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QUASI-UNIFORM CONVERGENCE ON DENSE SETS

Summary. In this paper quasi-uniform sequences of real-valued functions on topological space X are considered. Sufficient conditions are formulated under which the quasi-uniform convergence on a dense subset of X implies the quasi-uniform convergence on X.

If is well known that if a sequence of functions f_n, real-valued and continuous on some topological space X, is uniformly convergent on a dense subset of X then it is uniformly convergent on whole space X (to a continuous function f). In the above assertion one cannot replace the uniformity of convergence by the quasi-uniformity [1].

Let, for example, $\{f_{2n}\}$ be the sequence of cotinuous on $\langle 0,1\rangle$ functions with limit-function equal to 0 in irational and to $\frac{1}{q}$ in rational x of the form $\frac{p}{q}$ (with minimal integer positve q). Put yet $f_{2n+1}(x) = 0$ for $x \in \langle 0,1\rangle$ and $n = 0,1,2,\dots$. The sequence $\{f_n\}$ is then quasi-uniformly convergent on the set of irrational to zero (for every irrational x, $\{f_n(x)\}$ converges to zero and subsequence $\{f_{2n+1}\}$, as a constant, converges uniformly). On the other hand $\{f_n\}$ does not converge quasi-uniformly on $\langle 0,1\rangle$ because for rational numbers x, $\{f_n(x)\}$ does not converge.

In the present example essential role played the fact that $\{f_n\}$ was not convergent and (what is not so easy to seen) that the subsequencce $\{f_{2n}\}$ was not quasi-uniformly convergent on the set of irraional numbers. The aim of this paper is to prove that these two condition (convergence of $\{f_n\}$, and quasi-uniform convergence of its subsequences), which are not fulfill by the sequence in above example, are sufficient for the quasi-uniformity of convergence of a sequence, which converges quasi-uniformly on some dense set.

All functions in this paper will be real-vaued on topological space X.

Theorem 1. Let f_n be continuous on the topological space X, for $n = 1,2,3,\dots$ and let $\{f_n(x)\}$ converge to $f(x)$ for all $x \in X$. Let moreover $\{f_n\}$ converge quasi-uniformly on some dense subset Z of X.

Then $\{f_n\}$ is quasi-uniformly convergent on X ($f_n(x) \to f(x)$ on X and

$$\bigwedge_{\varepsilon>0} \bigwedge_{n} \bigvee_{p} \bigwedge_{x\in X} \bigvee_{1\leq j\leq p} |f_{n+j}(x) - f(x)| < \varepsilon \text{).}$$

Proof. Let $\varepsilon = 5\eta > 0$ and $n \in N$ be arbitrary. There exists a number p such that

$$\bigwedge_{x \in Z} \bigvee_{1 \leqslant j < p} |f_{n+j}(x) - f(x)| < \eta$$

Put

$$A_j = \{x \in Z: |f_{n+j}(x) - f(x)| < \eta\} \quad \text{for } j = 1,2,\ldots,p.$$

Let now $x_o \in X$. Since $f_n(x_o) \to f(x_o)$ there exists n_o (we may asume $n_o > n$) such that for $m > n_o$ is $|f_m(x_o) - f(x_o)| < \eta$. Because of quasi-uniformity of convergence of $\{f_n\}$ on Z there exists a number q such that

$$\bigwedge_{x \in Z} \bigvee_{1 \leqslant i \leqslant q} |f_{n_o+i}(x) - f(x)| < \eta$$

Put now

$$B_i = \{x \in Z: |f_{n_o+i}(x) - f(x)| < \eta\} \quad \text{for } i = 1,2,\ldots,q.$$

It follows from density of $Z = \bigcup_{j=1}^{p} \bigcup_{i=1}^{q} A_j \cap B_i$ that x_o belongs to or is an accumulation point of one (or more) of sets $A_j \cap B_i$. Choose the j, i for which $x_o \in \overline{A_j \cap B_i}$. It follows from continuity of f_{n+j} and f_{n_o+i} that there exists a neighbourhood U of x_o such that

$$|f_{n_o+i}(x) - f_{n_o+i}(x_o)| < \eta \quad \text{and} \quad |f_{n+j}(x) - f_{n+j}(x_o)| < \eta \quad \text{for } x \in U.$$

Let now $x_1 \in B_i \cap A_j \cap U \neq \emptyset$. Thus

$$|f_{n+j}(x_o) - f(x_o)| \leqslant |f_{n+j}(x_o) - f_{n+j}(x_1)| + |f_{n+j}(x_1) - f(x_1)| +$$

$$+ |f(x_1) - f_{n_o+i}(x_1)| + |f_{n_o+i}(x_1) - f_{n_o+i}(x_o)| +$$

$$+ |f_{n_o+i}(x_o) - f(x_o)| < 5\eta = \varepsilon .$$

We have proved that for every $x_o \in X$ there exists a number $j=1,\dots,p$ such that $|f_{n+j}(x_o) - f(x_o)| < \varepsilon$. Then finaly

$$\bigwedge_{n} \bigwedge_{\varepsilon>0} \bigvee_{p} \bigwedge_{x\in X} \bigvee_{1\leq j\leq p} |f_{n+j}(x) - f(x)| < \varepsilon ,$$

what means that the sequence $\{f_n\}$ is quasi-uniformly convergent on X.

Definition [3]. The sequence $\{f_n\}$ of functions is said to be strong quasi-uniformly convergent to f on A iff every its subsequence $\{f_{n_k}\}$ converges quasi-uniformly to f (we well then write $f_n \xrightarrow[s.q]{} f$)

Theorem 2. Let f and f_n $(n = 1,2,\dots)$ be continuous on X and let $\{f_n\}$ converge strong quasi-unformly to f on some dense set $Z \subset X$. Then $\{f_n\}$ is strong quasi-uniformly convergent on X.

Proof. According to the theorem 1 it suffice to prove that $\{f_n(x)\}$ converges to f(x) on X. Let then $x_o \in X$. We will prove that every subsequence $\{f_{n_k}(x_o)\}$ of the sequence $\{f_n(x_o)\}$ posses a subsequence converging to $f(x_o)$, from what will be follow that $\{f_n(x_o)\}$ converges to $f(x_o)$. Let $\{f_{n_k}\}$ be arbitrary subsequence of $\{f_n\}$, and put $g_k = f_{n_k}$. The sequence $\{g_{k_j}\}$, as a subsequence of $\{f_n\}$ also converges strong quasi-uniformly to f on Z. Thus for every $m \in N$ there exists p such that

$$\bigwedge_{x\in Z} \bigvee_{1\leq j\leq p} |f(x) - g_{m+j}(x)| < \frac{1}{m}$$

Put

$$A_j^m = \left\{x \in Z: |g_{m+j}(x) - f(x)| < \frac{1}{m}\right\}.$$

For density of Z, x_o is an accumulation point of the union of sets $\bigcup_{j=1}^{p} A_j^m = Z$ or belongs to Z. Then x_o belongs to the closure $\bar{B}$ of one of this sets, say $B = A_{j_m}^m$. Because of continuity of g_{j_m+m} and f there exists a neighbourhood U of x_o such that

$$|f(x) - f(x_o)| < \frac{1}{m} \text{ and } |g_{m+j_m}(x) - g_{m+j_m}(x_o)| < \frac{1}{m} \text{ for } x \in U.$$

Let $x_1 \in U \cap B$. Then

$$|g_{m+j_m}(x_o) - f(x_o)| \leq |g_{m+j_m}(x_o) - g_{m+j_m}(x_1)| +$$

$$+ |g_{m+j_m}(x_1) - f(x_1)| + |f(x_1) - f(x_o)| < \frac{3}{m}.$$

we have obtain a sequence $\{m+j_m\}$ of integer positive numbers such that $m+j_m \to \infty$ and $g_{m+j_m}(x_o) \to f(x_o)$. Let now $\{k_l\}$ denote arbitrary increasing subsequence of $m+j_m$. Then the sequence $\{g_{k_l}\}$ is simultanously a subsequence of each $\{g_k\} = \{f_{n_k}\}$ and $\{g_{m+j_m}\}$. Thus we have prove that for every subsequence $\{f_{n_k}(x_o)\}$ of the sequence $\{f_n(x_o)\}$ there exists a subsequence $\{f_{n_{k_l}}(x_o)\}$ converging to $f(x_o)$. This finishes the proof.

Theorem 3. Let functions f_n be continuous on X and let $\{f_n\}$ converge strong quasi-uniformly on some dense subset Z of X to a function $f: Z \to R$.

Then $\{f_n\}$ is (strong) quasi-uniformly convergent on X to a function F, obviously continuous, which restriction to Z is identical with f.

Proof. First we will show that for every x_o and increasing sequence $\{n_k\}$ of indices the sequence $\{f_{n_k}(x_o)\}$ posses a convergent subsequence. Put for simplicity of denotations $g_k = f_{n_k}$. The sequence $\{g_k\}$ converges quasi-uniformly on Z, then there exists a number p_1 such that

$$\bigwedge_{x \in Z} \bigvee_{1 \le j \le p_1} |g_j(x) - f(x)| < \frac{1}{2^1}$$

Put

$$A_j^1 = \left\{x \in Z: |g_j(x) - f(x)| < \frac{1}{2^1}\right\} \quad \text{for} \quad j = 1,\ldots,p_1.$$

For x_o belongs to closure X of dense set $Z = \bigcup_{j=1}^{p_1} A_j^1$ there exists $j_1 \in \{1,\ldots,p_1\}$ such that x_o belongs to the closure of $A_{j_1}^1$.

Put now $A_{j_1}^1 = A_1$ and $k_1 = j_1$.

Let us assume that we have define sets $A_1 \supset A_2 \supset \ldots \supset A_l$ and numbers $k_1,\ldots,k_l$ such that x_o belongs to closure of A_l, $A_l \subset Z$ and

$$\bigwedge_{x \in A_j} |g_{k_j}(x) - f(x)| < \frac{1}{2^j} \quad \text{for} \quad j = 1,\ldots,l.$$

For quasi-uniformity of convergence of $\{g_k\}$ on $A_l \subset Z$ there exists a number p_l such that

$$\bigwedge_{x \in A_l} \bigvee_{1 \le j \le p_l} |g_{k_l+j}(x) - f(x)| < \frac{1}{2^{l+1}}.$$

Put

$$A_j^{l+1} = \left\{ x \in A_l : |g_{k_l+j}(x) - f(x)| < \frac{1}{2^{l+1}} \right\} \quad \text{for} \quad j = 1,\dots,p_l$$

and let A_{l+1} denote such one of A_j^{l+1}, say $A_{j_{l+1}}^{l+1}$, that x_o belongs to its closure. Put yet $k_{l+1} = k_l + j_{l+1}$.

In this way have defined, by induction process, two sequences: a) the sequence of sets A_l such that $Z \supset A_1 \supset A_2 \supset \dots$ and x_o belongs to the closure of A_l for every l, b) the increasing sequence of indices $\{k_l\}$ such that for every l

$$|g_{k_l}(x) - f(x)| < \frac{1}{2^l} \qquad \text{for} \quad x \in A_l.$$

Thus for $l = 1,2,\dots$ we have inequality

$$|g_{k_{l+1}}(x) - g_{k_l}(x)| \leqslant |g_{k_{l+1}}(x) - f(x)| + |f(x) - g_{k_l}(x)| < \frac{1}{2^{l-1}} \quad \text{for } x \in A_l.$$

Since functions g_k are continuous on X and x_o belongs to the closure of A_l we obtain the inequalities

$$|g_{k_{l+1}}(x_o) - g_{k_l}(x_o)| < \frac{1}{2^{l-1}} \qquad \text{for} \quad l = 1,2,\dots,$$

from which follows convergence of $\{g_{k_l}(x_o)\}$ to some finite limit a.

We have proved that every subsequence of $\{f_n(x_o)\}$ posses a convergent subsequence. It remains to prove that limits of such convergent subsequences are equal.

Let then $a = \lim_{k\to\infty} g_k(x_o)$, where $x_o \in X$ and $\{g_k\}$ is a subsequence of $\{f_n\}$. Let furthermore $\varepsilon > 0$ be arbitrary. There exists a number m such that

$$|g_k(x_o) - a| < \varepsilon \qquad \text{for} \quad k = m, m+1, \dots .$$

Since $\{g_k\}$ is quasi-uniformly convergent on Z there exists a number q that $Z = \bigcup_{j=1}^{q} E_j$, where

$$E_j = \left\{ x \in Z : |g_{m+j}(x) - f(x)| < \varepsilon \right\}.$$

Functions $g_{m+1},\dots,g_{m+q}$ are continuous, hence there exists a neighborhood U of x_o such that

$$|g_{m+j}(x) - g_{m+j}(x_o)| < \varepsilon \quad \text{for} \quad x \in U \quad \text{and} \quad j = 1,2,\dots,q.$$

Let now $x \in Z \cap U$ and let j_x be such number that $x \in E_{j_x}$. Then

$$|f(x)-a| \leqslant |f(x) - g_{m+j_x}(x)| + |g_{m+j_x}(x) - g_{m+j_x}(x_o)| + |g_{m+j_x}(x_o) - a| < 3\varepsilon,$$

what means that there exists $\lim_{x \to x_o} f(x) = a$, when x_o is an accumulation point of Z or $a = f(x_o)$ when $x_o \in Z$. In both cases we obtain equality of limits of convergent sequences $\{f_{n_k}(x_o)\}$. We have then proved simultanously the convergence of sequences $\{f_n(x)\}$ for all $x \in X$, and existence of continuous extension F of function f from Z to whole X in this way that the extended function F fultills the following condition

$$\lim_{n\to\infty} f_n(x) = F(x) \quad \text{for} \quad x \in X.$$

Using now or theorem 1 ($f_n(x)$ converges for every $x \in X$) or theorem 2 (for continuity of function F) we obtain quasi-uniformity of convergence of $\{f_n\}$ to F. Since this is true also for every subsequence of $\{f_n\}$ thus $\{f_n\}$ converges strong quasi-uniformly.

Remark. Theorem 2 is simultanously a corollary from theorem 3 and, on the other hand, a lemma for its. By some modifications of the proof of theorem 3 it was possible to omit the theorem 2. We did not that for the following cause.

One can, without essential exchanges in the proofs, obtain for nets of functions [2], [3] results similar as presented in theorem 1 and 2 for usualy sequences, but it is not so easy in the case of theorem 3 (the proof of convergence of $\{g_k(x_o)\}$ are typical for usualy sequences - by using criterion of summability of series). In connection with this is the following question:

Problem. Does theorem 3 remain valid by replacing in it the usualy sequences by nets?

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QUASI-JEDNOSTAJNA ZBIEŻNOŚĆ NA GĘSTYCH ZBIORACH

Wyniki pracy zebrane są w poniższych twierdzeniach:

Twierdzenie 1. Niech ciąg $\{f_n\}$ funkcji ciągłych na X będzie zbieżny na X do funkcji ciągłej f. Jeżeli ponadto $\{f_n\}$ jest zbieżny quasi-jednostajnie na pewnym zbiorze gęstym $Z \subset X$, to $\{f_n\}$ jest zbieżny quasi-jednostajnie na X.

Twierdzenie 3. Niech funkcje $\{f_n\}$ będą ciągłe na X. Jeżeli ciąg $\{f_n\}$ jest silnie quasi-jednostajnie zbieżny (tzn. quasi-jednostajnie zbieżny jest każdy jego podciąg) na gęstym zbiorze $Z \subset X$, to jest on silnie quasi-jednostajnie zbieżny na X (do funkcji będącej ciągłym rozszerzeniem na X funkcji granicznej dla zbioru Z).

Problem. Czy twierdzenie 3 jest prawdziwe dla ciągów uogólnionych (Moore'a-Smitha)?

КВАЗИ-РАВНОМЕРНАЯ СХОДИМОСТЬ НА ПЛОТНЫХ МНОЖЕСТВАХ

Р е з ю м е

В работе рассматриваются квазиравномерно сходящиеся последовательности непрерывных функций $f: X \to R$, где X - топологическое пространство. Даются достаточные условия того, что из квазиравномерной сходимости на плотном подмножестве следует квазиравномерная сходимость на всём пространстве X.