ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1986

Seria: MATEMATYKA-FIZYKA z. 48 Nr kol. 853

To Professor Zygmunt Zahorski
who held the beautful lectures
on Theory of Real Functions at
the University in Łódź when the
Author was a student.

by Włodzimierz WALISZEWSKI

# ANALYTICAL PREMANIFOLDS

Summary. The concept of an analytical premanifold (a.p.) and the concept of a smooth mapping of an a. p. into another one are introduced. There is proved in the paper that if the Cartesian product of two a. p. is an analytical manifold, then those a. p. are analytical manifolds, too.

## 0. THE CONCEPT OF AN ANALYTICAL PREMANIFOLD

Consider a set of real functions defined on arbitrary sets. Denote the union of all domains $D_\alpha$ of functions $\alpha \in M$ by $\underline{M}$. On the set $\underline{M}$ we consider the smallest topology, TopM, such that all the sets of the form $\alpha^{-1}[B]$, where $B$ is open in $\mathbb{R}$ and $\alpha \in M$, are regarded as open. For every $S \subset \underline{M}$ we denote $M_S$ the set of all functions $\beta$ such that for any $p \in D_\beta$ there exist $\alpha \in M$ and a set $U \in$ TopM such that $U \subset D$, $p \in U \cap S \subset D_\beta$, and $\beta | U \cap S = \alpha | U \cap S$. It is easy to state that $\mathrm{Top}M_S$ concides with the topology induced from TopM to the set $S$. For any real analytic function $\varphi$ in an open subset of $\mathbb{R}^n$ and any $\alpha_1,\dots,\alpha_n$ M we consider the function $\varphi(\alpha_1,\dots,\alpha_n)$ defined on the set of all $p \in D_{\alpha_1} \cap \dots \cap D_{\alpha_n}$ for which $(\alpha_1(p),\dots,\alpha_n(p)) \in D_\varphi$ by the formula

$$\varphi(\alpha_1,\dots,\alpha_n)(p) = \varphi(\alpha_1(p),\dots,\alpha_n(p)).$$

The set of all functions $\varphi(\alpha_1,\dots,\alpha_n)$, where $\varphi$ is real analytic in the open subset of $\mathbb{R}^n$ and $\alpha_1,\dots,\alpha_n \in M$, $n = 1,2,\dots$, will be called the analytical closure of the set $M$ and will be denoted by anM. We have $M \subset M_{\underline{M}}$ and $M \subset \mathrm{an}M$. The set $M$ such that

$$M = M_{\underline{M}} = \mathrm{an}M$$

will be called an analytical premanifold (a.p.).

Let M and N be a.p. and f be a mapping from $\underline{M}$ into $\underline{N}$. We say that f smoothly maps M into N,

$$f: M \longrightarrow N, \tag{0}$$

iff for any $\beta \in N$ the function $\beta \circ f$ with the domain $f^{-1}[D_\beta]$ defined by the formula $(\beta \circ f)(p) = \beta(f(p))$ for $p \in f^{-1}[D_\beta]$ belongs to M. We say that (0) is a diffeomorphism iff f is one-one, $f[\underline{M}] = \underline{N}$ and we have the smooth mapping $f^{-1}: N \longrightarrow M$. It is easy to state that all a.p. treated as abjects together with all triplets (M, f, N), where we have (0), treated as morphisms constitute a category of Eilenberg and MacLane.

## 1. GENERATING OF AN ANALYTICAL PREMANIFOLD AND THE CARTESIAN PRODUCT OF TWO ANALYTICAL PREMANIFOLDS

We will prove the following

1.1. Proposition

For any set G of real functions the set $(\mathrm{an}G)_{\underline{G}}$ is the smallest among all a.p. containg G.

Proof. First, let us remark that assigning to any C of real functions defined on subsets of $\underline{G}$ the set anC we get an algebraic closure, i. e. there are satisfied the following conditions: (i) $C \subset \mathrm{an}C$; (ii) if $C' \subset C$, so $\mathrm{an}C' \subset \mathrm{an}C$; (iii) ananC = anC.

Similarly, assigning to every such a set C the set $C_{\underline{C}}$ we get an algebraic closure, too. We will prove now that

$$\mathrm{an}(C_{\underline{C}}) \subset (\mathrm{an}C)_{\underline{C}}. \tag{1}$$

Indeed, take any $\gamma \in \mathrm{an}(C_{\underline{C}})$ and any $p \in D_\gamma$. The function $\gamma$ is of the shape $\varphi(\beta_1,\ldots,\beta_n)$, where $\beta_1,\ldots,\beta_n \in C_{\underline{C}}$ and $\varphi$ is a real analytic function in the open subset $D_\varphi$ of $R^n$. We have then $p \in D_{\beta_1} \cap \ldots \cap D_{\beta_n}$ and $(\beta_1(p),\ldots,\beta_n(p)) \in D_\varphi$. So, there are functions $\alpha_1,\ldots,\alpha_n \in C$ and sets $U_1,\ldots,U_n \in \mathrm{Top}C$ such that $p \in U_i \subset D_{\alpha_i} \cap D_{\beta_i}$ and $\beta_i|U_i = \alpha_i|U_i$, $i = 1,\ldots,n$. There exist sets $B_i$ open in $R$ such that $\beta_i(p) \in B_i$, $i = 1,\ldots,n$, and $B_1 \times \ldots \times B_n \subset D_\varphi$. Setting

$$U = \bigcap_i (U_i \cap \beta_i^{-1}[B_i])$$

we get $p \in U \in \mathrm{Top}C$, $U \subset D_{\alpha_i} \cap D_{\beta_i}$ and $\beta_i|U = \alpha_i|U$. Thus, for any $q \in U$ we have $(\alpha_1(q),\ldots,\alpha_n(q)) = (\beta_1(q),\ldots,\beta_n(q)) \in D_\varphi$ and $\gamma(q) = \varphi(\beta_1(q),\ldots,\beta_n(q)) =$

$= \varphi(\alpha_1(q),\dots,\alpha_n(q)) = \varphi(\alpha_1,\dots,\alpha_n)(q)$. So, $\gamma|U = \varphi(\alpha_1,\dots,\alpha_n)|U$. Therefore, $\gamma \in (\mathrm{an}C)_C$. Hence it follows that (see [4], p. 264, 1.1) $(\mathrm{an}G)_G$ is the smallest of all sets M containing G and satisfying the condition $M = \mathrm{an}M = M_M$. Which ends the proof.

For any set G of real functions the set $(\mathrm{an}G)_G$ will be called the a.p. spanned by G, or generated by G. We have of course,

$$\mathrm{Top}G = \mathrm{Top}((\mathrm{an}G)_G). \tag{2}$$

Let $M_1$ and $M_2$ be a.p. Denote the Cartesian product of the sets $\underline{M_1}$ and $\underline{M_2}$ by P and put $\pi_i(p_1, p_2) = p_i$ for $(p_1, p_2) \in P$, $i = 1,2$. We have then the mappings $\pi_i : P \to \underline{M_i}$. Setting $\pi^*_i(\alpha) = \alpha \circ \pi_i$ for any real functions $\alpha$ we define the pull-backs $\pi^*_i$ from all real functions to the set of all real functions defined on subsets of P. We have then

$$D_{\pi^*_i(\alpha)} = \pi_i^{-1}[D_\alpha] \quad \text{and} \quad \pi^*_i(\alpha)(p_1, p_2) = \alpha(p_i)$$

for $(p_1, p_2) \in \pi_i^{-1}[D_\alpha]$. The a.p. generated by the set $\pi^*_1[M_1] \cup \pi^*_2[M_2]$ of the real functions will be called the Cartesian product of the spaces $M_1$ and $M_2$ and denoted by $M_1 \times M_2$. The equality (2) yields

$$\mathrm{Top}(M_1 \times M_2) = \mathrm{Top}(\pi^*_1[M_1] \cup \pi^*_2[M_2]). \tag{3}$$

We shall prove that

$$\mathrm{Top}(M_1 \times M_2) = \mathrm{Top}M_1 \times \mathrm{Top}M_2. \tag{4}$$

First, take any function $\gamma \in \pi^*_1[M_1]$ and any set B open in $\mathbb{R}$. We have $\gamma = \alpha \circ \pi_1$, where $\alpha \in M_1$. Thus, $\gamma^{-1}[B] = \pi_1^{-1}[\alpha^{-1}[B]]$. From the definition of $\mathrm{Top}M_1$ it follows that $\alpha^{-1}[B] \in \mathrm{Top}M_1$. So, $\gamma^{-1}[B] = \alpha^{-1}[B] \times \underline{M_2}$ belongs to $\mathrm{Top}M_1 \times \mathrm{Top}M_2$. Similarly, for any $\gamma \in \pi^*_2[M_2]$ we have $\gamma^{-1}[B] \in \mathrm{Top}M_1 \times \mathrm{Top}M_2$. This yields $\mathrm{Top}(M_1 \times M_2) \subset \mathrm{Top}M_1 \times \mathrm{Top}M_2$. Now, take any set U of the form $U_1 \times U_2$, where $U_1 \in \mathrm{Top}M_1$ and $U_2 \in \mathrm{Top}M_2$. Let $p = (p_1, p_2) \in U$. So, we have $p_1 \in U_1$ and $p_2 \in U_2$. Then there exist $\alpha_1,\dots,\alpha_m \in M_1$, $\beta_1,\dots,\beta_n \in M_2$ and sets $A_1,\dots,A_m$, $B_1,\dots,B_n$ open in R such that

$$p_1 \in \bigcap_i \alpha_i^{-1}[A_i] \subset U_1 \quad \text{and} \quad p_2 \in \bigcap_j \beta_j^{-1}[B_j] \in U_2.$$

We have $p_1 = \pi_1(p)$ and $p_2 = \pi_2(p)$. Therefore, $p \in \bigcap_i (\alpha_i \circ \pi_1)^{-1}[A_i] \subset \pi_1^{-1}[U_1]$, $p \in \bigcap_j (\beta_j \circ \pi_2)^{-1}[B_j] \subset \pi_2^{-1}[U_2]$, $\alpha_i \circ \pi_1 = \pi_1^*(\alpha_i) \in \pi_1^*[M_1]$ and $\beta_j \circ \pi_2 = \pi_2^*(\beta_j) \in \pi_2^*[M_2]$. Hence it follows that $\pi_1^{-1}[U_1]$ and $\pi_2^{-1}[U_2]$ belong to $\mathrm{Top}(\pi_1^*[M_1] \cup \pi_2^*[M_2])$. Therefore, by (3), we have

$$U = \pi_1^{-1}[U_1] \cup \pi_2^{-1}[U_2] \in \mathrm{Top}(M_1 \times M_2).$$

So, $\mathrm{Top}M_1 \times \mathrm{Top}M_2 \subset \mathrm{Top}(M_1 \times M_2)$. What ends the proof.

## 2. ANALYTICAL PREMANIFOLDS AND R-QUASI-ALGEBRAIC SPACES

In the paper [6] the concept of K-quasi-algebraic space (K-q.a.s.) has been introduced. K-q.a.s is meant as a set M of functions with values in the field K; M is assumed to be closed with respect to addition and multiplication of functions. For any set M of functions with values in K there has been defined topM as the smallest topology $\underline{M}$ such that all sets $D_\alpha$, $\alpha \in M$, are regarded as open. The concept of K-q.a.s. $M_0$ generated by M has been introduced, too. We have $\mathrm{top}M_0 = \mathrm{top}M$. In particular, taking as K the field R of all reals we get the concept of R-q.a.s. We have then $\mathrm{top}M \subset \mathrm{Top}M$. By an easy verification we obtain the following

2.1. Proposition

For any set M of real functions we have topM = TopM if and only if any function $\alpha \in M$ is continuous in topM.

We remark that if M is an a.p. such taht there is a constant function on $\underline{M}$ belonging to M, so M is an R-q.a.s. If M is any a.p., so R-q.a.s. $M \cup \{c_{\underline{M}};\ c \in R\}$, where $c_{\underline{M}}$ denotes the constant fuction taking the value c on $\underline{M}$, we will call the R-q.a.s. determined by M.

For any point p of an a.p. M the set of all $\alpha \in M$ such that $p \in D_\alpha$ will be denoted by M(p). We have then $\alpha + \beta$, $\alpha \cdot \beta \in M(p)$ for $\alpha, \beta \in M(p)$.

Consider the set $\underline{M_p}$ of all derivations on M(p), i.e. the set of all $v: M(p) \longrightarrow R$ such that $v(\alpha + \beta) = v(\alpha) + v(\beta)$, $v(c\alpha) = cv(\alpha)$ and $v(\alpha\beta) = \alpha(p)v(\beta) + \beta(p)v(\alpha)$ for $\alpha, \beta \in M(p)$ and $c \in R$. Setting $(v + w)(\alpha) = v(\alpha) + w(\alpha)$ and $(cv)(\alpha) = cv(\alpha)$ for $\alpha \in M(p)$, $c \in R$ and $v, w \in \underline{M_p}$ we obtain the vector space, $M_p$, with $\underline{M_p}$ as the set of all vectors. The space $M_p$ will be called the tangent space to M at the point p.

For any smooth mapping (O), any $p \in \underline{M}$, any $v \in \underline{M_p}$ and any $\beta \in N(f(p))$ we set $f_{*p}(v)(\beta) = v(\beta \circ f)$. So, we have the linear mapping

$$f_{*p}: M_p \longrightarrow N_{f(p)}$$

of tangent spaces; it will be caled the tangent to (O) at the point p.

For any set S and any nomber c, the constant function with value c defined on the set S will be denoted by $c_S$.

2.2. Proposition

If M is an a.p., $\alpha \in M$, $c \in \mathbb{R}$, so $c_{D_\alpha} \in M$; in particular, $1_{D_\alpha} \in M$.

Proof. For any $\alpha \in M$ we have $0_{D_\alpha} = \alpha - \alpha \in M$. Composing the function $0_{D_\alpha}$ with the analytic function $c_{\mathbb{R}}$ we get $c_{D_\alpha} = c_{\mathbb{R}} \circ 0_{D_\alpha} \in \operatorname{an}M = M$. The end of the proof.

2.3. Proposition

If $v \in M_p$, $\alpha, \beta \in M(p)$ and there exist $U \in \operatorname{Top}M$ such that $p \in U \subset D_\alpha \cap D_\beta$ and $\alpha | U = \beta | U$, so $v(\alpha) = v(\beta)$.

Proof. There exist $\gamma_1, \dots, \gamma_m \in M$ and sets $B_1, \dots, B_m$ open in $\mathbb{R}$ such that $p \in \bigcap_i \gamma_i^{-1}[B_i] \subset U$. Let us set $\varphi_i(x) = x$ for $x \subset B_i$ and $\eta_i = \varphi_i \circ \gamma_i$, $i = 1, \dots, m$. So, we have analytic functions $\varphi_i$ such that $D_{\varphi_i} = B_i$ and functions $\eta_i \in M$ such that $D_{\eta_i} = \gamma_i^{-1}[B_i]$. Hence, by 2.2, we have $1_{D_{\eta_i}} \in M$. Setting $V = \bigcap_i \gamma_i^{-1}[B_i]$ we get $p \in V \subset U$ and

$$\alpha 1_V = 1_{D_{\eta_1}} \cdots 1_{D_{\eta_m}} \in M.$$

Thus, $\alpha 1_V = \alpha | V = \beta | V = \beta \circ 1_V$. This yields, $v(\alpha 1_V) = \alpha(p)v(1_V) + 1_V(p)v(\alpha) = v(\alpha)$, because of $v(1_V) = 0$. Similarly, $v(\beta 1_V) = v(\beta)$. Therefore $v(\alpha) = v(\beta)$. What ends the proof.

2.4. Proposition

If M is an a.p., $\varphi$ is a real analytic function on an open subset of $\mathbb{R}^m$, $\alpha_1, \dots, \alpha_m \in M$ and $\varphi(\alpha_1, \dots, \alpha_m) \in M(p)$, then for any tangent vector v to M at p we have

$$v(\varphi(\alpha_1, \dots, \alpha_m)) = \sum_{i=1}^{m} \partial_i \varphi(\alpha_1(p), \dots, \alpha_m(p)) v(\alpha_i),$$

where $\partial_i$ denotes the partial derivative with respect to i-th variable.

Proof. Let $p \in D_{\alpha_1} \cap \dots \cap D_{\alpha_m}$, $(\alpha_1(p), \dots, \alpha_m(p)) \in D_\varphi$. Then there exist neighbourhoods $B_i$ of numbers $\alpha_i(p)$ open in $\mathbb{R}$ and analytic functions $\varphi_i$ in $B_1 \times \dots \times B_m$ such that $(\alpha_1(p), \dots, \alpha_m(p)) \in B_1 \times \dots \times B_m \subset D_\varphi$ and $\varphi(x_1, \dots, x_m) - \varphi(\alpha_1(p), \dots, \alpha_m(p)) = \sum_{i=1}^{m} \varphi_i(x_1, \dots, x_m)(x_i - \alpha_i(p))$.

when $x_1 \in B_1, \dots, x_m \in B_m$, and $\varphi_i(\alpha_1(p), \dots, \alpha_m(p)) = \partial_i \varphi(\alpha_1(p), \dots, \alpha_m(p))$. Hence if follows that for any $q \in U = \bigcap_i \alpha_i^{-1}[B_i]$ we have

$$\varphi(\alpha_1(q), \dots, \alpha_m(q)) - \varphi(\alpha_1(p), \dots, \alpha_m(p)) =$$

$$= \sum_{i=1}^{m} \varphi_i(\alpha_1(q), \dots, \alpha_m(q))(\alpha_i(q) - \alpha_i(p)).$$

Setting $c = \varphi(\alpha_1(p), \dots, \alpha_m(p))$ and $c_i = \alpha_i(p)$ we get

$$\varphi(\alpha_1|U, \dots, \alpha_m|U) - c_U = \sum_i \varphi_i(\alpha_1|U, \dots, \alpha_m|U)(\alpha_i|U - c_{iU}).$$

Hence, because of the equalities $v(c_U) = 0 = v(c_{iU})$ and $(\alpha_i|U)(p) = c_{iU}(p)$, according to 2.2 we get the required equality. What ends the proof.

2.5. Proposition

If $M$ is an a.p. generated by the set $G$ of real functions, $v \in \underline{M_p}$ and $v(\alpha) = 0$ for any $\alpha \in G$ such that $p \in D_\alpha$, so $v = 0$.

Proof. Let us take any $\gamma \in M(p)$. We have then $p \in D_\gamma$, $\gamma \in (\mathrm{an}G)_{\underline{G}}$. Then there exist $\beta \in \mathrm{an}G$ and $U \in \mathrm{Top}M$ such that $p \in U \subset D_\beta \cap D_\gamma$ and $\gamma|U = \beta|U$. Hence by 2.3 we get $v(\gamma) = v(\beta)$. Next, we have $\beta = \varphi(\alpha_1, \dots, \alpha_m)$, where $\varphi$ is analytic and $\alpha_1, \dots, \alpha_m \in G$. According to 2.4 we obtain

$$v(\beta) = \sum_i \partial_i \varphi(\alpha_1(p), \dots, \alpha_m(p)) v(\alpha_i).$$

So, $v(\gamma) = 0$, because of $\alpha_i \in G$. What ends the proof.

2.6. Proposition

For any a.p. $M_1$ and $M_2$ and any points $p_1 \in \underline{M_1}$ and $p_2 \in \underline{M_2}$ the pair of mappings

$$(\pi_{1*p}, \pi_{2*p}) : (M_1 \times M_2)_p \longrightarrow M_{1p_1} \oplus M_{2p_2}, \tag{2.1}$$

where $p = (p_1, p_2)$, is an isomorphism of vector spaces.

Proof. Assume $(\pi_{1*p}, \pi_{2*p})v) = 0$, where $v$ is a vector of the space $(M_1 \times M_2)_p$. So, we have $\pi_{1*p}(p) = 0$ and $\pi_{2*p}(v) = 0$. Taking any functions $\alpha_1 \in M_1(p)$ and $\alpha_2 \in M_2(p)$ we have $0 = \pi_{i*p}(v)(\alpha_i) = v(\alpha_i \circ \pi_i) = v(\pi_i^*(\alpha_i))$. So, $v(\alpha) = 0$ for any $\alpha \in \pi_1^*[M_1] \cup \pi_2^*[M_2]$ such that $p \in D_\alpha$.

According to 2.5 we have $v = 0$. So, (2.1) is a monomorphism. Now, take any $v_1$ in $M_{1p_1}$ and any $v_2$ in $M_{2p_2}$. For any $\alpha \in (M_1 \times M_2)(p)$ the functions $\alpha_1$ and $\alpha_2$ defined by the equalities: $\alpha_1(t) = \alpha(t, p_2)$ when $(t, p_2) \in D_\alpha$, and $\alpha_2(u) = \alpha(p_1, u)$ when $(p_1, u) \in D_\alpha$, belong to $M_1$ and $M_2$, respectively. Let us set

$$v(\alpha) = v_1(\alpha_1) + v_2(\alpha_2). \tag{2.2}$$

It is easy to check that (2.2) defines a vector od $(M_1 \times M_2)_p$. Moreover, for $\beta \in M_1(p)$ we have the function $\alpha = \beta \circ \pi_1$ such that $\alpha_1(t) = \beta(t)$ when $t \in D_\beta$ and $\alpha_2(u) = \beta(p_1)$ when $u \in M_2$. This yields $v_1(\alpha_1) = v_1(\beta)$ and $v_2(\alpha_2) = 0$. Hence it follows that $\overline{\pi_{1*p}}(v)(\beta) = v(\beta \circ \pi_1) = v(\alpha) = v_1(\alpha_1) = v_1(\beta)$. Thus, $\pi_{1*p}(v) = v_1$. Similarly, $\pi_{2*p} = v_2$. So, the vector defined by (2.2) satisfies the equality $(\pi_{1*p}, \pi_{2*p})(v) = (v_1, v_2)$.
What ends the proof.

## 3. AN ANALYTICAL MANIFOLD AS AN ANALYTICAL PREMANIFOLD

The set $R^n$ has the standard structure, $R_n$, of an a.p. compound of all real analytic functions on open subsets of $R^n$. Any real analytical n-dimensional manifold may be considered as an a.p. locally diffeomorphic to $R_n$, i. e. such that for any point p in M there exist a neighboorhood U of p open in TopM, a set V open in $\mathbb{R}^n$ and a diffeomorphism $f: M_U \longrightarrow R_{nV}$.

If M is an analytical subpremanifold of $R_n$, i. e. $\underline{M} \subset \mathbb{R}^n$ and $M = R_{nM}$, so for any $p \in \underline{M}$ we have the tangent space $M_p$, and, on the other hand, we have (see [5] ) the tangent hyperplane, $\underline{M}^a_p$, to the set $\underline{M}$ at the point p. In [5] there is defined a natural isomorphism the vector space $M_p$ with the vector subspace $M^a_p - p$ of the vector space $\mathbb{R}^n$ . $M^a_p - p$ is the vector space obtained form $\underline{M}^a_p$ by the translation $x \mapsto s - p$ In what follows we may identify $M_p$ and $\underline{M}^a_p$. The following theorems will be useful.

Theorem A. ( [5] ) If $M \subset \mathbb{R}^n$, dim $M^a_p = m$ and the mapping

$$p^{\perp}_M: \mathbb{R}^n \longrightarrow M^a_p, \tag{4}$$

which to every q in $\mathbb{R}^n$ assigns its orthogonal projection $p^{\perp}_M(q)$ onto the hyperplane $M^a_p$, is open at the point p, then M is m-dimensional analytical submanifold of $R^n$.

The mapping $p_M$ is said to be open at p iff there exists a neighbourhood $V \in$ TopM of p such that the image $p^{\perp}_M[U]$ of any $U \in$ TopM, $U \subset V$, is open in $M^a_p$.

Theorem B. ([6] p. 260 and [5] p. 226) If $M \subset R^n$ and $\dim M_p^a > 1$, so there exists a mapping $\mu$ satisfying the following conditions:

(i) the domain $D_\mu$ of $\mu$ is contained in $M_p^a$ and dense in itself;

(ii) there exists $r > 0$ such that $\mu[D_\mu] = M \cap B(p, r)$, where $B(p, r)$ denotes the ball in $R^n$ with the center p and the radius r;

(iii) for any $x \in D_\mu$ the orthogonal projection of the point $\mu(x)$ onto $M_p^a$ is equal to x;

(iv) $\mu$ is continuous in its domain.

3.1. Theorem

If the Cartesian product $M \times N$ of a.p. M and N is an analytical manifold, so M and N are analytical manifolds, too.

Proof. To prove that M is an analytical manifold it sufficies to prove that M is locally diffeomorphic to an analytical submanifold of $R^n$. Let us take any $p \in M$ and $o \in N$. Then there exists a diffeomorphism

$$f: M_U \times N_V \longrightarrow R_{nW}, \tag{3.1}$$

where $p \in U \in \mathrm{Top}M$, $o \in V \in \mathrm{Top}N$ and $W = f[U \times V]$ is an open subset of $R^n$. Consider diffeomorphisms

$$a: M_U \longrightarrow (M \times N)_{U \times \{o\}}, \qquad b: N_V \longrightarrow (M \times N)_{\{p\} \times V}, \tag{3.2}$$

where $a(s) = (s, o)$ for $s \in U$ and $b(t) = (p, t)$ for $t \in V$, and the mapping

$$g: U_o \times V_o \longrightarrow W \tag{3.3}$$

defined by the formula

$$g(s, t) = (f(a^{-1}(f^{-1}(u))), b^{-1}(f^{-1}(v))) \text{ for } (u, v) \in U_o \times V_o,$$

where $U_o = f[U \times \{o\}]$ and $V_o = f[\{p\} \times V]$. It is easy to check that (3.3) is one-one and

$$g^{-1}(x) = (f(a(pr_1 f^{-1}(x))), f(b(pr_2 f^{-1}(x)))) \text{ for } x \in W, \tag{3.4}$$

where $pr_1$ and $pr_2$ are the standard projections form $M_U \times N_V$ onto $M_U$ and $N_V$, respectively. From (3.4) and the fact that (3.1) and (3.2) are diffeomorphisms it follows that we have the diffeomorphism

$$g: R_{nU_o} \times R_{nV_o} \longrightarrow R_{nW}.$$

In particular, this mapping is a homeomorphism.

Let us set $P = R_{nU_o}$ and $Q = R_{nV_o}$. So, we have the diffeomorphism

$$g: P \times Q \longrightarrow R_{nW}. \tag{3.5}$$

Taking for any $(u, v) \in U_o \times V_o$ the tangent mapping to (3.5) we get the isomorphism

$$g_{*(u,v)}: (P \times Q)_{(u,v)} \longrightarrow (R_{nW})_{g(u,v)}.$$

From the fact that $(P \times Q)_{(u,v)}$ is isomorphic to $P_u \oplus Q_v$ and W is open in $R^n$ it follows that $\dim P_u + \dim Q_v = n$ for $u \in U_o$, $v \in V_o$. We obtain then the constant function $U_o \ni u \mapsto \dim P_u$; denote its value by m. Thus,

$$\dim P_u = m \text{ for } u \in U_o. \tag{3.6}$$

According to Theorem A it remains to prove that the orthogonal projection

$$u^{\perp}_{\underline{P}}: R^n \longrightarrow \underline{P}^a_u \tag{3.7}$$

from $R^n$ onto the hyperplane $\underline{P}^a_u$ is open for $u \in U_o$. Here $\underline{P} = U_o$. By Theorem B there exist a mapping $\mu$ continuous in its domain $D_\mu \subset \underline{P}^a_u$ and $r > 0$ such that $\mu[D_\mu] = \underline{P} \cap B(u, r)$ and $u^{\perp}_{\underline{P}} \circ \mu = \mathrm{id}$.

Similarly, for any $v \in \underline{Q}$ there exist a mapping $\vartheta$ continuous in its domain $D_\vartheta \subset \underline{Q}_v$ and $r' > 0$ such that $\vartheta[D_\vartheta] = \underline{Q} \cap B(v, r')$ and $v^{\perp}_{\underline{Q}} \circ \vartheta = \mathrm{id}$. Set

$$h(x, y) = g(\mu(x), \vartheta(y)) \text{ for } (x, y) \in D_\mu \times D_\vartheta \tag{3.8}$$

and $W_o = g[(U_o \times V_o) \cap (B(u, r) \times B(v, r'))]$. So, we get one-one mapping

$$h: D_\mu \times D_\vartheta \longrightarrow W \tag{3.9}$$

with the inverse mapping given by the formula

$$h^{-1}(z) = (u_{\underline{P}}(pr_1 g^{-1}(z)), v_{\underline{Q}}(pr_2 g^{-1}(z))) \text{ for } z \in W. \tag{3.10}$$

From (3.8) and (3.10) it follows that (3.9) is a homeomorphism. The mapping (3.3) as a homeomorphism transforms any open subset of the set $U_o \times V_o$ onto open subsets of the set W. Hence it follows that $W_o$ is an open subset of $R^n$. From the equality $\dim P_u + \dim Q_v = n$, according to theorem of Brouwer on open sets, we conclude that the set $D_\mu \times D_\vartheta$ is open in $P_u \times Q_v$. So, the set $D_\mu$ is open in $P_u$. This yields the mapping (3.7) is open at the point u. What ends the proof.

## REFERENCES


[1] MacLane S.: Differentiable spaces, Notes for Geometrical Mecanics, Winter 1970, p. 1-9 (unpublished).

[2] Postnikov M.M.: Vviedenie v tieoriu Morsa, Moskow 1971 (Russian).

[3] Sikorski R.: Abstract covariant derivative, Colloq. Math. 18 (1967), p. 251-272.

[4] Waliszewski W.: Regular and coregular mappings of differential spaces, Ann. Polon. Math. 30 (1975), p. 263-281.

[5] Waliszewski W.: A characterization of a regular analytic m-dimensional hypersurface of the Euclidean space, Czechoslovakian-GDR-Polish Scientific School on Differential Geometry, Boszkowo, Poland, February 1-10, Scientific communications, Part 2, p. 222-231.

[6] Waliszewski W.: On differential subspace of Cartesian space, Colloq. Math. 45, 2 (1981), p. 257-265.

[7] Waliszewski W.: K-quasi-algebraic spaces, Zeszyty Naukowe Akademii Górniczo-Hutniczej w Krakowie, The volume devoted to memory of Professor Antoni Hoborski (to appear).

[8] Waliszewski W.: On differential spaces the Cartesian product of which is a differentiable manifold, Colloq. Math. (to appear).


Recenzent: Prof. dr hab. Tadeusz Świątkowski

Wpłynęło do redakcji: 27.XII.1983 r.

## PRE-ROZMAITOŚCI ANALITYCZNE


S t r e s z c z e n i e

W pracy zostało wprowadzone pojęcie prerozmaitości analitycznej oraz pojęcie odwzorowania gładkiego takich prerozmaitości. Udowodniono, że jeśli produkt kartezjański dwu prerozmaitości analitycznych jest rozmaitością analityczną, to te prerozmaitości są rozmaitościami analitycznymi.


## АНАЛИТИЧЕСКИЕ ПРЕДМНОГООБРАЗИЯ


Р е з ю м е

Автор предлагает понятие аналитического предмногообразия и понятие гладкого отображения одного аналитического предмногообразия в другое. Доказана теорема: Если декартовое произведение двух аналитических предмногообразий - аналитическое многообразие, то эти предмногообразия - аналитические многообразия.