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A NON-MEASURABLE FUNCTION OF TWO VARIABLES * WITH MEASURABLE SECTIONS

Summary. This paper contains a proof of the following theorem: There exists a nonmeasurable (in the sense of Lebesgue) function f defined on $[0,1] \times [0,1]$ such that every section $f_{x_0}(y) = f(x_0,y)$ and $f^{y_0}(x) = f(x,y_0)$, $x_0,y_0 \in [0,1]$

1) belongs to Baire* class I,
2) has the Darboux property,
3) is approximately non-continous in at most one point.

Let f be a real function on $I^2 = I \times I$ where $I = \langle 0,1 \rangle$. Some authors considered the measurability of f under assumption that all the horisontal sections $f^y(x) = f(x,y)$ and all the vertical sections $f_x(y) = f(x,y)$ belong to the first class of Baire or to some subclasses of the class. The review of those results can be found in the expository part of the paper [5] by M. Laczkovich and Gy. Petruska. They point out that the measurability of f very delicaty depends on the measurability of the sections. There exist non-measurable function such that all sections are the Darboux functions of the first class, approximately continuous with at most one exceptional point for each section [6]. On the other hand each function sectionwise approximately continuous is in the second Baire class [3]. Another example of non-measurable function with the sections in the first class was given by Z. Grande [4]. All sections of his function are Darboux and belong to Baire* 1 class which is a proper subclass of the first class. From thisppoint of view his function is better than of the paper [6]. From another side the sets of approximately discontinuity points of some sections of Grande's function may have the power of continuum.

The aim of this paper is to prove the existance of a non-measurable function which joins the adventages of both the above-mentioned functions.

Let us recall the definition of Baire* class 1. A function $g : I \longrightarrow I$ belongs to this class if for any perfect set $P \subset I$ and any open set U containing points of P, there exists an open set $U^* \subset U$, such that $P \cap U^* \neq \emptyset$ and the restriction of g to $P \cap U^*$ is continuous. We shall use the following Császár theorem. A function $g : I \longrightarrow I$ has the Darboux property if and only if each point of I is a Darboux point of g [2]. The definition of Darboux point can be found in [1].

Theorem. There exists a function f defined on I^2, non-measurable in the sens of Lebesgue, such that every section f_x and f^y belongs to Baire* class 1, has the Darboux property and is approximately non-continuous in at most one point.

Proof. In his paper [8] W. Sierpiński has shown the existence of a flat and non-measurable set (in the sens of Lebesgue) of interior measure zero, possessing at most two common points with each straight line. A set like this can be easily presented as a sum of four disjoint sets each of which has at most one common point with every straight line parallel to one of the axes of the coordinate system. At least one of these four sets must be nonmeasurable. Hence there exists in the XOY - plane a non-measurable set A such that on each straight line $x_0 = x$ and $y_0 = y$ there exists at most one point belonging to this set. With no loss of generality we can assume $A \subset I^2$.

Let ε be a positive number such that $\varepsilon < m_e A$ where m_e denotes the exterior measure. Let P be a perfect and nowhere-dense subset of the open interval (0,2) such that $|(0,2)\setminus P| < \varepsilon$. Then $(0,2)\setminus P = \bigcup_{n=1}^{\infty} (\alpha_n,\beta_n)$ where the components of the union are pairwise disjoint. Denote by a_n, b_n and c_n the solutions of the equations $\alpha_n + \beta_n = a_n + b_n = 2c_n$ and $\beta_n - \alpha_n = 2^n(b_n - a_n)$. Obviously $\alpha_n < a_n < c_n < b_n < \beta_n$ and c_n is the joint center of the intervals $\langle\alpha_n,\beta_n\rangle$ and $\langle a_n,b_n\rangle$. Let us define the function g on $\langle 0,2\rangle$ in the following way: $g(x) = 0$ for $x \in P \cup \{0\} \cup \{2\} \cup \bigcup_{n=1}^{\infty}\langle\alpha_n,a_n\rangle \cup \bigcup_{n=1}^{\infty}\langle b_n,\beta_n\rangle$, $g(c_n) = 1$ and take g as linear on each interval $\langle a_n,c_n\rangle$ adn $\langle c_n,b_n\rangle$. The function g has the Darboux property and is approximately continuous at each point. The restrictions of g to P and to $(0,2)\setminus P$ are continuous. Thus g belongs to Baire* class 1.

Let us consider the function h defined on I^2 by $h(x,y) = g(x+y)$. The sections of h belong to Baire* class 1, have the Darboux property and are approximately continuous. $h(x,y) = 0$ for $x+y \in P$. Define the function f in the following way: $f(x,y) = 1$ for $(x,y) \in A \cap \{(x,y) : x+y \in P\} = D$, and $f(x,y) = h(x,y)$ for $(x,y) \in I^2 \setminus D$.

The set D is non-measurable. Indeed. Let us denote $F = \{(x,y)\in I^2 : x,y \notin P\}$. We have $|F| < \varepsilon$ and $m_e A - |F| > m_e A - \varepsilon > 0$. Obvieusly the interior measure of D is equal to zero. So the set $\{(x,y) : f(x,y) = 1\} = D \cup \{(x,y)\in I^2 : x+y = c_n\}$ and the function f are non-measurable.

Let us now consider the section f_{x_0} for a fixed x_0. If there is no point belonging to D on the straight line $x_0 = x$ then $f_{x_0} = h_{x_0}$ and has all properties mentioned in the theorem. If there is a point belonging to D on the straight line $x_0 = x$, then it can be only one. Let (x_0,ξ) be the point of D. Then $f_{x_0}(y) = h_{x_0}(y)$ for $y \neq \xi$ and $f_{x_0}(\xi) = 1 \neq h_{x_0}(\xi) = 0$. The section belongs hence ot Baire* class 1. The point ξ is the only point of approximate discontinuity of the section. By the Császár theorem every point of $I \setminus \{\xi\}$ is a Darboux point of f_{x_0}. We have $f_{x_0}(\langle \xi-h, \xi)) = h_{x_0}(\langle \xi-h, \xi)) = I$ and $f_{x_0}((\xi, \xi+h\rangle) = h_{x_0}((\xi, \xi+h\rangle) = I$ for each positive number h. Thus ξ is a Dorboux point of f_{x_0}. All points of the domain of f_{x_0} are Darboux points of the section. It follows from the Császár theorem that the section has the Darboux property.

The proof of properties of f^y can be carried out analogously.

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NIEMIERZALNA FUNKCJA DWÓCH ZMIENNYCH MIERZALNA PO KAŻDEJ ZMIENNEJ

S t r e s z c z e n i e

Praca zawiera dowód następującego twierdzenia: Istnieje funkcja niemierzalna f (w sensie Lebesque'a) w kwadracie $[0,1] \times [0,1]$ taka, że wszystkie funkcje 1 zmiennej postaci $f_{x_0}(y) = f(x_0,y)$ oraz $f^{y_0}(x) = f(x,y_0)$, $x_0,y_0 \in [0,1]$

1) należą do I klasy Baire'a*,
2) mają własność Darboux,
3) są aproksymatywnie ciągłe, z wyjątkiem co najwyżej jednego punktu.

НЕИЗМЕРИМАЯ ФУНКЦИЯ ДВУХ ПЕРЕМЕННЫХ, ИЗМЕРИМАЯ ПО КАЖДОЙ ПЕРЕМЕННОЙ

Р е з ю м е

В работе доказывается существование неизмеримой по Лебегу функции f на $[0,1] \times [0,1]$, такой, что все функции вида $f_{x_0}(y) = f(x_0,y)$ $f^{y_0}(x) = f(x,y_0)$, $x_0,y_0 \in [0,1]$

1. Принадлежат к I классу Бэра*,
2. Обладают свойством Дарбу,
3. Являются аппроксимативно непрерывными за исключением, может быть, одной точки.