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Ernest PŁONKA

SOME REMARKS ON INVERTIBLE ELEMENTS IN GROUP ALGEBRA OF DISCRETE GROUPS

Summary. The invertible elements in group algebra of discrete groups are investigated , some methods of construction such elements are presented. It is proved that the groups of invertible elements of group algebra of discrete Heisenberg group does not satisfy the Engel condition.

0. Let G be a discrete group and let $\ell^1(G)$ be the group algebra of G. This is the set of all elements of the form $\sum_{g\in G} a_g g$ such that $a_g \in \mathbb{C}$, $\sum_{g\in G} |a_g| < \infty$ together with usual addition given by $\sum a_g g + \sum b_g g = \sum (a_g+b_g)g$ and multiplication $\sum a_g g \sum b_g g = \sum c_g g$ where $c_g = \sum_{hk=g} a_h b_k$. $\ell^1(G)$ is a Banach algebra with involution $*$ defined by $(\sum a_g g)^* = \sum \bar{a}_{g^{-1}} g$. The totality of all elements $\sum a_g g$ such that all but finite number of a_g, equal 0 is the group ring $\mathbb{C}(G)$ over the complex number $\mathbb{C}$ I.Kaplansky in [4] has raised a problem to describe the invertible elements in the group ring. It is obvious that the elements cg,

$g \in G$, $c \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ are invertible. It turned out that for a large class of groups G (e.g. nilpotent) there are no other invertible elements in $\mathbb{C}(G)$.

In this note we gave some information about the group of invertible elements in the group algebra of the group $N_z = \{x^a y^b z^c \ : \ a, \ b, \ c \in \mathbb{Z}, \ z = y^{-1} x^{-1} y x \}$, where the multiplication is given by the formula

$$x^a y^b z^c x^{a'} y^{b'} z^{c'} = x^{a+a'} y^{b+b'} z^{c+c' + ba'}$$

The group N_z is called discrete Heisenberg group.

1. Let $\ell^1(G)$ be the group algebra of a group G. An element $p \in \ell^1(G)$ is called projection if $p = p^*$ and $p^2 = p$. We begin with simple

<u>Theorem 1.</u>

Let u be an invertible element in the group algebra $\ell^1(G)$ of a group G such that $u^n = 1$ for some $n \in \mathbb{N}$. Then the element

$$p = \frac{1}{n}(1 + u + u^2 + \dots + u^{n-1})$$

is a projection. Conversely, if $e \in \ell^1(G)$ is such that $e^2 = e$ then element $u = 1 - 2e$ is an invertible element with $u^2 = 1$.

<u>Proof</u>

Straightforward.

If G is a torsion-free Abelian group, then the only projections in $\ell^1(G)$ are 0 and 1. Indeed, if $pp = p$ then

$\hat{p}(\hat{p}-1) = 0$, where $\hat{p}$ is the Fourier transform of p. Thus $\hat{p}(\chi) \in \{0,1\}$ i.e. $\hat{p}$ is a characteristic function of a subset of dual group $\hat{G}$, which is connected (see[6]). Therefore $\hat{p} = 1$ or $\hat{p} = 0$.

The same is true for free group and for discrete Heisenberg group (cf.[5]).

Observe that the group algebra $\ell^1(G)$ of a group G in oposition to the situation in the group ring $\mathbb{C}G$ has lot of nontrivial invertible elements. Indeed, since $\ell^1(G)$ is a Banach algebra, any element $x = \sum_{g\in G} a_g g$ with $\sum_{g\neq 1} |g|+a_1-1 < 1$ is invertible. This because the series $x^{-1} = \sum_{n=1}^{\infty}(1-x)^n$ is absolutly convergent (cf.[2]).

Hence we get

<u>Proposition 1</u>

The group of invertible elements of $\ell^1(G)$, $G \neq 1$, contains of continuum elements other than cg, $c\in\mathbb{C}^*$, $g\in G$.

Other family of invertible elements produces the exponential mapping exp: $\ell^1(G) \rightarrow \ell^1(G)$ defined by

$$\exp(a) = \sum_{n=1}^{\infty} \frac{a^n}{n!}$$

It is known [2] that if a and b commute then

$$\exp(a)\exp(b) = \exp(a+b).$$

Hence, in particular, we get $\exp(a)\exp(-a) = 1$ i.e. $\exp(a)$, $a \in \ell^1(G)$, invertible element in $\ell^1(G)$.

The series exp(a) is specially simple in the case of elements which satisfy the condition $aa = 0$ (nilpotent element) H. Behncke in [1] has proved that such elements exist in any non-commuttive group algebra. For such element we have $\exp(a) = 1+a$.

Let $U(\ell^1(G))$ denote the group of invertible elements in and let $gr(\exp(\ell^1(G)))$ is a subgroup of $U(\ell^1(G))$ generated by the image of exp. Observe that the image need not form a group unless G is Abelian.

We have $g^{-1}\exp(a)g = \exp(g^{-1}ag)$, $g \in G$, $a \in \ell^1(G)$ which implies.

<u>Proposition 2</u>

The group generated by $\exp(\ell^1(G))$ is a normal subgroup of $U(\ell^1(G))$.

2. It is clear that $\ell^1(G)$ is commutative if G is so One may conjecture that the group of invertible elements $U(\ell^1(N_2))$ is "not far" from nilpotent groups. Unexpectedly enough it is not the case.

<u>Theorem 2</u>

Let $u_1 = [\exp(y), x] = \exp(-y)x^{-1}\exp(y)x$, $u_{n+1} = [u_n, x]$, for $n>1$, be elements from $U(\ell^1(N_2))$. Then we have $u_n = \exp(y(z-1)^n)$ for all $n \in \mathbb{N}$ and consequently the group $gr\langle\exp(y), x\rangle$ does not satysfy the Engel condition (cf. [3]).

Proof

We have

$u_1 = \exp(-y)x^{-1}\exp(y)y = \exp(-y)\exp(x^{-1}yx) = \exp(-y)\exp(yz) =$

$\exp(-y+yz) = \exp(y(1-z))$

because the elements $-y$ and yz, $z = [y,x]$, commute.

Furthe we get

$$u_{n+1} = [u_n, x] = \exp(-y(z-1)^n)x^{-1}\exp(y(z-1)^n)x =$$

$$\exp(-y(z-1)^n)\exp(x^{-1}y(z-1)^n x) =$$

$$\exp(-y(z-1)^n)\exp(yz(z-1)^n)) =$$

$$\exp(y(z-1)^{n+1}),$$

which completes the proof.

Altrough the group $gr\langle\exp(y),x\rangle$ is far to be nilpotent group we schow

Theorem 3

The commutator group $gr\langle\exp(y),x\rangle$ is Abelian. Thus $gr\langle\exp(y),x\rangle$ is metabelian group.

Proof

Any element in $gr\langle\exp(y),x\rangle$ is of the form

$$g = x^{n_1}\exp(m_1 y) \ \dots \ x^{n_k}\exp(m_k y).$$

Observe that

$$\exp(-my)x^{-n}\exp(my)x^n = \exp(-my)\exp(myz^n) =$$

$$\exp(y((mz^n - m)).$$

$$\exp(my)x^n = x^n\exp(my)\exp(y(mz^n - m))$$

Using this we can step by step by collecting process rewrite g in the form

$$g = x^n \exp(y \sum_i a_i z^i)$$

for some n, a_i and i from $\mathbb{Z}$.

Hence we obtain

$$[g,g'] = \exp(-y \sum a_i z^i) x^{-n} \exp(-y \sum a_i' z^i) x^{-n'+n} \exp(y \sum a_i z^i) x^n$$
$$\exp(y \sum a_i' z') = \exp(y \sum b_i z^i)$$

for some $b_i \in \mathbb{Z}$. Now it is clear that arbitrary commutators from gr(exp(y),x) and theorem 3 follows.

R E F E R E N C E S

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Recenzent: Dr hab. Józef Dudek

UWAGI O ELEMENTACH ODWRACALNYCH W ALGEBRZE GRUPOWEJ GRUPY DYSKRETNEJ

Streszczenie.

W pracy bada się elementy odwracalne w algebrze grupowej grupy dyskretnej. Podano kilka sposobów konstrukcji takich elementów. Dowiedziono, że w przypadku dyskretnej grupy Heisenberga elementy odwracalne tworzą grupę "daleką" od nilpotentnej, bo nie spełnia ona nawet warunku Engle'a.

ЗАМЕЧАНИЯ ОБ ОБРАТИМЫХ ЭЛЕМЕНТАХ В ГРУПОВЫХ АЛГЕБРАХ ДИСКРЕТНЫХ ГРУПП

Р е з ю м е

В этой работе исследованы обратимые элементы в групповой алгебре дискретной группы. Указаны некоторые методы конструкции таких элементов. Доказано, что группа всех обратимых еленентов в групповой алгебре дискретной группы Гейзенберга не является энгелевой.