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A METHOD FOR CONSTRUCTING INVARIANT FUNDAMENTAL SOLUTIONS FOR $P(\Delta_m)$

Dedicated to Prof. Z. Zahorski

Summary. This paper contains a method of constructing of fundamental solution of the operator $P(\Delta)$, where Δ is the Laplace operator and P is a polynomial. First, the authors construct a fundamental solution in the case $P = x^n$, and then using the classical method of Frobenius the general case is solved.

Introduction. In this paper we present a method for determining fundamental solutions for the operator $P(\Delta_m)$ where

$$\Delta_m = \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2}, \qquad m \geqslant 2$$

is the Laplace operator and P is an arbitrary polynomial.
The method is based on the invariance of the operator $P(\Delta_m)$ (cf. Def. 1 and Th. 1) which allows us to reduce the multidimensional problem to a one-dimensional. In this way we find a fundamental solution for the homogeneous operator $(\Delta_m)^r$ and the results obtained are then applied for finding a fundamental solution for an arbitrary operator $P(\Delta_m)$, in the form of a suitably constructed series. The convergence of those series results from the well-known Frobenius theorems concerning ordinary differential operators with regular singularities [2].

The fundamental solutions construced are rotation invariant (see Remark at the end of the paper) and in some cases they are homogeneous (u^0 in Th. 2 and u_1 in Th. 3) in other inhomogeneous (u^1 in Th. 2 and u_2, u_3 in Th. 3) depending on the degree of the polynomial P and the dimension m of the space.

A similar method has been applied in our paper [4] for determining fundamental solutions for the operator $P(\Delta_{mn})$ where P is an arbitrary polynomial and $\Box_{mn} = \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} \frac{\partial^2}{\partial y_i^2}$, and in paper [3] for a wide class of invariant operators.

1. NOTATION AND DEFINITIONS

R^m will denote the m-dimensional Eclidean space. N_o stands for the set of non-negative integers, N - for the set of positive itegers. We apply the notation commonly used in the theory of distributions and of differential operators. In particular $C_o^k(\Omega)$ stands for the set of compactly supported C^k $(0 \leqslant k \leqslant \infty)$ functions with support in an open set $\Omega \subset R^m$. The value of a distribution u on a test function $\varphi \in C_o^\infty(\Omega)$ will be written an $u[\varphi]$. By δ we denote the Dirac measure at zero.

In this paper we assume $m \geqslant 2$ and put $\bar{m} = \frac{m-2}{2}$. By S_m we denote the set $S_m = \{(x_1,\dots,x_m): x_1^2 + \dots + x_m^2 = 1\}$, ω_m is the Lebesque measure on this surface end

$$|S_m| = \int_{S_m} d\omega_m.$$

By R_+ we denote $R_+ = R_+^1 = \{s \in R^1 : s > 0\}$ and $\bar{R}_+$ stands for $\bar{R}_+ = \{s \in R^1 : s \geqslant 0\}$. Let $k \in N_o$. We say that a function defined on $\bar{R}_+$ is of class $C^k(\bar{R}_+)$ if it extends to a function in $C^k(R^1)$. We denote by F the function $F(x) = |x|^2 = x_1^2 + \dots + x_m^2$ for $x \in R^m$, playing a fundamental role in the study of the operator $\Delta_m = \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2}$ (or its iterations). By χ we denote an arbitrary function in $C_o^\infty(R^1)$ equal to 1 in a neighbourhood of zero.

We shall relate to the function F a linear operation F_* called the operation of averaging. The name is motivated by condition (1) which appears in Lemma 1 in which the existence of the operation F_* and an asymptotic expansion for $F_*\varphi$ with $\varphi \in C_o^\infty(R^m)$ are established.

<u>Lemma 1</u>. There exists a linear operation F_*

$$C_o^\infty(R^m) \ni \varphi \longrightarrow F_* \varphi \in C^0(\bar{R}_+)$$

such that for every function $f \in C^0(\bar{R}_+)$ and $\varphi \in C_o^0(R^m)$

$$\int_{R^m} (f \circ F)(x)\varphi(x)dx = \int_0^\infty (F_* \varphi)(s)f(s)ds. \tag{1}$$

supp $F_* \varphi$ is bounded i.e. there exists $A > 0$ such that supp $F_* \varphi \subset 0, A$. Moreover

$$C_o^\infty(R^m) \ni \varphi \to F_* \varphi \in C^\infty(R_+)$$

and for every $N \in \mathbb{N}$

$$(F_*\varphi)(s) = \chi(s) \sum_{i=0}^{N} s^{\bar{m}+i} c_i(\varphi) + (R_N\varphi)(s) \quad \text{for} \quad s \geq 0. \tag{2}$$

where: $R_N\varphi$ is a function with compact support of class $N + 1 + [\bar{m}]$ and flat at zero up to order $N + 1 + [\bar{m}]$; $c_i(\varphi)$ $(i = 1,\dots,N)$ are linear functionals, $c_0(\varphi) = |S_m|\,\varphi(0)$. Consequently putting for $s \leq 0$: $(F_*\varphi)(s)$ equal to zero if $\bar{m} > 0$ and equal to $\chi(s)c_0(\varphi)$ if $\bar{m} = 0$ we get $F_*\varphi \in D^{[\bar{m}]}(R^1)$. Moreover if $\varphi_\vartheta \xrightarrow[\vartheta\to\infty]{} 0$ in $D(R^m)$, then $F_*\varphi_\vartheta \xrightarrow[\vartheta\to\infty]{} 0$ in $D^{[\bar{m}]}(R^1)$

Proof. For $f \in C^0(\overline{R_+})$, $\varphi \in C_0^0(R^m)$ we have the following relations

$$\int_{R^m} (f \circ F)(x)\varphi(x)dx = \int_0^\infty f(r^2)r^{n-1}\left\{\int_{S_m} \varphi(r\omega)d\omega_m\right\}dr =$$

$$= \frac{1}{2}\int_0^\infty f(s)\left\{\int_{S_m}\varphi(\sqrt{s}\omega)d\omega_m\right\}s^{\bar{m}}ds = \int_0^\infty f(s)(F_*\varphi)(s)ds,$$

where

$$(F_*\varphi)(s) = \frac{1}{2}s^{\bar{m}}\int_{S_m}\varphi(\sqrt{s}\omega)d\omega_m.$$

To prove (2) take $\varphi \in C_0^\infty(R^m)$ and put

$$(H\varphi)(\mu) = \int_{S_m}\varphi(\mu\omega)d\omega_m \quad \text{for } \mu \in R^1.$$

$$(\Phi\varphi)(s) = (H\varphi)(\sqrt{s}) \quad \text{for } s \geq 0.$$

Observe that $H\varphi$ is a $C^\infty(R^1)$ even function and that

$$(F_*\varphi)(s) = \frac{1}{2}\cdot s^{\bar{m}}(\Phi\varphi)(s) \quad \text{for} \quad s \geq 0, \tag{3}$$

$$\Phi\varphi \in C(\overline{R_+}) \cap C^\infty(R_+).$$

It follows that $(H_1\varphi)(\mu)$ equal to $\frac{1}{2\mu}\frac{d(H\varphi)}{d\mu}(\mu)$ is a $C^{\infty}(R^1)$ even function and that

$$(\Phi_1\varphi)(s) = \frac{d(\Phi\varphi)(s)}{ds} = (H_1\varphi)(\sqrt{s}) \quad \text{for} \quad s \geq 0.$$

We conclude as before that $\Phi_1\varphi \in C(\overline{R}_+) \cap C^{\infty}(R_+)$, hence $\Phi\varphi \in C^1(\overline{R}_+)$. By induction $\Phi\varphi \in C^{\infty}(\overline{R}_+)$ and for every $N \in \mathbb{N}$

$$\Phi\varphi(s) - \sum_{i=0}^{N} \frac{(\Phi\varphi)^{(i)}(0)}{i!} s^i = \frac{1}{N!}\int_0^s (\Phi\varphi)^{(N+1)}(t)(s-t)^N dt.$$

Hence from (3) follows the assertion (2) wiht $C_i(\varphi) = \frac{1}{2}\frac{(\Phi\varphi)^{(i)}(0)}{i!}$ $(i = 0,1,\ldots,N)$, $C_0(\varphi) = \frac{1}{2}(\Phi\varphi)(0) = \frac{1}{2}|S_m|\,\varphi(0)$ and the continuity of the operation $D(R^m) \ni \varphi \longrightarrow F_*\varphi \in D^{[\bar{m}]}(R^1)$.

<u>Definition 1</u>. Let P be a linear differential operator of finite order M with smooth coefficients defined on R^m. We say that P is F-invariant if there exists on ordinary fifferential operator L defined on $\overline{R}_+ = F(R^m)$ such that

$$P(f \circ F) = Lf \circ F \quad \text{for} \quad f \in C^M(\overline{R}_+).$$

In this paper we shall consider the operator

$$P_r = \sum_{\vartheta=0}^{r} a_\vartheta(\Delta_m)^\vartheta, \quad (\Delta_m)^0 = \mathrm{Id} \text{ in } D'(R^m), \quad a_r = 1$$

with constant coefficients a_ϑ.

<u>Theorem 1</u>. The operator P_r of order $2r$ is F-invariant. More precisely

$$P_r(f \circ F) = (L_r f) \circ F \quad \text{for} \quad f \in C^{2r}(\overline{R}_+) \tag{4}$$

where

$$L_r = \sum_{\vartheta=0}^{r} a_\vartheta L^\vartheta, \; L^0 = \mathrm{Id} \text{ in } D'(R^1)$$

$$L = 2m\frac{d}{ds} + 4s\frac{d^2}{ds^2}.$$

Moreover for every function $\varphi \in C_o^\infty(R^m)$ we have

$$F_*((P_r)^{tr}\varphi)(s) = (L_r)^{tr}(F_*\varphi)(s) \quad \text{for} \quad s > 0, \tag{5}$$

where $(P_r)^{tr}$ denote the formal transpose of the operator P_r

$$(L_r)^{tr} = \sum_{\vartheta=0}^{r} a_\vartheta (L^\vartheta)^{tr} = \sum_{\vartheta=0}^{r} a_\vartheta (L^{tr})^\vartheta, \; L^{tr} = 2(4-m)\frac{d}{ds} + 4s\frac{d^2}{ds^2} \tag{6}$$

Proof. We restrict ourselves to the proof of (5) because the other assertions are very easy to verify. Take $\varphi \in C_o^\infty(R^m)$ and $f \in C_o^{2r}(\bar{R}_+)$. Then from Lemma 1 we get

$$\int_0^\infty f(s)F_*((P_r)^{tr}\varphi)(s)ds = \int_{R^m} (f \circ F)(x)(P_r)^{tr}\varphi(x)dx =$$
$$= \int_{R^m} P_r(f \circ F)(x)\varphi(x)dx,$$
$$\tag{7}$$
$$\int_0^\infty f(s)(L_r)^{tr}(F_*\varphi)(s)ds = \int_0^\infty (L_r f)(s)(F_*\varphi)(s)ds =$$
$$= \int_{R^m} (L_r f \circ F)(x)\varphi(x)dx.$$

Using the invariance relation (4) we deduce from (7) assertion (5).

In Section 2 we construct a fundamental solution of the homogeneous operator $P_r = (\Delta_m)^r$ (i.e. $a_\vartheta = 0$ for $= 0,1,\ldots,r-1$) and then in Section 3 we consider the general case.

2. FUNDAMENTAL SOLUTION OF THE OPERATOR $(\Delta_m)^r$, $r \geq 1$

Let $P = (\Delta_m)^r$, $r \geq 1$. Observe that in this case the corresponding one-dimensional operators $L_r = L^r$ and $(L_r)^{tr} = (L^r)^{tr} = (L^{tr})^r$ are homogeneous of order r. This means that for any real number λ

$$L^r(s^\lambda) = p(\lambda)s^{\lambda-r}, \; (L^{tr})^r(s^\lambda) = w(\lambda)s^{\lambda-r}$$

where

$$p(\lambda) = 4^r\lambda(\lambda-1)\ldots(\lambda-r+1)(\lambda+\bar{m})(\lambda+\bar{m}-1)\ldots(\lambda+\bar{m}-r+1),$$

$$w(\lambda) = 4^r\lambda(\lambda-1)\ldots(\lambda-r+1)(\lambda-\bar{m})(\lambda-m-1)\ldots(\lambda-m-r+1).$$

The polynomials p, w are called characteristic polynomials of the operators L^r adn $(L^{tr})^r$ respectively. If λ is a root of the polynomial w of multiplicity k, then $-\lambda+r-1$ is a root of p of the same multiplicity.

Let $\chi \in C_0^\infty(R)$, $\chi = 1$ in a neighbourhood of zero. Then there exist constants $c_\vartheta(\lambda)$ ($\vartheta = 1,\ldots,2r$) such that

$$(L^r)^{tr}(s^\lambda\chi(s)) = s^{\lambda-r}\chi(s)w(\lambda) + \sum_{\vartheta=1}^{2r} c_\vartheta(\lambda)\chi^{(\vartheta)}s^{\lambda-r+\vartheta}. \tag{8}$$

Note that $\lambda = \bar{m}$ is a root of the polynomial w, hence $r-\bar{m}-1$ is a root of p. There are precisely two possibilities:

(i) $\bar{m}$ is a simple root of the polynomial w,

(ii) $\bar{m}$ is a root of w of multiplicity 2 precisely.

Define formally a functional E^h $(h = 0,1)$ putting[1)]

$$E^h[\beta] = \mathrm{Pf}\int_0^\infty s^{r-\bar{m}-1}(\ln s)^h\beta(s)ds \quad \text{for } \beta \in D^{[\bar{m}]}(R)$$

Clasearly E^h is a distribution of order $[\bar{m}]$ with support in $[0,+\infty)$.
We begin by considering the case (i). In this case

$$w(\lambda) = (\lambda-\bar{m})v^0(\lambda)$$

[1)] For $\alpha \geq 1$ we define

$$\mathrm{Pf}\int_0^\infty \frac{(\ln s)^k}{s^\alpha}\beta(s)ds = \lim_{\varepsilon\to 0}\left\{\int_\varepsilon^\infty \frac{(\ln s)^k}{s^\alpha}\beta(s)ds + \sum_{i=0}^{[\alpha]-1}\frac{\beta^{(i)}(0)}{i!}v_{i(\varepsilon)}^{\alpha,k}\right\}$$

where

$$v_i^{\alpha k}(s) = \sum_{j=0}^{k}(-1)^j\frac{k!}{(k-j)!}\frac{s^{i-\alpha+1}}{(i-\alpha+1)^{j+1}}(\ln s)^{k-j} \quad \text{for } i < \alpha - 1$$

$$= \frac{1}{k+1}(\ln s)^{k+1} \quad \text{for } i = \alpha-1,\ \alpha \text{ and integer}$$

In $\alpha < 1$ the symbol Pf in the definition of E^h can be neglected.

where

$$v^{o}(\bar{m}) = 4^{r}\ \bar{m}(\bar{m}-1)\ldots(\bar{m}-r+1)(-1)^{r-1}(r-1)! \neq 0. \tag{9}$$

We shall show that

$$L^{r}E^{o}[F_{*}\varphi] = -\frac{1}{2}|S_{m}|\,v^{o}(\bar{m})\varphi(0) \quad \text{for} \quad \varphi \in C_{o}(R^{m}). \tag{10}$$

Observe first that $L^{r}(s^{-\bar{m}+r-1}) = 0$ for $s > 0$. Thus the distribution E^{o} satisfies the equation

$$L^{r}E[\beta] = 0 \quad \text{for} \quad \beta \in C_{o}^{[\bar{m}]+2r}(R^{1}) \text{ flat at zero up to order } [\bar{m}]+2r \tag{11}$$

and therefore by Lemma 1 it verifies also the relation

$$L^{r}E[R_{N}\varphi] = 0 \tag{12}$$

for $\varphi \in C_{o}(R^{m})$ and $N \in \mathbb{N}$ sufficiently large.
Thus by Lemma 1[1)]

$$L^{r}E^{o}[F_{*}\varphi] = \sum_{i=0}^{N} c_{i}(\varphi)L^{r}E^{o}\left[s^{\bar{m}+i}\chi(s)\right]. \tag{13}$$

To compute this sum take $\lambda \geq \bar{m}$ and observe that in view of (8) we have

$$L^{r}E^{o}\left[s^{\lambda}\chi(s)\right] = w(\lambda)\,\mathrm{Pf}\int_{0}^{\infty} s^{\lambda-\bar{m}-1}\chi(s)ds +$$

$$+\sum_{\vartheta=1}^{2r} c_{\vartheta}(\lambda)\int_{0}^{\infty}\chi^{(\vartheta)}(s)s^{\lambda-\bar{m}+\vartheta-1}ds.$$

Integrating the last integral by parts $(\vartheta-1)$-times we get for a suitable constant $b(\lambda)$:

$$L^{r}E^{o}\left[s^{\lambda}\chi(s)\right] = w(\lambda)\,\mathrm{Pf}\int_{0}^{\infty} s^{\lambda-\bar{m}-1}\chi(s)ds + b(\lambda)\int_{0}^{\infty}\chi'(s)s^{\lambda-\bar{m}}ds. \tag{14}$$

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1) According to (5) $(L^{r})^{tr}(F_{*}\varphi)$ is computed outside zero.

Suppose $\lambda > \bar{m}$. Then in the second integral integration by parts can be performed once again leading to

$$L^r E^o[s^\lambda \chi(s)] = w(\lambda) \int_0^\infty s^{\lambda-\bar{m}-1}\chi(s)ds - b(\lambda)(\lambda-\bar{m}) \int_0^\infty \chi(s)s^{\lambda-\bar{m}-1}ds$$

$$= (w(\lambda) - b(\lambda)(\lambda-\bar{m})) \int_0^\infty s^{\lambda-\bar{m}-1}\chi(s)ds. \tag{15}$$

The left-hand side of (15) is independent of χ because E^o is a solution of (11). For the right-hand side to be independent of χ the relation $w(\lambda) = b(\lambda)(\lambda-\bar{m})$ must hold. Then from (15) we get

$$L^r E^o[s^\lambda \chi(s)] = 0 \quad \text{for} \quad \lambda > \bar{m}. \tag{16}$$

On the other hand $b(\lambda) = v^o(\lambda) = \frac{w(\lambda)}{\lambda-\bar{m}}$, $b(\bar{m}) = v^o(\bar{m})$, $w(\bar{m}) = 0$ and therefore (14) yields the following formula:

$$L^r E^o[s^{\bar{m}}\chi(s)] = -v^o(\bar{m}). \tag{17}$$

From Lemma 1, formulas (12), (16) and (17) we derive easily assertion (10).

Let us consider now the case (ii) when $\bar{m}$ is a double root of the polynomial w. In this case $\bar{m} = j$ where $j \in N_o$, $0 \leq j \leq r-1$, $w(\lambda) = (\lambda-\bar{m})^2 v^1(\lambda)$, $v^1(\bar{m}) \neq 0$ and $r-\bar{m}-1$ is a double root of p. Hence $L^r(s^{r-\bar{m}-1}\ln s) = 0$ for $s > 0$ and consequently E^1 satisfies equation (11) which together with Lemma 1 implies that E^1 satisfies also (12) and consequently

$$L^r E^1[F_* \varphi] = \sum_{i=0}^{N} c_i(\varphi) L^r E^1[s^{\bar{m}+i}\chi(s)]. \tag{18}$$

As in case (i) using (8) and integration by parts we obtain for every $\lambda \geq \bar{m}$ the relation

$$L^r E^1[s^\lambda \chi(s)] = w(\lambda)\,\mathrm{Pf} \int_0^\infty s^{\lambda-\bar{m}-1}\chi(s)\ln s\, ds +$$

$$+ b_1(\lambda)\,\mathrm{Pf} \int_0^\infty s^{\lambda-\bar{m}}\ln s\, \chi'(s)ds + b_2(\lambda)\,\mathrm{Pf} \int_0^\infty s^{\lambda-m}\chi'(s)ds \tag{19}$$

with some constants $b_1(\lambda)$, $b_2(\lambda)$.

Suppose $\lambda > \bar{m}$. Then in the last two summands the integration by parts can be performed once again leading to the formula

$$L^r E^1[s^\lambda \chi(s)] = (w(\lambda) - b_1(\lambda)(\lambda-\bar{m})) \int_0^\infty s^{\lambda-\bar{m}-1}\chi(s)\ln s\, ds +$$

$$- (b_1(\lambda) + b_2(\lambda)(\lambda-\bar{m})) \int_0^\infty \chi(s) s^{\lambda-\bar{m}-1} ds. \quad (20)$$

By the same argument as in case (i) the left-hand side of (20) is independent of χ, which leads to the following relations:

$$w(\lambda) - b_1(\lambda)(\lambda-\bar{m}) = 0, \quad b_1(\lambda) + b_2(\lambda)(\lambda-\bar{m}) = 0.$$

Consequently

$$L^r E^1[s^\lambda \chi(s)] = 0 \quad \text{for } \lambda > \bar{m} \quad (21)$$

and on the other hand

$$w(\bar{m}) = b_1(\bar{m}) = 0, \quad b_2(\bar{m}) = -v^1(\bar{m}) \neq 0.$$

Hence in view of (19) we obtain[1)]

$$L^r E^1[s^{\bar{m}}\chi(s)] = v^1(\bar{m})$$

and so by Lemma 1 and formulas (18) and (21) we get

$$L^r E^1[F_* \varphi] = c_0(\varphi) v^1(\bar{m}) = \tfrac{1}{2}|S_m|\, v^1(\bar{m})\varphi(0) \quad (22)$$

where[2)]

$$v^1(\bar{m}) = 4^r \bar{m}(\bar{m}-1)\ldots(\bar{m}-j+1)(\bar{m}-j-1)\ldots(\bar{m}-r+1)(-1)^{r-1}(r-1)! \quad (23)$$

We shall construct now a fundamental solution of the operator $P = (\Delta_m)^r$. To this end put

$$u^i[\varphi] = b^i E^i[F_* \varphi] \quad \text{for} \quad \varphi \in D(R^m), \quad (i = 0,1)$$

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1) Note that $L^r E^0[s^{\bar{m}}\chi(s)] = 0$ in case (ii).

2) If $\bar{m} = 0$ then $v^1(\bar{m}) = v^1(0) = 4^r(-1)^{r-1}(r-1)!$, if moreover $r = 1$ then $v^1(0) = 4$.

where

$$b^0 = \frac{-2}{|S_m| v^0(\bar{m})}, \quad b^1 = \frac{2}{|S_m| v^1(\bar{m})} \tag{24}$$

and $v^0(\bar{m})$, $v^1(\bar{m})$ are given by (9) and (23) correspondingly. From the last assertion of Lemma 1 it follows easily that $u^i \in D(R^m)$ $(i = 0,1)$ and from (5), (10) and (22) we get for $P_r = (\Delta_m)^r$, $L_r = L^r$:

$$(\Delta_m)^r u^i[\varphi] = P_r u^i[\varphi] = u^i[(P_r)^{tr}\varphi] = b^i E^i[F_*(P_r)^{tr}]$$

$$= b^i E^i[(L_r)^{tr}(F_*\varphi)] = b^i L^r E^i[F_*\varphi] = \delta\,[\varphi]$$

for $\varphi \in D(R^m)$.

Observe further that

$$E^0[F_*\varphi] = \int_0^\infty s^{r-\bar{m}-1}(F_*\varphi)(s)ds = \frac{1}{2}\int_0^\infty s^{r-1}\Big(\int_{S^m}\varphi(s\omega)d\omega\Big)ds =$$

$$= \int_0^\infty \rho^{2r-1}\Big(\int_{S^m}\varphi(\rho\omega)d\omega\Big)d\rho = \int_{R^m}\frac{\varphi(x)dx}{|x|^{m-2r}} \quad \text{for } \varphi \in D(R^m).$$

Analogously

$$E^1[F_*\varphi] = 2\int_{R^m}\frac{\varphi(x)}{|x|^{m-2r}}\ln|x|\,dx \quad \text{for } \varphi \in D(R^m).$$

Thus we have proved the following theorem:

<u>Theorem 2</u>. Let $m \geq 2$, $\bar{m} = \frac{m-2}{2}$, $r \in \mathbb{N}$. Then a fundamental solution of the operator $(\Delta_m)^r$ is given by

$$u^0[\varphi] = b^0\int_{R^m}\frac{\varphi(x)}{|x|^{m-2r}}dx \quad \text{for } \varphi \in D(R^m)$$

if $\bar{m} \neq j$, $j \in N_0$, $0 \leq j \leq r-1$, and by

$$u^1[\varphi] = 2b^1\int_{R^m}\frac{\varphi(x)}{|x|^{m-2r}}\ln|x|\,dx \quad \text{for } \varphi \in D(R^m)$$

if $\bar{m} = j$, $j \in N_0$ $0 \leq j \leq r-1$. The constants b^0, b^1 are defined by (24).

3. FUNDAMENTAL SOLUTION OF THE OPERATOR $P(\Delta_m)$

Let P_r be an operator with constant coefficients a_ϑ:

$$P_r = \sum_{\vartheta=0}^{r} a_\vartheta(\Delta_m)^\vartheta, \ (\Delta_m)^0 = \mathrm{Id} \ \text{ in } \ D'(R^m), \ a_r = 1. \tag{25}$$

Retain the notation L and L_r of Theorem 1 and denote by p_h the characteristic polynomial of the operator L^h:

$$p_h(\alpha) = 4^h\alpha(\alpha-1)\ldots(\alpha-h+1)(\alpha+\bar{m})(\alpha+\bar{m}-1)\ldots(\alpha+\bar{m}-h+1)$$

$$h = 1,\ldots,r.$$

Observe that p_r coincides with the polynomial p from Section 2 and that

$$t_0 = r-\bar{m}-1$$

is a root of the polynomial p_r.

Frist we shall find a classical solution of the equation

$$L_r y = 0 \ \text{ in } \ R^1_+. \tag{26}$$

Following a method of Frobenius [2] we look for a solution of (26) of the form:

$$y(s,t) = \sum_{i=0}^{\infty} c_i(t)s^{t+i} \quad \text{for} \quad s > 0, \tag{27}$$

where t is a parameter. Let us substitute formally series (27) to $L_r y$ and arrange it with respect to the powers of s:

$$\sum_{\vartheta=0}^{r} a_\vartheta L^\vartheta y(s;t) = \sum_{\vartheta=0}^{r} a_\vartheta L^\vartheta \left(\sum_{i=0}^{\infty} c_i(t)s^{t+i}\right) =$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{\min(k,r)} a_{r-j}c_{k-j}(t)L^{r-j}s^{t+k-j} =$$

$$= \sum_{k=0}^{\infty} s^{t+k-r}\left(\sum_{j=0}^{\min(k,r)} a_{r-j}c_{k-j}(t)p_{r-j}(t+k-j)\right). \tag{28}$$

By equating to zero the expressions thus obtained we arrive at the following system of equations

$$c_o(t)p_r(t) = 0, \tag{29}$$

$$\sum_{j=0}^{\min(k,r)} a_{r-j}c_{k-j}(t)p_{r-j}(t+k-j) = 0 \qquad (k = 1,2,\dots). \tag{30}$$

Neglecting the equation (29) which for $t = t_o$ is satisfied by an arbitrary c_o we can write the system (30) in the following equivalent form:

$$\sum_{i=\max(0,k-r)}^{k} a_{r-k+i}c_i(t)p_{r-k+i}(t+i) = 0, \quad k = 1,2,\dots \tag{31}$$

Looking for a non zero solution $c_1(t)$, $c_2(t),\dots$ of the system (31) we distinguish three cases. We begin with the simplest one.

Case 1. t_o is not an integer. In this case $p_r(t_o+k) \neq 0$ for $k = 1,2,\dots$ and we compute $c_k(t_o)$ from the k-th equation of the system (31) with t_o instead of t and $c_o(t) = 1$. By the theorem of Frobenius the radius of convergence of the series $\sum_{i=0}^{\infty} c_i(t_o)s^i$ is $+\infty$ and the function

$$y(s;t_o) = \sum_{i=0}^{\infty} c_i(t_o)s^{t_o+i}$$

satisfies equation (26). We define a distribution $E_1 \in D'(R^1)$ by putting[1)]

$$E_1[\alpha] = \sum_{i=0}^{\infty} c_i(t_o)s_+^{t_o+i}[\alpha] \quad \text{for } \alpha \in C_o^\infty(R^1).$$

Case 2. t_o is a negative integer. In this case all the roots of the characteristic polynomial p_r are simple and $p_r(t_o+j) = 0$ when $j = |t_o|, |t_o|+1,\dots,|t_o|+r-1$. Put

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1) We apply here the notation used by Gel'fand Shilov [1] in which $s_+^t\ln^q s_+[\alpha]$ denotes pf $\int_0^\infty s^t\ln^q s\,\alpha(s)ds$ and is meromorphic extension to the complex plane of the distribution (function) $\alpha \to \int_0^\infty s^t\ln^q s\,\alpha(s)ds$ defined for $\mathrm{Re}\, t > -1$.

$$\begin{cases} A = p_r'(0)p_r'(1)\dots p_r'(r-1), \\ c_0(t) = p_r(t-t_0)p_r(t-t_0+1)\dots p_r(t-t_0+r-1) \end{cases} \tag{32}$$

and observe that

$$c_0^{(\vartheta)}(t)\Big|_{t=t_0} = 0 \quad \text{for} \quad \vartheta = 0,1,\dots r-1, \ c_0^{(r)}(t)\Big|_{t=t_0} = A \neq 0. \tag{33}$$

We compute $c_k(t)$ from the k-th equation of the system (31) successively for $k = 1,2,\dots$. Denote by $y_q(s;t)$, $q = 0,1,\dots,r$ the series obtained from $\sum_{i=0}^{\infty} c_i(t)s^{t+i}$ by a formal differentiation $\frac{\partial^q}{\partial t^q}$ term by term. Following Frobenius the radius of convergence of the series $\sum_{i=0}^{\infty} c_i^{(\vartheta)}(t_0)s^i$, $\vartheta = 0,1,\dots,r$ is $+\infty$ and for every $q = 0,1,\dots,r$ the series

$$y_q(s;t_0) = \sum_{i=0}^{\infty} \sum_{\vartheta=0}^{q} \binom{q}{\vartheta} c_i^{(\vartheta)}(t_0)(\ln^{q-\vartheta} s)s^{t_0+i}$$

is a solution of equation (26). We define a distribution E_2 by putting

$$E_2[\alpha] = \sum_{\vartheta=0}^{r} \binom{r}{\vartheta} \sum_{i=0}^{\infty} c_i^{(\vartheta)}(t_0)s_+^{t_0+i} \ln^{r-\vartheta} s_+ [\alpha] \quad \text{for} \quad \alpha \in C_0^{\infty}(R^1).$$

<u>Case 3</u>. t_0 is an integer, $0 \leqslant t_0 \leqslant r-1$. In this case t_0 is a double root and $t_0+1,\dots,$ $t_0+(r-1-t_0) = r-1$ are simple roots of the characteristic polynomial p_r and $p_r(t_0+k) \neq 0$ if $k \geqslant r-t_0$. Put

$$\begin{aligned} &B = p_r'(t_0+1)\dots p_r'(r-1) \quad \text{if} \quad t_0 < r-1, \ B = 1 \quad \text{if} \quad t_0 = r-1, \\ &c_0(t) = p_r(t+1)\dots p_r(t-t_0+r-1) \quad \text{if} \quad t_0 < r-1, \\ &c_0(t) = 1 \quad \text{if} \quad t_0 = r-1. \end{aligned} \tag{34}$$

It follows that

$$B \neq 0, \ c_0^{(\vartheta)}(t_0) = 0 \quad \text{if} \quad \vartheta < r-1-t_0, \quad t_0 < r-1 \quad \text{and} \quad c_0^{(r-1-t_0)}(t_0) = B. \tag{35}$$

Proceeding as in Case 2 we find $c_k(t)$ from the k-th equation of the system (31) successively for $k = 1,2,\dots$. Denote by $y_{r-t_o}(s;t)$ the series obtained from $\sum_{i=0}^{\infty} c_i(t)s^{t+i}$ by formal defferentiation $\frac{\partial^{r-t_o}}{\partial t^{r-t_o}}$ term by term. Then

$$y_{r-t_o}(s;t_o) = \sum_{i=0}^{\infty} \sum_{\vartheta=0}^{r-t_o} \binom{r-t_o}{\vartheta} c_i^{(\vartheta)}(t_o)(\ln s)^{r-t_o-\vartheta} s^{t_o+i}.$$

We define a distribution E_3 putting

$$E_3[\alpha] = \sum_{i=0}^{\infty} \sum_{\vartheta=0}^{r-t_o} \binom{r-t_o}{\vartheta} c_i^{(\vartheta)} s_+^{t_o+i} \ln^{r-t_o-\vartheta} s_+ [\alpha].$$

We shall show that for $\alpha = F_* \varphi$

$$L_r E_1[\alpha] = L^r E^o[\alpha], \tag{36}$$

$$L_r E_2[\alpha] = AL^r E^o[\alpha], \tag{37}$$

$$L_r E_3[\alpha] = (r-t_o) BL^r E^1[\alpha]. \tag{38}$$

The proof will be based on the following Lemma.

<u>Lemma 2</u>. Let $\beta \in C_o^{2r}(R^1)$. Put $\alpha(s) = s^{\bar{m}}\beta(s)$ for $s > 0$ and suppose that $t > r-\bar{m}-2$. Thus for all $k = 1,2,\dots$ we have

$$\sum_{j=0}^{\min(k,r)} a_{r-j} c_{k-j}(t) L^{r-j} s_+^{t+k-j} [\alpha] = 0.$$

<u>Proof</u>. Define $K(\beta)$:

$$K(\beta) = \sum_{j=0}^{\min(k,r)} a_{r-j} c_{k-j}(t) s_+^{t+k-j} \left[(L^{r-j})^{tr} (\beta(s) s^{\bar{m}}) \right].$$

Using (6) we get for suitable constants $q_\vartheta, \vartheta = 0,1,\dots,2r-2j$

$$K(\beta) = \sum_{j=0}^{\min(k,r)} a_{r-j}c_{k-j}(t) \sum_{\vartheta=0}^{2r-2j} q_\vartheta \int_0^\infty s^{t+k+\bar{m}-r+\vartheta}\beta^{(\vartheta)}(s)ds$$

because in view of the inequality $t+k+\bar{m}-r+\vartheta > -1$ valid for all $k = 1,2,\dots$ the symbol Pf can be omitted. Then in the integrals in the right-hand side integration by parts can be performed ϑ times leading to the relation:

$$K(\beta) = Q(t) \int_0^\infty s^{t+k+\bar{m}-r}\beta(s)ds \tag{39}$$

with an adequate function Q. Take $\beta = \chi$. We shall show that $K(\chi)$ is independent of the choice of χ equal to one in a neighbourhood of zero. In fact if $\chi_1, \chi_2 \in C_0^\infty(R^1)$ are two such functions and if $\tilde{\alpha}(s) = (\chi_1(s) - \chi_2(s))s^{\bar{m}}$ for $s > 0$ then $\tilde{\alpha} \in C_0^\infty(R_+^1)$ and

$$K(\chi_1) - K(\chi_2) = K(\chi_1 - \chi_2) = \sum_{j=0}^{\min(k,r)} a_{r-j}c_{k-j}(t) \int_0^\infty L^{r-j}(s^{t+k-j})\tilde{\alpha}(s)ds$$

$$= \int_0^\infty \Big(\sum_{j=0}^{\min(k,r)} a_{r-j}c_{k-j}(t)p_{r-j}(t+k-j)\Big)s^{t+k-r}\tilde{\alpha}(s)ds = 0 \tag{40}$$

in view of (30). In order that the right-hand side of (39) with $\beta = \chi$ be independent of χ the function Q must be equal to zero, hence $K(\beta) = 0$.

We shall now prove (38). The proof of identities (36) and (37) is simpler and therefore omitted.

Take $\beta \in C^{2r}(R^1)$, $\alpha(s) = \beta(s)s^{\bar{m}}$ for $s > 0$. In view of (40), (28) Lemma 2, (35) and foot-note 1 on p.155 we get successively

$$L_rE_3[\alpha] = \sum_{\vartheta=0}^{r} a_\vartheta L^\vartheta \frac{\partial^{r-t_0}}{\partial t^{r-t_0}}\Big(\sum_{i=0}^\infty c_i(t)s_+^{t+i}\Big)\Big|_{t=t_0}[\alpha]$$

$$= \frac{\partial^{r-t_0}}{\partial t^{r-t_0}} \sum_{k=0}^\infty \sum_{j=0}^{\min(k,r)} a_{r-j}c_{k-j}(t)L^{r-j}s_+^{t+k-j}\Big|_{t=t_0}[\alpha]$$

$$= \frac{\partial^{r-t_o}}{\partial t^{r-t_o}} (C_o(t) L^r s_+^t) \Big|_{t=t_o} [\alpha]$$

$$= \sum_{\vartheta=0}^{r-t_o} \binom{r-t_o}{\vartheta} c_o^{(\vartheta)}(t) \frac{\partial^{r-t_o-\vartheta}}{\partial t^{r-t_o-\vartheta}} L^r s_+^t \Big|_{t=t_o} [\alpha]$$

$$= (r-t_o) B L^r s_+^{t_o} \ln s_+ [\alpha] + c_o^{(r-t_o)}(t_o) L^r s_+^{t_o} [\alpha]$$

$$= (r-t_o) B L^r E^1 [\alpha] + c_o^{(r-t_o)}(t_o) L^r E^o [\alpha]$$

$$= (r-t_o) c_o^{(r-t_o-1)}(t_o) L^r E^1 [\alpha] .$$

Thus Lemma 1 implies (38) for $\alpha = F_* \varphi$, $\varphi \in C_o^\infty(R^m)$.

From (36), (37) and (10) we obtain

$$L_r E_1 [R_* \varphi] = -\frac{1}{2} |S_m| v^o(\bar{m}) \varphi(0)$$

$$\text{for} \quad \varphi \in C_o^\infty(R^m)$$

$$L_r E_2 [F_* \varphi] = -\frac{1}{2} A |S_m| v^o(\bar{m}) \varphi(0)$$

with $v^o(\bar{m})$ defined by (9) and A given by (33). Analogously (38) and (22) leads to the formula

$$L_r E_3 [F_* \varphi] = \frac{1}{2} (r-t_o) B |S_m| v^1(\bar{m}) \varphi(0)$$

where $v^1(\bar{m})$ is given by (23) and B by (34), (35).

Put

$$b_1 = \frac{-2}{|S_m| v^o(\bar{m})}, \quad b_2 = \frac{-2}{A |S_m| v^o(\bar{m})}, \quad b_3 = \frac{2}{(r-t_o) B |S_m| v^1(\bar{m})} \tag{41}$$

and define:

$$u_i [\varphi] = b_i E_i [F_* \varphi] \quad \text{for} \quad \varphi \in C_o^\infty(R^m), \quad i = 1,2.$$

It follows from (5) that

$$P_r u_i[\varphi] = u_i[(P_r)^{tr}\varphi] = b_i E_i[F_*(P_r)^{tr}\varphi] =$$

$$= b_i E_i[(L_r)^{tr} F_* \varphi] = b_i L_r E_i[F_* \varphi] = \varphi(0) = \delta[\varphi] \quad (i = 1,2,3).$$

This proves that in the case i (i = 1,2,3) u_i is a fundamental solution of the operator P_r. We formulate this result in the form analogous to Theorem 2 stated for $(\Delta_m)^r$.

<u>Theorem 3</u>. Let $m > 2$, $\bar{m} = \frac{m-2}{2}$, $r \in N$, $t_o = r-\bar{m}-1$ and P_r be the differential operator defined by (25). The fundamental solution of P_r is given by different formulas depending on which of the three possible cases occurs: 1) t_o is not an integer; 2) t_o is a negative integer; 3) t_o is an integer $0 \leqslant t_o \leqslant r-1$. Denote by u_i the fundamental solution of P_r in the case i (i = 1,2,3). We have for $\varphi \in C_o^\infty(R^m)$:

$$u_1[\varphi] = b_1 \sum_{i=0}^{\infty} c_i(t_o) \int_{R^m} \frac{\varphi(x)}{|x|^{m-2r-2i}}\, dx,$$

$$u_2[\varphi] = b_2 \sum_{\vartheta=0}^{r} \binom{r}{\vartheta} \sum_{i=0}^{\infty} c_i^{(\vartheta)}(t_o) 2^{r-\vartheta} \int_{R^m} \frac{(\ln|x|)^{r-\vartheta}}{|x|^{m-2r-2i}} \varphi(x)dx,$$

$$u_3[\varphi] = b_3 \sum_{\vartheta=0}^{r-t_o} \sum_{i=0}^{\infty} \binom{r-t_o}{\vartheta} c_i^{(\vartheta)}(t_o) 2^{r-t_o-\vartheta} \int_{R^m} \frac{(\ln|x|)^{r-t_o-\vartheta}}{|x|^{m-2r-2i}} \varphi(x)dx$$

where b_1, b_2, b_3 are constants defined by (41) with A, B given by (33) and (35), correspondingly. The coefficients c_i (i = 1,2,...) are solutions of the system (30) with $c_o = 1$ in case 1 and c_o given by (32) in case 2 and by (34) in case 3.

Remark. The fundamental solutions given by Theorem 2 and 3 are rotation invariant since they are expressible in terms of the operation F_* which is clearly rotation invariant.

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METODA KONSTRUKCJI NIEZMIENNICZYCH ROZWIĄZAŃ PODSTAWOWYCH DLA $P(\Delta_m)$

S t r e s z c z e n i e

Praca zawiera metodę konstrukcji rozwiązania fundamentalnego operatora $P(\Delta)$, gdzie Δ jest operatorem Laplace'a, zaś P wielomianem. Pierwszym krokiem jest zbudowanie rozwiązania podstawowego w przypadku $P(x) = x^n$, a następnie, stosując metodę Frobeniusa rozwijania w szereg, wykazuje się istnienie rozwiązania w przypadku ogólnym.

МЕТОД КОНСТРУКЦИИ ИНВАРИАНТНОГО ОСНОВНОГО РЕШЕНИЯ ДЛЯ $P(\Delta_m)$

Р е з ю м е

В работе даётся метод конструкции фундаментального решения оператора $P(\Delta)$, где Δ - оператор Лапласа а P - произвольный многочлен. Первый шаг заключается в конструкции решения для случая $P(x) = x^n$. Общий случай использует классический метод Фробениуса.