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SEPARATE J-APPROXIMATE CONTINUITY IMPLIES THE BAIRE PROPERTY

Summary. The main result of this paper is a theorem which asserts that any real function of two real variables which is J-approximateby continuous with respect of each of its variables has the Baire property.

In [3] one can find the definition and basic properties of so called τ_y topology, which is a category analogue of the density topology. The behaviour of real functions which are continuous with respect to this topology is under many aspects similar to the behaviour of approximately continuous functions. This paper will show one more similarity. The proof of the main theorem is going along the lines described in [2] with necessary modifications.

We shall present now all necessary definitions and facts about τ_y topology. For proofs see [3].

Let R be the real line, S the σ-algebra of subsets of R having the Baire property, J the σ-ideal of sets of the first category on the real line. If $A \subseteq R$ and $x_0 \in R$, denote $x_0 \cdot A = \{x_0 \cdot x : x \in A\}$ and $A - x_0 = \{x - x_0 : x \in A\}$; χ_A will mean the characteristic function of the set A; N will always be the set of natural numbers. The number 0 is an J-density point of a set $A \in S$ if and only if the sequence $\{\chi_{(nA)\cap[-1,1]}\}_{n \in N}$ converges to 1 with respect to the σ-ideal J (it means that for every invreasing sequence $\{n_m\}_{m \in N}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p \in N}$ such that $\{\chi_{(n_{m_p} A)\cap[-1,1]}\}_{p \in N}$ converges to 1 except on a set belonging to J, shortly J-almost everywhere (J-a.e.)). Further, x_0 is an J-density point of $A \in S$ if and only if 0 is an J-density point of the set $A - x_0$. If $\Phi(A) = \{x \in R : x$ is an J-density point of $A\}$ and $\tau_y = \{A \in S : A \subseteq \Phi(A)\}$, then τ_y is a topology. Each function $f : R \longrightarrow R$ which is continuous as a function from (R, τ_y into R equipped with the natural topology) is called an J-approximately continuous function. Every J-approximately cintinuous function is in the first Baire class so it obviously has the Baire property. J-almost all points of a set having the Baire property are J-density points of this set. J-density point of two sets is an J-density point of the intersection of these sets.

Theorem: If $f: R^2 \to R$ is separately J-approximately continuous, then it has the Baire property.

The theorem is an immediate consequence of lemmas 2 and 4 below (obviously we may consider I^2 in place of R^2). First two lemmas are formulated in more general setting.

Lemma 1. Let (X,S) be a measurable space and let $J \subseteq S$ be a σ-ideal such that (S,J) fulfills the countable chain condition. If $D \subseteq S$ has the property that for each $E \in S - J$ there exists $D_0 \in D - J$ such that $D_0 \subset E$, then there axists a sequence $\{D_n\}_{n\in N}$ of disjoint sets in D such that $X - \bigcup_n D_n \in J$.

Proof: Let $E_0 = X$. From the assumption it follows that there exists a set $D_0 \in D - J$, $D_0 \subset E_0$. Suppose that we have defined sets D_β for $\beta < \alpha$, where α is a countable ordinal. Consider the set $E_\alpha = X - \bigcup_{\beta<\alpha} D_\beta$. This set belongs to S. If $E_\alpha \in J$, then obviously the family $\{D_\beta\}_{\beta<\alpha}$ (after renumeration) fulfills all requirements. If $E_\alpha \notin J$, then we choose $D_\alpha \subset E_\alpha$, $D_\alpha \in D - J$. There exists a countable ordinal α_0 such that $E_{\alpha_0} \in J$. Indeed, in the contrary case we should obtain a disjoint non-denumerable family $\{D_\alpha\}_{\alpha<\Omega}$ of sets belonging to $S - J$. It is impossible because of the c.c.c. Then the family $\{D_\alpha\}_{\alpha<\alpha_0}$ is good (again after renumeration).

Lemma 2. Let (X,S) be a measurable space and let $J \subset S$ be a σ-ideal such that (S,J) fulfills the countable chain condition. Suppose that for every $\varepsilon > 0$ the clase $D_\varepsilon = \{D \in S: \operatorname{osc} f \leqslant \varepsilon \text{ on } D\}$ satisfies the conditions of lemma 1. Then f is S-measurable.

Proof: By lemma 1, for each m we can find a sequence of disjoint sets $\{D_{mn}\}_{n\in N}$ such that $\operatorname{osc} f < m^{-1}$ on D_{mn} for every $n \in N$ and $X - \bigcup_n D_{mn} \in J$. Let $D = \bigcap_m \bigcup_n D_{mn}$. Then $X - D \in J$. If $f_m : X \to R$ for $m \in N$ is defined in the following way: $f_m(x) = \sup\{f(z): z \in D_{mn}\}$ for $x \in D_{mn} \cap D$, $n \in N$ and $f_m(x) = 0$ for $x \in X - D$, then clearly f_m is S-measurable and $f_m \to f$ J - a.e., so f is S-measurable.

From now we shall supposie that $X = (0,1)$ or R, S is the σ-algebra of sets having the Baire property and J is a σ-ideal of sets of the first category. Findly J^2 will denote the σ-ideal of plane sets of the first cathegory. Finally I^2 will denate the σ-ideal of plane sets of the first category.

Lemma 3. If g is J-approximately continuous on $I = (0,1)$ and F is a second category subset of I having the Baire property then for every $\varepsilon > 0$ there exists an open interval $J \subset I$ such that $\operatorname{osc} g \leqslant \varepsilon$ on J and $J \cap F$ is residual in J.

Proof: Let $\varepsilon > 0$ be a given number. From the assumption it follows that g has the Baire property (company [3], th. 11). Hence there exists a set $E \subset I$ which is residual in I and such that the restriction of g to E is continuous. Consider a set $F \cap E$. It is a second category set having the Baire property. Let $J_0 \subset I$ be an interval on which $F \cap E$ is residual. From the continuity of g on E it follows that there exists an interval $J \subset J_0$ such that osc $g \leqslant \varepsilon$ on $J \cap E$. Obviously $J \cap F$ is residual in J. We shall prove that osc $g \leqslant \varepsilon$ on J. There exists an interval $[c,d]$ such that $d - c < \varepsilon$ and $g(J \cap E) \subset [c,d]$. Suppose there exists a point $x_0 \in J$ such that $g(x) \notin [c,d]$. To fix the ideas let $g(x_0) > d$. Put $\eta = \frac{1}{2}(g(x_0)-d)$. From the J-approximate continuity ot follows that the set $\{x: g(x) > g(x_0) - \eta\}$ is of the second category in each neighbourhood of x_0, so also in J. This is a contradiction, because on residual subset of J we have $g(x) \leqslant d$.

Corollary. Under the above conditions, osc $g \leqslant \varepsilon$ on the set of J-density points of F in J.

Lemma 4. Let $f: I^2 \to R$ be separately J-approximately continuous and let E be a second category subset of I^2 having the Baire property. Then for each $\varepsilon > 0$ there exists a second category set $H \subset E$ having the Baire property on which osc $f \leqslant \varepsilon$.

Proof: Let $\varepsilon > 0$ be a given number. Let Q denote the set of points $x \in I$ such that the section E_x has the Baire property and is of the second category; then Q is a second category subset having the Baire property (on the real line). Let $\{J_n\}_{n \in N}$ be a sequence of all open subintervals of I with rational end-points, and let $\{K_n\}_{n \in N}$ be a sequence of all closed intervals in R with rational end-points x and of length not exceeding ε. In view of lemma 3 and the corollary we have $Q = \bigcup_{r,s} Q_{r,s}$, where $Q_{r,s}$ is the set of points $x \in Q$ such that $J_r \cap E_x$ is residual in J_r and $f(x,y) \in K_s$ for every, J-density point of E_x in J_r. Therefore $Q_{r,s}$ is of the second category for some pair of indices (henceforth fixed).

Let $P_{r,s} \supset Q_{r,s}$ be a Baire cover of $Q_{r,s}$, that is, a set having the Baire property such that if $A \subset P_{r,s} - Q_{r,s}$ has the Baire property, then A is of the first category. If x is a point of J-density of $P_{r,s}$, then we shall say that x is an outer J-density point of $Q_{r,s}$. Let P denote the set of all points $x \in I$ which are outer J-density points of $Q_{r,s}$. Then P is a second category set having the Baire property. Let $F = E \cap (P \times J_r)$. This is obviously a second category set having the Baire property. As is known (compare [1], th. 4) the set G of points $(x,y) \in F$ such that F_x has the Baire property and y is an J-density point of F_x has the Baire property and consists of J^2-almost all points of F, and the set H of points $(x,y) \in G$ such that G^y has the Baire property and x is an J-density point of G^y has also the Baire property and consits of I^2-almost all points of G. In particular H is a second category subset of E having the Baire property, and it will therefore be sufficient to prove that $f(t,u) \in K_s$ for every point $(t,u) \in H$.

Let $\vartheta > 0$ be an arbitrary number. Observe that t is an J-density point of the following three sets: G^u, $\{x: |f(x,u) - f(t,u)| < \vartheta\}$ and $P_{r,s}$. Hence t is also an J-density point of the intersection of these sets, so this intersection is the set of the second category having the Baire property. From that it follows immediately that the set $G^u \cap Q_{r,s} \cap \{x: |f(x,u) - f(t,u)| < \vartheta\}$ is non-empty. Let x_0 be an element of this set. We have $(x_0,u) \in G$, therefore F_x has the Baire property and u is an J-density point of F_x, so also of E_x. Moreover $u \in J_r$ and $x_0 \in Q_{r,s}$, and therefore $f(x_0,u) \in K_s$. Then $f(t,u)$ is within ϑ of a value in K_s. Since δ was arbitrary, it follows $f(t,u) \in K_s$ as required.

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J-APROKSYMATYWNA CIAGŁOŚĆ PO KAŻDEJ ZMIENNEJ POCIAGA WŁASNOŚĆ BAIRE'a

S t r e s z c z e n i e

Głównym wynikiem pracy jest twierdzenie orzekające, że funkcja dwóch zmiennych J-aproksymatywnie ciągła względem każdej z nich ma własność Baire'a.

J-АППРОКСИМАТИВНАЯ НЕПРЕРЫВНОСТЬ ПО КАЖДОИ ПЕРЕМЕННОЙ ВЕДЁТ К СВОЙСТВУ БЭРА

Р е з ю м е

Главным результатом работы является теорема о том, что каждая функция двух действительных переменных должна обладать свойством Бэра, если она обладает свойством J-аппроксимативной непрерывности относительно каждой переменной.