A$, $y_n \in B$ and $|x_n| + |y_n| \to \infty$, then $|x_n+y_n| \to \infty$.

It is easy to see that conditions (i) and (ii) are equivalent, respectively, to the following ones:

(i′) for each bounded set $K \subset R^q$ the set $(K-A) \cap B$ is bounded;

(ii″) for each bounded set $K \subset R^q$ the set $A \cap (K-B)$ is bounded.

Conditions (i′), (ii″) and (iii) appear usually in literature in a little modified form, fitted to the assumption (usually made) that A, B are closed: the words "bounded" are replaced by the words "compact" (cf. [10], p. 170; [4], p. 383). The proof that conditions (i′), (ii′), (iii) are equivalent is easy in both formulations (cf. [4], p. 383). In [1] (p.125), it is proved that condition (iii) is equivalent to conditions (i), (ii).

Sets $A,B \subset R^q$ satisfying one of conditions (i)-(iv), (i′), (ii′) are called compatible.

Supports of functions (distributions) considered above in 1^o and 2^o are compatible, but they do not exhaust all examples of compatible supports both unbounded in all directions (cf. [5], p. 71-84).

Let us recall some results on the convolution of distributions and functions with compatible supports.

The convolution of distributions is defined for many different ways. The first general definition was given by C. Chevalley [2]. Several general definitions produced later by other authors appeared to be equivalent to the definition of Chevalley (cf. [3] and [7]). The sequential definition of the convolution given in [1] (p. 153; 131) is embraced by a more general sequential definition, equivalent to the definition of Chevalley (see [6] and [7]). In the theorems recalled below (Theorems II and III) the convolution can be meant in the sense of any definition equivalent to Chevalley's definition or in the sense of the definition in [1] (p.153; 131).

<u>Theorem II</u> (cf. [10], p. 170; [4], p. 384; [1], p. 158).

If the supports of distributions f, g in R^q are compatible, then the convolution $f * g$ exists.

In [8] (p. 77), it is shown that Theorem II can be conversed in the following way:

<u>Theorem III</u> (cf. [8], p. 77).

Let $A, B \subset R^q$ and suppose that the convolution $f * g$ exists for all distributions (or less: for all non-negative measures) f, g whose supports are contained in A and B, respectively. Then A and B are compatible.

In the case of locally integrable functions, we have the following analogue of Theorem II:

<u>Theorem IV</u> (see [1], p. 124).

If the supports of locally integrable functions f, g in R^q are compatible, then the convolution $f * g$ exists almost everywhere in R^q and represents a locally integrable function in R^q.

The analogue of Theorem III for locally integrable functions is not true as a simple example shows (see section 2). The condition of compatible supports appears to be too strong for the class of locally integrable functions.

In section 2, we shall introduce in three equivalent forms a weaker condition on supports of locally integrable functions, called almost compatibility.

The notion of almost compatible sets allows us to formulate in section 3 an analogue of Theorem III for locally integrable functions.

The notation in the paper is typical. For $x = (\xi_1, \ldots, \xi_q) \in R^q$ and $A, B \subset R^q$, we adopt $|x| = \sqrt{\xi_1^2 + \ldots + \xi_q^2}$, $A^c = R^q \setminus A$ and $A \pm B = \{x \pm y : x \in A,\ y \in B\}$. By $L(A)$ we denote the q-dimensional Lebesgue measure of $A \subset R^q$.

2. LET US START FROM THE FOLLOWING EXAMPLE

<u>Example</u>. Let $A = B = B^q$, where B^q denotes, as in [1], the set of points in R^q whose all coordinates are integers. Every locally integrable function in R^q whose support is contained in B^q is equal to 0

almost everywhere. Hence the convolution $f * g$ of such functions f, g exists and equals to 0. On the other hand, the sets A, B are not compatible, because $|x_n| + |y_n| \to \infty$ and $|x_n + y_n| \to 0$ for $x_n = (n, \ldots, n) \in A$ and $y_n = (-n, \ldots, -n) \in B$.

This example shows that Theorem III is not true if the class of distributions (non-negative measures) is replaced by the class of locally integrable functions.

We shall introduce now the concept of almost compatible sets in R^q. Given a measurable set $A \subset R^q$ denote

$$\sup A = \sup \operatorname{ess} \{|x| : x \in A\},$$

i.e.,

$$\sup A = \inf \{a > 0 : L(Z_a) = 0\}, \tag{1}$$

where $Z_a = \{x \in A : |x| > a\}$; if the set $\{a > 0 : L(Z_a) = 0\}$ is empty, we adopt $\sup A = \infty$. It is clear that

$$\sup A = 0 \Longleftrightarrow L(A) = 0. \tag{2}$$

A set $A \subset R^q$ for which $\sup A < \infty$ will be called essentially bounded.

Theorem 1. Let A, B be measurable sets in R^q. The following conditions are equivalent:

(i) for each interval $I \subset R^q$ there exists an interval $J \subset R^q$ such that

$$L(\sigma_{AB}(x) \cap J^c) = 0$$

for $x \in I$;

(ii) for each interval $I \subset R^q$ there exists an interval $J \subset R^q$ such that

$$L(\{x \in I : L(\sigma_{AB}(x) \cap J^c) > 0\}) = 0;$$

(iii) for arbitrary sequences A_n and B_n of non-empty measurable (or, equivalently, measurable and essentially bounded) subsets of A and B, respectively, the following implication holds:

$$\sup A_n + \sup B_n \to \infty \Longrightarrow \sup(A_n + B_n) \to \infty .$$

The sets $A, B \subset R^q$ will be called almost compatible if one of conditions (i)-(iii) holds.

It is evident that if measurable sets $A,B \subset R^q$ are compatible, then they are almost compatible. The sets A, B in Example are almost compatible, but not compatible.

To prove Theore 1 we shall need two lemmas:

Lemma 1. Let A be a measurable set in R^q such that $0 < \sup A < \infty$. Then for arbitrary $\varepsilon, \varrho > 0$ there exist a measurable set $B \subset A$ and a ball $K_\varrho \subset R^q$ with the radius ϱ such that $B \subset K_\varrho$ and $\sup B \geqslant \sup A - \varepsilon$.

Proof. Let $\sup A = \alpha$ and $\sigma = \min(\varepsilon, \frac{\varrho}{2})$. By (1), the set $\tilde{A} = \left\{x \in A : \alpha - \sigma < |x| \leqslant \alpha\right\}$ is of positive measure. It is clear that there are balls $K_\varrho^1, \dots, K_\varrho^n$ with the radius ϱ such that $\tilde{A} \subset \bigcup_{i=1}^{n} K_\varrho^i$. Since $L(\tilde{A}) > 0$, there is an index i_o such that $L(\tilde{A} \cap K_\varrho^{i_o}) > 0$ and the set $B = \tilde{A} \cap K_\varrho^{i_o}$ satisfies the assertion of the lemma.

Lemma 2. Let A and B be measurable sets in R^q such that $L(A) > 0$, $L(B) > 0$. One can find $x \in R^q$ and a set $C \subset B$, $L(C) > 0$ that for every $y \in C$ there exists a set D_y, $L(D_y) > 0$ such that

$$x + y - D_y \subset A \quad \text{and} \quad D_y \subset B \qquad (y \in C).$$

Proof. Let χ_A, χ_B and χ_{-B} be the characteristic functions of the set A, B and -B, respectively and put

$$f = \chi_A * \chi_B * \chi_{-B}.$$

Of course, $f \geqslant 0$ and $\int f = \int \chi_A \cdot \int \chi_B \cdot \int \chi_{-B} > 0$, so

$$f(x) = \int_{R^{2q}}\int \chi_A(x-y+z)\chi_B(y)\chi_B(z)\,dy\,dz > 0$$

for some $x \in R^q$. The last inequality means that the set

$$\left\{(y,z) \in R^{2q} : x-y+z \in A,\ y \in B,\ z \in B\right\}$$

is of positive measure. By the Fubini theorem, there exists a set $C \subset B$ with $L(C) > 0$ such that for each $y \in C$ the set

$$D_y = \left\{z \in R^q : x-y+z \in A,\ z \in B\right\}$$

is of positive measure, which completes the proof of Lemma 2.

Proof of Theorem 1. It is obvious that (i) implies (ii).

To prove the implication (ii) $\Longrightarrow$ (iii) suppose that (iii) does not hold, i.e., there exist sequences A_n and B_n of nonempty measurable subsets of A and B, respectively, and a constant $M > 0$ such that

$$\sup A_n \to \infty, \quad \sup B_n \to \infty \tag{3}$$

and

$$\sup(A_n+B_n) < M \qquad (n = 1,2,\ldots).$$

This implies that the sets A_n, B_n are essentially bounded. In fact. Let $\sup A_n = \infty$ for fixed n Then, for arbitrary $y_o \in B_n$ and $a > 0$, we have $L(Z) > 0$, where

$$Z = \left\{x \in A_n : \ |x| > a + |y_o|\right\}.$$

Since $Z \subset \left\{x \in A_n : \ |x+y_o| > a\right\}$ and

$$\left\{u \in A+y_o : \ |u| > a\right\} \subset \left\{u \in A_n+B_n : \ |u| > a\right\},$$

we infer that

$$L(\left\{u \in A_n+B_n : \ |u| > a\right\}) > 0,$$

i.e., $\sup(A_n+B_n) = \infty$. Similarly $\sup B_n = \infty$ implies $\sup(A_n+B_n) = \infty$.

In view of Lemma 1, we can assume that the sets A_n and B_n are contained in balls with the same radius 1.

Since, by (3) and (2), $L(A_n) > 0$ and $L(B_n) > 0$, we conclude from the Steinhaus theorem that the sets A_n+B_n contain intervals and, in particular, $L(A_n+B_n) > 0$.

Let $I = [-M,M]^q \subset R^q$. Since $L(A_n+B_n \backslash I) = 0$, for each n there is $z_n \in (A_n+B_n) \cap I$. Clearly, $|x-z_n| < 4$ for all $x \in A_n+B_n$ and

$$A_n+B_n \subset \tilde{I}, \tag{4}$$

where $\tilde{I} = [-M-4, M+4]^q$.

For every interval $J \subset R^q$ we find an index n, for which $\sup B_n > \sup J + 2$, i.e.,

$$z \notin J \qquad \text{for } z \in B_n. \tag{5}$$

By Lemma 2, there exists a $x_o \in R^q$ and a set $C \subset B_n$ with $L(C) > 0$ such that for any $y \in C$ we can find a set D_y, $L(D_y) > 0$ satisfying the relations

$$x_o+y-z \in A_n \subset A \quad \text{and} \quad z \in B_n \subset B \qquad \text{for } z \in D_y. \tag{6}$$

Consequently,

$$x_0+y \in A_n+D_n \subset A_n+B_n \qquad (y \in C). \tag{7}$$

Let $X = x_0+C$. Clearly, $L(X) = L(C)>0$ and $X \subset \tilde{I}$, by virtue of (7) and (4). Moreover, by $x \in X$ we have

$$L(\sigma_{AB}(x) \cap J^c) \geqslant L(D_y) > 0,$$

in view of (5) and (6). The contradiction with condition (ii) proves the implication (ii) $\Rightarrow$ (iii).

It remains to show that (iii) implies (i). Suppose that (i) does not hold, i.e., there exist an interval $I \subset R^q$, a sequence x_n of points in I and sequence Y_n of measurable sets such that $L(Y_n)>0$ and

$$x_n-y \in A, \quad y \in B, \quad |y|>n \qquad \text{for } y \in Y_n$$

and for $n = 1,2,\ldots$. By Lemma 1, we can assume that $Y_n \subset K_\varrho^n$, where K_ϱ^n are balls whose radii are equal to ϱ .

Defining $X_n = x_n-Y_n$, we have $X_n \subset A$, $Y_n \subset B$ and $\sup X_n + \sup Y_n \to \infty$. Moreover, if $z \in X_n+Y_n$, then $z = x_n-u+v$ with $u,v \in Y_n$ and

$$|z| \leqslant |x_n| + 2\varrho \leqslant \sup I + 2\varrho$$

for $n = 1,2,\ldots$. The last equality contradicts condition (iii) and thus the implication (iii) $\Longrightarrow$ (i) and the whole theorem is proved.

3. IN THEOREM IV THE ASSUMPTION ABOUT COMPATIBILITY CAN BE REPLACED BY THE ASSUMPTION THAT THE SUPPORTS ARE ALMOST COMPATIBLE

Namely, we have

Theorem 2. The convolution $f * g$ of arbitrary locally integrable functions f and g in R^q whose supports are almost compatible exists and represents a locally integrable function.

Proof. Assume that the supports A, B of the locally integrable functions f, g, respectively, are almost compatible.

By condition (i), for every interval $I \subset R^q$ there exists an interval $J \subset R^q$ such that

$$L(\sigma_{AB}(x) \cap J^c) = 0 \tag{8}$$

for every $x \in I$.

Let K be an interval in R^q such that $I-J \subset K$ and $J \subset K$. We define $\tilde{f} = f \cdot \chi_K$ and $\tilde{g} = g \cdot \chi_K$, where χ_K is the characteristic function of the interval K.

Obviously,

$$(f * g)(x) = \int_{\sigma_{AB}(x)} f(x-t)g(t)dt. \tag{9}$$

If $x \in I$, then

$$\int_Z f(x-t)g(t)dt = \int_Z \tilde{f}(x-t)\tilde{g}(t)dt,$$

where $Z = \sigma_{AB}(x) \cap J$, and, in view of (8),

$$\int_{Z'} f(x-t)g(t)dt = \int_{Z'} \tilde{f}(x-t)\tilde{g}(t)dt = 0,$$

where $Z' = \sigma_{AB}(x) \cap J^c$. Hence, by (9),

$$(f * g)(x) = \int_{\sigma_{AB}(x)} \tilde{f}(x-t)\tilde{g}(t)dt = (\tilde{f} * \tilde{g})(x), \tag{10}$$

because if $t \notin \sigma_{AB}(x)$ then $f(x-t) = 0$ or $g(t) = 0$, so $\tilde{f}(x-t) = 0$ or $\tilde{g}(t) = 0$.

Since the functions $\tilde{f}$ and $\tilde{g}$ are integrable, we infer from (10) that $f * g$ exists almost everywhere in I and represents there an integrable function. But the interval I was taken arbitrarily and this means that $f * g$ exists almost everywhere in R^q, being a locally integrable function in R^q. The proof is finished.

Almost compatibility of two sets in R^q is a necessary condition for the existence of the convolution for all locally integrable functions with the supports included in these sets. In other words:

Theorem 3. Let A, B be measurable sets in R^q. Suppose that the convolution $f * g$ exists and represents a locally integrable function in R^q for any locally integrable functions f, g whose supports are contained in A, B, respectively. Then A and B are almost compatible.

Proof. Suppose that the sets A and B are not almost compatible. Then there are sequences A_n, B_n of non-empty measurable subsets of A, B, respectively, and a constant $M > 0$ such that $\sup A_n \to \infty$, $\sup B_n \to \infty$ and $\sup(A_n+B_n) < M$ for $n = 1,2,\ldots$.

By Lemma 1, we can assume that all the sets A_n, B_n are contained in balls with the radius 1. Moreover, we can assume that the sets A_n are disjoint and the sets B_n are disjoint. As in the proof of Theorem 1, one can show that

$$A_n + B_n \subset I \qquad (n = 1,2,\dots), \tag{11}$$

where $I = [-M-4, M+4]^q$.

Let

$$f(x) = \begin{cases} L(A_n)^{-1} & \text{if } x \in A_n \\ 0 & \text{if } x \in \bigcup\limits_{n=1}^{\infty} A_n \end{cases}$$

and

$$g(x) = \begin{cases} L(B_n)^{-1} & \text{if } x \in B_n \\ 0 & \text{if } x \in \bigcup\limits_{n=1}^{\infty} B_n. \end{cases}$$

Suppose that the convolution $f * g$ exists and is a locally integrable function in R^q. Then the integral

$$S = \int\limits_{R^q} (f * g)(x)\Psi(x)dx$$

exists for an arbitrary measurable bounded function Ψ, vanishing outside some interval, e.g., for $\Psi(x) = \chi_I(x)$. But then, in view of (11), we have

$$S = \int\limits_{R^q} \chi_I(s)dx \int\limits_{R^q} f(x-t)g(t)dt \geqslant \sum_{n=1}^{\infty} \int\limits_{A_n}\int\limits_{B_n} f(x)g(y)\chi_I(x+y)dxdy$$

$$\geqslant \sum_{n=1}^{\infty} L(A_n)^{-1}L(B_n)^{-1} \int\limits_{A_n}\int\limits_{B_n} dxdy = \infty ,$$

which contradicts our assumption. The theorem is thus proved.

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O PRAWIE ZGODNYCH NOŚNIKACH FUNKCJI LOKALNIE CAŁKOWALNYCH

S t r e s z c z e n i e

W pracy wprowadza się na trzy równoważne sposoby pojęcie zbiorów zgodnych w R^q. Pojęcie to gra analogiczną rolę dla splotu funkcji lokalnie całkowalnych, jak pojęcie zbiorów zgodnych dla splotu dystrybucji. Dowodzi się, że jeśli A i B są mierzalnymi zbiorami w R^q, to splot $f * g$ istnieje dla wszystkich funkcji lokalnie całkowalnych f i g, o nośnikach zawartych odpowiednio w zbiorach A i B, wtedy i tylko wtedy, gdy A i B są prawie zgodne.

О ПОЧТИ СОГЛАСОВАННЫХ НОСИТЕЛЯХ ЛОКАЛЬНО ИНТЕГРИРУЕМЫХ ФУНКЦИЙ

Р е з ю м е

В работе вводится понятие почти согласованных множеств в R^q. Это понятие играет аналогичную роль для свертки локально интегрируемых функций, как понятие согласованных множеств для свертки обобщенных функций. Доказана следующая теорема. Свертка $f * g$ существует для произвольных локально интегрируемых функций f и g с носителями в фиксированных измеримых множествах A и B тогда и только тогда, когда A и B почти согласованные.