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# CONVERSE APPROXIMATION THEOREMS IN THE SPACES $L_P(-\pi,\pi;-\pi,\pi)$ WHERE $P=(p_1,p_2)$ AND $0<p_1<1$, $0<p_2\leqslant 1$

Abstract. In the first part of this paper there is proved that, for every function $f\in L_P$, and for each integers $m\geqslant 0$, $n\geqslant 0$, there exists a trigonometric polynomial of the best approximation with respect to the metric of $L_P$. In the second part some converse approximation theorems are proved, in which functions of two variables from the space $L_P$ are approximated by trigonometric polynomials.

## 1. INTRODUCTION

Let $P=(p_1,p_2)$, where $0<p_1<1$, $0<p_2\leqslant 1$ and by $L_P = L_P(-\pi,\pi;-\pi,\pi)$ we denote the space of all $2\pi$ periodic functions f of two variables with respect to each variable separately, measurable in the square $\langle -\pi,\pi;-\pi,\pi\rangle$, for which

$$//f//_P = \int_{-\pi}^{\pi}\Big[\int_{-\pi}^{\pi}|f(x,y)|^{p_1}\,dx\Big]^{p_2} dy < \infty .$$

In the paper [2] it has been proved that $//f//_P$ is a F-norm in the space $L_P$ and $\langle L_P, //\ //\rangle$ is a Fréchet space.

Let $T_{m,n}$ be a trigonometric polynomial of two variables and

$$T_{m,n}(x,y) = \sum_{k=0}^{m}\sum_{l=0}^{n}(a_{kl}\cos kx\cos ly + b_{kl}\sin kx\cos ly + c_{kl}\cos kx\sin ly + d_{kl}\sin kx\sin ly).$$

Let

$$\Lambda(T_{m,n}) = \sum_{k=0}^{m}\sum_{l=0}^{n}(|a_{kl}| + |b_{kl}| + |c_{kl}| + |d_{kl}|)$$

be a norm of $T_{m,n}$. Consequently, for the polynomial

$$T_{m,n}(x,y) = \left[\frac{1}{2} a_0 + \sum_{k=1}^{m} (a_k \cos kx + b_k \sin kx)\right]\left[\frac{1}{2} c_0 + \right.$$

$$\left. + \sum_{l=1}^{n} (c_l \cos ly + d_l \sin ly)\right]. \tag{1}$$

we get

$$\Lambda(T_{m,n}) = \left[\frac{1}{2}|a_0| + \sum_{k=1}^{m} (|a_k| + |b_k|)\right]\left[\frac{1}{2}|c_0| + \sum_{l=1}^{n} (|c_l| + |d_l|)\right].$$

and for the polynomial

$$T_m(x) = \frac{1}{2} a_0 + \sum_{k=1}^{m} (a_k \cos kx + b_k \sin kx)$$

there will be $\Lambda(T_m) = \frac{1}{2}|a_0| + \sum_{k=1}^{m} (|a_k| + |b_k|)$ (see [4] p. 28).

A set of the polynomials (1) will be called $\Lambda$-bounded, if there exists positive number A such that $\Lambda(T_{m,n}) \leq A < \infty$.

Of course, if a set of the polynomials is $\Lambda$-bounded, then

$$\max_{\substack{-\pi \leq x \leq \pi \\ -\pi \leq y \leq \pi}} |T_{m,n}(x,y)| \leq A. \tag{2}$$

Let the polynomials (1) (m,n-constants) be uniformly bounded. Then the set of these polynomials is $\Lambda$-bounded, because by [4] p. 28 we get $\Lambda(T_{m,n}) = \Lambda(T_m)\,\Lambda(T_n) \leq 2A\sqrt{2m+1}\,\sqrt{2n+1} = B$.

A set of all trigonometric polynomials of the form (1) (m,n-constants) will be denoted by $H_{m,n}$.

Applying the method used in the proof of Lemma 1.1 [4] p. 29, we can prove.

<u>Lemma 1</u>. If $\{T_{m,n,\vartheta}\}_{\vartheta=1}^{\infty}$ is a sequence of the trigonometric polynomials from $H_{m,n}$ uniformly bounded in $\langle -\pi,\pi; -\pi,\pi \rangle$, then there exist a subsequence $\{T_{m,n,\vartheta_r}\}$ converging to certain polynomial $T^* \in H_{m,n}$ with respect to the metric of the space $L_p$; and $\Lambda(T^*) \leq B$.

Lemma 2. If $f \in L_p$, then there exists non-negative number $M = M(f)$ such that for every polynomial $T \in H_{m,n}$ with norm $\Lambda(T) > M$, the following inequality is satisfied $\|f-T\|_p > 2\|f\|_p$.

Proof. It is known from the first part of the proof of Lemma 1.3 [4] p. 30, that there exist numbers $\varepsilon > 0$ and $\delta > 0$ such that, for every $T \in H_{m,n}$, the set $D_T = G_{T_m} \times G_{T_n}$ of the points $(x,y) \in \langle -\pi,\pi; -\pi,\pi\rangle$ for which $|T(x,y)| \geqslant \Lambda(T)\delta$, has the Lebesgue measure greater than $\varepsilon$.

Hence, for every $T \in H_{m,n}$ $\|f-T\|_p \geqslant \|T\|_p - \|f\|_p \geqslant$

$$\geqslant \int_{G_{T_n}} \left[\int_{G_{T_m}} |T(x,y)|^{p_1} dx\right]^{p_2} dy - \|f\|_p >$$

$$> \int_{G_{T_n}} \left[\int_{G_T} |\Lambda(T)\delta|^{p_1} dx\right]^{p_2} dy - \|f\|_p \geqslant$$

$$\geqslant (\Lambda(T)\delta)^{p_1p_2} \int_{G_{T_n}} \left[\int_{G_{T_m}} dx\right]^{p_2} dy - \|f\|_p =$$

$$= (\Lambda(T)\delta)^{p_1p_2} c_{p_2} - \|f\|_p.$$

$$\text{If } (\Lambda(T)\delta)^{p_1p_2} c_{p_2} > 3\|f\|_p \text{ i.e. } \Lambda(T) > \frac{1}{\delta}\left(\frac{3\|f\|_p}{c_{p_2}}\right)^{1/p_1p_2} =$$

$= M(f)$, then $\|f-T\|_p > 2\|f\|_p$.

Let be $E_{m,n}(f)_p = \inf\limits_{T_{m,n} \in H_{m,n}} \|f-T_{m,n}\|_p$.

Similarly as in theorem 1.4 [4] p.31, by lemma 1 and 2, we obtain

Theorem 1. For every function $f \in L_p$ and for every integers $m \geqslant 0$, $n \geqslant 0$, there exists a trigonometric polynomial of the best approximation with respect to the metric of $L_p$, i.e. an element $T^* \in H_{m,n}$, that $\|f-T^*\|_p = E_{m,n}(f)_p$.

Property 1. If $f \in L_p$, then $E_{m,n}(f+g)_p \leqslant E_{m,n}(f)_p + E_{m,n}(g)_p$.

Proof. Let $T^*$, $t^*$ be a trigonometric polynomials of the best approximation of the functions f and g respectively. Then

$$E_{m,n}(f+g)_p \leqslant \|(f+g) - (T^* + t^*)\|_p \leqslant \|f-T^*\|_p + \|g-t^*\|_p =$$

$$= E_{m,n}(f)_p + E_{m,n}(g)_p$$

Let be $u \geqslant 0$, $v \geqslant 0$, $k$, $l$ positive integers and $f \in L_p$, then we denote

$$\omega_{k,l}(f;u,v)_p = \sup_{\substack{|h|\leqslant u\\|r|\leqslant v}} \Big\| \sum_{i=0}^{k} \sum_{j=0}^{l} (-1)^{k+l-i-j} \binom{k}{i}\binom{l}{j} f(x+ih, y+jr) \Big\|_p =$$

$$= \sup_{\substack{|h|\leqslant u\\|r|\leqslant v}} \| \Delta_h^k \Delta_r^l f(x,y) \|_p .$$

From the above definition the property 2 proves to be true.

Property 2. $\omega_{k,l}(f+g;u,v)_p \leqslant \omega_{k,l}(f;u,v)_p + \omega_{k,l}(g;u,v)_p$.

Lemma 3. If $f \in L_p$ and $T_{m,n}$ is a polynomial such that

$E_{m,n}(f)_p = \|f-T_{m,n}\|_p$, then $\omega_{k,l}(f-T_{m,n};u,v)_p \leqslant 2^{k+l} E_{m,n}(f)_p$.

Proof. $\omega_{k,l}(f-T_{m,n};u,v)_p =$

$$= \sup_{\substack{|h|\leqslant u\\|r|\leqslant v}} \Big\| \sum_{i=0}^{k} \sum_{j=0}^{l} (-1)^{k+l-i-j} \binom{k}{i}\binom{l}{j} \left[ f(x+ih,y+jr) - T_{m,n}(x+ih,y+jr) \right] \Big\|_p \leqslant$$

$$\leqslant \sum_{i=0}^{k} \sum_{j=0}^{l} \left[ \binom{k}{i}\binom{l}{j} \right]^{p_1 p_2} E_{m,n}(f)_p \leqslant E_{m,n}(f) \sum_{i=0}^{k} \sum_{j=0}^{l} \binom{k}{i}\binom{l}{j} = 2^{k+l} E_{m,n}(f)_p .$$

Lemma 4. If $T_{m,n} \in H_{m,n}$, then

$$\| \Delta_h^k \Delta_r^l T_{m,n} \|_p \leqslant C_{p_1,p_2,k,l} (m|h|)^{kp_1p_2} (n|r|)^{lp_1p_2} \| T_{m,n} \|_p .$$

Proof. By lemma 5 [1] $\| \Delta_h^k \Delta_r^l T_{m,n} \|_p =$

$$= \int_{-\pi}^{\pi} \left[ \int_{-\pi}^{\pi} |\Delta_h^k \Delta_r^l T_{m,n}(x,y)|^{p_1} dx \right]^{p_2} dy =$$

$$= \int_{-\pi}^{\pi} \left[ |\Delta_r^l T_n(y)|^{p_1} \int_{-\pi}^{\pi} |\Delta_h^k T_m(x)|^{p_1} dx \right]^{p_2} dy \leqslant$$

$$\leqslant C_{p_1,p_2,k,l}(m|h|)^{kp_1p_2}(n|r|)^{lp_1p_2} \int_{-\pi}^{\pi} |T_n(y)|^{p_1p_2} dy \left[ \int_{-\pi}^{\pi} |T_m(x)|^{p_1} dx \right]^{p_2} =$$

$$= C_{p_1,p_2,k,l}(m|h|)^{kp_1p_2}(n|r|)^{lp_1p_2} \| T_{m,n} \|_p .$$

Corollary

$$\omega_{k,l}(T_{m,n};u,v)_p \leqslant C_{p_1,p_2,k,l}(m|u|)^{kp_1p_2}(n|v|)^{lp_1p_2} \| T_{m,n} \|_p .$$

## 2. CONVERSE APPROXIMATION THEOREMS

Theorem 2. Let $f \in L_p(-\pi,\pi;-\pi,\pi)$, and $l,k$ be positive integers, then for $n \geqslant 1$, $m \geqslant 1$ $\omega_{k,l}(f;\frac{1}{m},\frac{1}{n})_p \leqslant$

$$\leqslant \frac{C_{p_1,p_2,k,l}}{m^{kp_1p_2} n^{lp_1p_2}} \sum_{i=0}^{m} \sum_{j=0}^{n} (i+1)^{kp_1p_2-1} (j+1)^{lp_1p_2-1} E_{i,j}(f)_p .$$

Proof. Let $T_{m,n}$ be a polynomial of the best approximation then, by property 2, for any integers $r_1 \geqslant 0$, $r_2 \geqslant 0$

$$\omega_{k,l}(f;\tfrac{1}{m},\tfrac{1}{n})_p \leqslant \omega_{k,l}(f - T_{2^{r_1+1},2^{r_2+1}};\tfrac{1}{m},\tfrac{1}{n})_p +$$

$$+ \omega_{k,l}(T_{2^{r_1+1},2^{r_2+1}};\tfrac{1}{m},\tfrac{1}{n})_p \leqslant 2^{k+l} E_{2^{r_1+1},2^{r_2+1}}(f)_p +$$

$$+ \omega_{k,l}(T_{2^{r_1+1},2^{r_2+1}};\tfrac{1}{m},\tfrac{1}{n})_p . \qquad (2)$$

By corollary of lemma 4

$$\omega_{k,l}(T_{2^{r_1+1},2^{r_2+1}};\tfrac{1}{m},\tfrac{1}{n})_p \leqslant \omega_{k,l}(T_{1,1};\tfrac{1}{m},\tfrac{1}{n})_p +$$

$$+ \sum_{i=0}^{r_1-1} \omega_{k,l}(T_{2^{i+1},1} - T_{2^i,1})_p + \sum_{j=1}^{r_2-1} \omega_{k-l}(T_{1,2^{j+1}} - T_{1,2^j})_p +$$

$$+ \sum_{i=0}^{r_1-1} \sum_{j=0}^{r_2-1} \omega_{k,l}(T_{2^{i+1},2^{j+1}} - T_{2^{i+1},2^j} - T_{2^i,2^{j+1}} + T_{2^i,2^j})_p +$$

$$+ \omega_{k,l}(T_{2^{r_1+1},2^{r_2+1}} - T_{2^{r_1},2^{r_2}})_p \leqslant$$

$$\leqslant C^{(1)}_{p_1,p_2,k,l} \Big\{ \left(\tfrac{1}{m}\right)^{kp_1p_2} \left(\tfrac{1}{n}\right)^{lp_1p_2} E_{0,0}(f)_p +$$

$$+ \sum_{i=0}^{r_1-1} \left(2^{i+1} \tfrac{1}{m}\right)^{kp_1p_2} \left(\tfrac{1}{n}\right)^{lp_1p_2} E_{2^i,1}(f)_p +$$

$$+ \sum_{j=0}^{r_2-1} \left(\tfrac{1}{m}\right)^{kp_1p_2} \left(2^{j+1} \tfrac{1}{n}\right)^{lp_1p_2} E_{1,2^j}(f)_p +$$

$$+ \sum_{i=0}^{r_1-1} \sum_{j=0}^{r_2-1} \left(2^{i+1} \tfrac{1}{m}\right)^{kp_1p_2} \left(2^{j+1} \tfrac{1}{n}\right)^{lp_1p_2} E_{2^i,2^j}(f)_p +$$

$$+ \left(2^{r_1+1} \tfrac{1}{m}\right)^{kp_1p_2} \left(2^{r_2+1} \tfrac{1}{n}\right)^{lp_1p_2} E_{2^{r_1},2^{r_2}}(f)_p \Big\} . \qquad (3)$$

Since for any $p$ $2^{kp(r+1)} E_{2^r,1}(f)_p \leqslant C_{p,k} \sum_{s=2^{r-1}+1}^{2^r} s^{kp-1} E_{s,1}(f)_p$,

then from the above and (3)

$$\omega_{k,l}(T_{2^{r_1+1},2^{r_2+1}}; \tfrac{1}{m}, \tfrac{1}{n})_p \leqslant$$

$$< \frac{C^{(2)}_{p_1,p_2,k,l}}{m^{kp_1p_2} n^{lp_1p_2}} \sum_{i=0}^{2^{r_1-1}} \sum_{j=0}^{2^{r_2-1}} (i+1)^{kp_1p_2-1} (j+1)^{lp_1p_2-1} E_{i,j}(f)_p. \quad (4)$$

Let us choose $r_1$, $r_2$ so that $2^{r_1-1} < m \leqslant 2^{r_1}$, $2^{r_2-1} < n \leqslant 2^{r_2}$.

By (2) and (4) $\omega_{k,l}(f;\frac{1}{m},\frac{1}{n})_p <$

$$< \frac{C_{p_1,p_2,k,l}}{m^{kp_1p_2} n^{lp_1p_2}} \sum_{i=0}^{m} \sum_{j=0}^{n} (i+1)^{kp_1p_2-1} (j+1)^{lp_1p_2-1} E_{i,j}(f)_p.$$

Similarly as in paper [3] p.100, we can prove the following

<u>Corollary</u>. If $f \in L_p$, and for some $\alpha > 0$, $\beta > 0$

$E_{m,n}(f)_p \leqslant K(\frac{1}{(m+1)^{\alpha}} + \frac{1}{(n+1)^{\beta}})$, then for

1. $kp_1p_2 < \alpha$, $lp_1p_2 < \beta$ $\omega_{k,l}(f;\delta,\gamma)_p = O(\delta^{kp_1p_2} + \gamma^{lp_1p_2})$,

2. $kp_1p_2 > \alpha$, $lp_1p_2 > \beta$ $\omega_{k,l}(f;\delta,\gamma)_p = O(\delta^{\alpha} + \gamma^{\beta})$,

3. $kp_1p_2 = \alpha$, $lp_1p_2 = \beta$ $\omega_{k,l}(f;\delta,\gamma)_p = O(\delta^{\alpha}(1+|\ln\delta|)+\gamma^{\beta}(1+|\ln\gamma|))$,

4. $kp_1p_2 > \alpha$, $lp_1p_2 < \beta$ $\omega_{k,l}(f;\delta,\gamma)_p = O(\delta^{\alpha} + \gamma^{lp_1p_2})$,

5. $kp_1p_2 = \alpha$, $lp_1p_2 > \beta$ $\omega_{k,l}(f;\delta,\gamma)_p = O(\delta^{\alpha}(1+|\ln\delta|) + \gamma^{\beta})$,

and so on.

## REFERENCES

[1] Ivanov V.I.: Direct and indirect theorems of the approximation theory in the metric $L_p$ for $0 < p < 1$, Mat. Zametki (18) 5 (1975) p.641-658 (in Russian).

[2] Kozłowska G.: Aproksymacja funkcjami całkowitymi dwóch zmiennych w przestrzeniach $L_p(D)$, gdzie $P = (p_1,p_2)$ i $0 < p_1 < 1$, $0 < p_2 \leqslant 1$, cz. I, Zeszyty Naukowe Pol. Śl. z. 31, Gliwice 1980, p. 79-88.

[3] Kozłowska G.: Aproksymacja funkcji dwóch zmiennych funkcjami całkowitymi, Zeszyty Naukowe Pol. Śl. seria Mat.-Fiz. z. 24, 1974, p.79-111.

[4] Taberski R.: Aproksymacja funkcji wielomianami trygonometrycznymi, UAM Poznań 1979.

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ODWROTNE TWIERDZENIA APROKSYMACYJNE W PRZESTRZENIACH $L_p(-\pi,\pi;-\pi,\pi)$, GDZIE $P = (p_1,p_2)$ I $0 < p_1 < 1$, $0 < p_2 \leq 1$

S t r e s z c z e n i e

W pierwszej części pracy dowodzi się, że dla każdej funkcji $f \in L_p$ i dla dowolnych całkowitych $m \geq 0$, $n \geq 0$ istnieje wielomian trygonometryczny najlepszego przybliżenia w sensie metryki przestrzeni $L_p$.
W części drugiej dowodzi się pewnych odwrotnych twierdzeń aproksymacyjnych, w których funkcje dwóch zmiennych, należące do przestrzeni $L_p$, aproksymowane są wielomianami trygonometrycznymi.

ОБРАТНЫЕ АППРОКСИМАЦИОННЫЕ ТЕОРЕМЫ В ПРОСТРАНСТВАХ $L_p(-\pi,\pi;-\pi,\pi)$, ГДЕ $P = (p_1,p_2)$ и $0 < p_1 < 1$, $0 < p_2 \leq 1$

Р е з ю м е

В первой части роботы доказано, что для любой функции $f \in L_p$ и для всех целых чисел $m \geq 0$, $n \geq 0$ существует тригонометрический полином наилучшего приближения в смысле метрики пространства $L_p$.
Во второй части доказывается некоторые обратные аппроксимационные теоремы в которых функции с пространства $L_p$ аппроксимированы тригонометрическими полиномами.