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REMARKS ON PROJECTIONS IN GROUP ALGEBRA
OF SOME DISCRETE GROUPS

Summary. Some informations on existing of non-trivial projections in the group algebra $L(G)$ of the Heisenberg group and of the abelian group are given.

Using the Pontryagin duality theory we show that the answer to this problem in the case of G being abelian is negative.

Another example of the group algebra without non-trivial projections in the algebra $L_1(G)$ where G is a free group on two generators, [1].

In the case of the Heisenberg group we consider some classes or functions of the algebra $L_1(G)$ which have not non-trivial projections. In the case the general problem remains open, however.

I. Introduction

Let G be a discrete group and let us denote by $L_1(G)$ group algebra of G i.e. of all functions $f: G \rightarrow \mathbb{C}$, such that $\sum_{g \in G} |f(g)| < +\infty$.

We say that $f \in L_1(G)$ is a projection of $f * f = f$ and $f^* = f$, where $f * f_1(t) = \sum_{g \in G} f(tg^{-1}) f_1(g)$, $f^*(g) = \overline{f(g^{-1})}$. It follows that $V = \text{supp } f = \{g \in G : f(g) \neq 0\}$ is a symmetrical subset of G (i.e. $V = V^{-1}$). It is clear that, if G has an element g of finite (say n) order, then the function f defined by the formula

$$f(x) = \begin{cases} 1/n & \text{if } x \in \{1, g, \dots, g^{n-1}\}, \\ 0 & \text{otherwise} \end{cases}$$

is a non-trivial projection of $L_1(G)$ and therefore it is of interest to ask on projection in group algebra of torsion-free group. Using the Pontryagin duality theory we solve this problem in the case G being abelian and give some information in the case G being discrete Heisenberg group. In the second case the general problem remains open, however.

II. Abelian case

Let G be a discrete, abelian group, and \widehat{G} the dual object of this group. From the duality theory of G , Pontryagin [3] follows that \widehat{G} is compact. We have also

(i) Let G be a compact, Hausdorff abelian group, and \widehat{G} the dual object of this group. The following conditions are equivalent

- (a) \widehat{G} is connected,
- (b) G is torsion-free group.

(ii) The mapping $L_1(G) \ni f \rightarrow \widehat{f}$, where $\widehat{f}(x) = \sum f(g)x(g)$ is an $*$ -isomorphism of $L_1(G)$ into the Banach algebra $C(\widehat{G})$ of all continuous functions on \widehat{G} , [2].

Theorem 1. In the group algebra of abelian, discrete, torsion-free group, the only projections are 0 and δ_1 (Kronecker delta at 1) functions.

Proof. Suppose that there is idempotent f in the group algebra $L_1(G)$ of G . Then $\widehat{f}^2 = \widehat{f}$, and thus $\widehat{f}(\widehat{f} - 1) = 0$. It follows that the set $\{0, 1\}$ is the range of function \widehat{f} . Since $\widehat{f}^{-1}(\{0\})$, $\widehat{f}^{-1}(\{1\})$ are disjoint, closed sets of the connected space \widehat{G} , one of them should be whole space and the other the empty set. Thus f is constant function 0 or 1. Hence $f = 0$ or $f = \delta_1$.

Another example of the group algebra without non-trivial projections is the algebra $L_1(G)$, where G is a free group on two generators, [1].

III. The group algebra of Heisenberg group N_2

Let N_2 be the free, nilpotent group of two generators x, y . Elements this of group we will denote in forms $x^\alpha y^\beta z^\tau$, $\alpha, \beta, \tau \in \mathbb{Z}$, $z = [x, y]$. The group multiplication is given by the formula

$$x^\alpha y^\beta z^\tau \cdot x^{\alpha'} y^{\beta'} z^{\tau'} = x^{\alpha+\alpha'} y^{\beta+\beta'} z^{\tau+\tau'+\alpha'\beta}$$

We introduce in the group N_2 a relation $<$. Namely $x^\alpha y^\beta z^\tau < x_1^{\alpha_1} y_1^{\beta_1} z_1^{\tau_1}$ if and only if one of the following conditions are fulfilled

- (1) $\alpha < \alpha_1$ or
- (2) $\alpha = \alpha_1$, $\beta < \beta_1$ or else
- (3) $\alpha = \alpha_1$, $\beta = \beta_1$, $\tau \leq \tau_1$.

Let \mathcal{H}_0 be the class of functions $f \in L_1(G)$ such that $\overline{\text{supp } f} < \infty$. We prove the following.

Theorem 2. The class functions \mathcal{K}_0 have no projections.

Proof. Suppose there is a function $f \in \mathcal{K}_0$ such that $f * f = f$ and $f = f^*$. Let $V = \text{supp } f$, then $V = V^{-1}$. Let us order this set using the relation $<$

$$\begin{aligned} x^{\alpha_n} y^{\beta_n} z^{\gamma_n} &< x^{\alpha_{n-1}} y^{\beta_{n-1}} z^{\gamma_{n-1}} < x^{\alpha_n} y^{\beta_n} z^{\gamma_n} < \dots \\ \dots &< x^{\alpha_{n-1}} y^{\beta_{n-1}} z^{\gamma_{n-1}} < x^{\alpha_n} y^{\beta_n} z^{\gamma_n}, \end{aligned}$$

where $0 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n$.

Consider the element $t = x^{\alpha_n} y^{\beta_n} z^{\gamma_n + \alpha_n \beta_n}$ of N_2 . Then we have

$$f * f(t) = \sum_{g \in N_2} f(tg^{-1})f(g) = \sum_{x^{\alpha_n} y^{\beta_n} z^{\gamma_n} \in N_2} f(x^{\alpha_n - \alpha} y^{\beta_n - \beta} z^{\gamma_n + \alpha_n \beta_n - \gamma + \alpha \beta - 2\alpha_n \beta_n})$$

$f(x^{\alpha_n} y^{\beta_n} z^{\gamma_n}) = (f(x^{\alpha_n} y^{\beta_n} z^{\gamma_n}))^2$ because for $x^{\alpha_n} y^{\beta_n} z^{\gamma_n} \in V$, $2\alpha_n - \alpha > \alpha_n$.

If $\alpha_{n-1} = \alpha_n$, then $2\beta_n - \beta \geq \beta_n$ for $x^{\alpha_n} y^{\beta_n} z^{\gamma_n} \in V$. If $\alpha_{n-1} = \alpha_n$, $\beta_{n-1} = \beta_n$, then $2\gamma_n - \gamma > \gamma_n$ for $x^{\alpha_n} y^{\beta_n} z^{\gamma_n} \in V$. Hence

$$[f(x^{\alpha_n} y^{\beta_n} z^{\gamma_n})]^2 = f(x^{\alpha_n} y^{\beta_n} z^{\gamma_n + \alpha_n \beta_n}) = 0,$$

which is a contradiction with the fact that $x^{\alpha_n} y^{\beta_n} z^{\gamma_n} \in V$.

Remark 1. If \mathcal{K}_1 is the set of all functions $f \in L_1(G)$ (here G is an arbitrary discrete group) such that $f(1) = 0$ or 1, then there is no projections in \mathcal{K}_1 .

In fact, we have

$$\begin{aligned} f * f(1) &= \sum_{g \in G} f(g^{-1})f(g) = \sum_{g \in G} \overline{f(g)}f(g) = \\ &= |f(1)|^2 + 2 \sum_{g \in G \setminus \{1\}} |f(g)|^2 = f(1) = 0. \end{aligned}$$

Hence $f(g) = 0$ for all $g \in G$. The case $f(1) = 1$ is analogous.

Remark 2. If \mathcal{K}_2 is the set of all functions $f \in L_1(G)$ such that $\text{supp } f \subset H$, where H is an abelian subgroup of G , then there is no projections in \mathcal{K}_2 .

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ЗАМЕЧАНИЯ ПО ПОВОДУ ПРОЕКТИРОВАНИЯ В НЕКОТОРОЙ ДИСКРЕТНОЙ АЛГЕБРЕ ГРУПП

Р е з ю м е

В настоящей работе авторами даётся некоторая информация о нетривиальных проекторах в групповых алгебрах $L_1(G)$ дискретных абелевых и групповой алгебре группы Гейзенберга.

В случае, когда G дискретна абелева группа, в групповой алгебре $L_1(G)$ нет G нетривиальных проекторов. Второй пример группы с этим свойством это дискретная свободная группа с двумя образующими [1].

В случае дискретной группы Гейзенберга проблема ещё не решена. В работе показаны некоторые классы функций, принадлежащих к алгебре $L_1(G)$ которые не имеют нетривиальных проекторов этой алгебры.

S t r e s z c z e n i e

W pracy zostały podane informacje na temat istnienia nietrywialnych projektów w algebrze grupowej $L_1(G)$. Dotyczą one dwóch przypadków - gdy G jest dyskretną grupą abelową oraz gdy G jest dyskretną grupą Heisenberga.

Wykorzystując podstawowe fakty z teorii dualności Pontriagina pokazujemy, że w przypadku dyskretnej grupy abelowej G w algebrze grupowej $L_1(G)$ nie istnieją nietrywialne projektorzy. Innym przykładem algebry grupowej o tej samej własności jest algebra odpowiadająca wolnej grupie dyskretnej o dwóch generatorach [1].

W przypadku dyskretnej grupy Heisenberga N_2 problem nadal jest nie rozstrzygnięty. W pracy są rozpatrywane pewne klasy funkcji należących do algebry $L_1(N_2)$, które nie zawierają nietrywialnych projektów tej algebry.