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Olga MACEDOŃSKA

ANOTHER FORMULATION OF HANNA NEUMANN'S 5'th PROBLEM

Summary. In the paper an equivalence of the fifth problem of H. Neumann and a problem of an existance of a nonabelian variety with every abelian subgroup in the centre of its normalizer has been proved. Moreover a condition for positive solution of the problem has been found which says that it is sufficient to find a nonabelian variety with a transitive relation of normality.

The problem in the title is: "Does there exist a non-abelian variety without a non-abalian metabelian subvariety?". The article shows that an example of a non-abelian variety of t-groups (with the transitivity of the relation of normality) would give a positive solution of the problem. It is shown also that the problem is equivalent to: "Does there exist a non-abelian variety where every abelian subgroup belongs to the centre of its normalizer?".

A group G is called a t-group if for any given subgroups A, B, C of G, the relation $A \lhd B \lhd C$ implies $A \lhd C$. Obviously all subgroups and homomorphic images of a t-group are t-grcups.

THEOREM 1. If a variety $\mathfrak{M}$ consists of t-groups then all finite groups and all soluble groups in $\mathfrak{M}$ are abelian.

Proof. According to ([4], Theorem 3.3.1) a finitely generated soluble t-group is finite or abelian. Thus we need to consider only the case of a finite group G in $\mathfrak{M}$. Let $|G| = p^s n$, $(p,n) = 1$, with p prime. We shall denote by P a Sylow p-subgroup in G and by N its normalizer. Since P is a nilpotent t-group, it is abelian ([4], Corollary to Theorem 6.1.1). We take a direct product $N \times N$ of two copies of N and denote by D the diagonal subgroup in $P \times P$. Then, since $N \times N \in \mathfrak{M}$, $D \lhd (P \times P) \lhd (N \times N)$ implies $D \lhd (N \times N)$, which is possible if and only if elements of P and N commute. Thus P belongs to the centre of its normaliser and by ([2], Theorem 14.3.1) there exists a normal subgroup H in G, such that $|H| = n$, $G/H \cong P$. Since P is abelian $G' \leq H$, and $(|G'|,p) = 1$ for every prime diviser of $|G|$ which gives $|G'| = 1$ and G is abelian. Theorem is proved.

From the proof of the Theorem we can see that if a variety consists of t-groups, then every abelian subgroup is in the centre of its normalizer. The autor does not know if a class of groups with this property which is closed with respect to taking subgroups and homomorphic images is a variety.

THEOREM 2. The variety $\mathfrak{M}$ consists of t-groups if and only if it satisfies the law:

$$[x,y] = u(x,y), \quad \text{where} \quad u(x,y) \in \big[[\langle x\rangle, \langle y\rangle], \langle x\rangle\big]. \tag{1}$$

Proof. We show first that G is a t-group if and only if for every two subgroups A, B of G

$$A^B = A^{A^B}, \tag{2}$$

where A^B is a subgroup generated by all the conjugates $b^{-1}ab$, $a \in A$, $b \in B$. Let G be a t-group, then $A^{A^B} \lhd A^B \lhd \langle A,B\rangle$ implies $A^{A^B} \lhd \langle A,B\rangle$. Since A^B is the smallest normal subgroup of $\langle A,B\rangle$ containing A, we get $A^B \subseteq A^{A^B}$ and hence (2) holds. Conversely, suppose $A \lhd C \lhd B$ in G. Then $A^B \lhd C$ and $A \lhd A^B \lhd C$. This implies $A^{A^B} = A$ which together with (2) gives A B and hence, G is a t-group.

Since the commutator subgroup $[A,B]$ is normal in $\langle A,B\rangle$, we can get with the use of commutator calculus $A^B = A[A,B]$ and $A^{A^B} = A[A^B,A] = A\big[[A,B],A\big]$. Thus G is a t-group if and only if for every two subgroups A, B of G the relation

$$[A,B] \subseteq A\big[[A,B],A\big] \tag{3}$$

holds. If now G is a relatively free t-group then it follows from (3) that G satisfies the law (1). Conversely, if G satisfies the law (1) then, for every two subgroups in G, the relation (3) holds, and G is a t-group which finishes the proof.

The simplest example of the law of the type (1) $[x,y] = [x,y,y,\dots,y]$ is considered in [1]. It is shown that for y repeated n times for n = = 2 or 3 the correspondent variety is abelian. For $n \geqslant 4$ the problem is open.

Now by F we shall denote a free group of rank two with generators x, y. By V we denote a verbal subgroup of a variety $\mathfrak{M}$ under consideration. By A and N(A) we denote an abelian subgroup of a group G in $\mathfrak{M}$ and its normalizer in G respectively.

THEOREM 3. A variety $\mathfrak{M}$ does not contain a non-abelian metabelian subvariety if and only if every abelian subgroup of any group in $\mathfrak{M}$ belongs to the centre of its normalizer.

Proof. If $\mathfrak{M}$ does not contain a non-abelian metabelian subvariety then $F' \leqslant F''V$ and hence $[x,y] = u(x,y)$, where $u(x,y) \in \big[[\langle x\rangle, \langle y\rangle], [\langle x\rangle, \langle y\rangle]\big]$ is a law in $\mathfrak{M}$. If now $A \subseteq G \in \mathfrak{M}$, then for $a \in A$, $n \in N(A)$, $[a,n] \in \big[[A,N],[A,N]\big] \subseteq [A,A] = 1$.

Conversely, let in a variety $\mathfrak{M}$ every abelian subgroup belong to the centre of its normalizer. We take than $G = F/[x^F, x^F]V$. Since the image x^F in G is an abelian normal subgroup, we get $[x,y] \in [x^F, x^F]V = [\langle x\rangle[\langle x\rangle,\langle y\rangle], \langle x\rangle[\langle x\rangle,\langle y\rangle]]V = [\langle x\rangle,\langle y\rangle,\langle x\rangle]F''V$. This means that the law (1) holds in every metabelian subgroup in $\mathfrak{M}$ and hence, by Theorem 2 the subvariety of metabelian subgroups in $\mathfrak{M}$ consists of t-groups, and by Theorem 1 it is abelian, which finishes the proof.

REFERENCES

[1] Gupta N.: Some Group-Laws Equivalent to the Commutative Law. Arch. Math. Vol XVII, 1966, 97-102.

[2] Hall M.Jr.: The theory of groups. The Macmillan Co. N.Y. 1959.

[3] Neumann H.: Varieties of Groups. Springer-Verlag, 1967.

[4] Robinson D.: Groups in which normality is atransitive relation. Proc. Camb. Phil. Soc. 1964, 60, 21-38.

INNE SFORMUŁOWANIE 5-TEGO PROBLEMU HANNY NEUMANN

S t r e s z c z e n i e

Wykazano, że piąty problem H. Neumann jest równoważny problemowi istnienia nieabelowej rozmaitości, gdzie każda podgrupa abelowa leży w centrum swego normalizatora. Pokazano również, że dla pozytywnego rozwiązania problemu, wystarczy znaleźć nieabelową rozmaitość z tranzytywnością pojęcia dzielnika normalnego.

ДРУГАЯ ФОРМА 5-ТОЙ ПРОБЛЕМЫ ХАННЫ НЕЙМАНН

Р е з ю м е

Показано, что пятая проблема Ханны Нейманн равносильна вопросу о существовании неабелевого многообразия, где каждая абелева подгруппа лежит в центре своего нормализатора. Показано также, что для положительного решения проблемы достаточно найти неабелево многообразие с транзитивностью понятия нормального делителя.