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BASIC CONCEPTS OF KLEIN GEOMETRIES

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PREFACE

Although almost a hundred years passed from the famous Erlangen Program of Felix Klien, it is still not being used in full. The main reason of this situation lies, among others, in the fact that it was not presented precisely enough. The original definition of geometry, as formulated by F. Klein (see [6]), may be stated as follows: *Geometry* of the set M with respect to a group of transformations $\mathcal{G}(M)$ of this set or, simply, $\mathcal{G}(M)$-geometry is the set af all properties of geometric figures which do not change under the transformations of the group $\mathcal{G}(M)$. Such properties are called *invariants* or *geometric properties*. When the necessity appeared to study geometries based on the sets of transformations not necessarily forming groups, the spaces with a group of transformations were called *Klein spaces*. The present paper deals only with Klein spaces. Precise definition of concepts of geometric spaces which do not allow groups of transformations and studying their properties is much more difficult. Formulation of Klein's ideas in a precise way forms the base for more general studies.

R. Sulanke defines Klein space (see [21], [22]) as a transitive, left Lie group of transformations, i.e. the triplet (M, G, f), where M is a manifold, G - Lie group and f a transitive left operation of the group G on M, whereas G-geometry is some category connected with Lie group G. It seems that the definition of geomety as a category coincides with original Klein's definition. Invariants, Klein's definition describes are simply morphisms of a proper category.

E. J. Jasińska and M. Kucharzewski defined G-geometry in [4] as an abstract object (M, G, f). M. Kucharzewski in his further papers (see [12], [13]) derived some ideas from papers by R. Sulanke (cf. [21], [22]) and defined Klein space, geometric object and geometry as in §2, Section 1 of present paper. Definitions of these concepts, however, arouse some reservations. The main deficiency of the definition of geometric object is that there is no correlation between the fibre of object and the fibre of space, as well as between transformation formulas of object and space. Moreover, it follows from the definition of geometry as a category of geometric objects that, in some cases, non-equivalent Klein spaces have the same geometry.

The aim of this paper is to present in a precise way some concepts of the theory of Klein spaces and to discuss some of their properties. Section I is a kind of introduction. Presented there are some basic notions, necessary to clear further part of the paper.

Section II contains new definitions of geometric object and Klein geometry. Undefined till now, the notion of equivalence of two Klein geometries is also introduced, as well as the necessary and sufficient condition for two geometries to be equivalent.

Section III is devoted to methods of construction of geometric objects. Two new objects are defined there, i.e. the object of transformations and the disjoint union of objects. It is proved that the objects of transformations and factor objects as well as G-products and disjoint unions of geometric objects of a given Klein space are geometric objects of this space. There are also presented some necessary and sufficient conditions for the objects of category of abstract objects supperted by the same group to be geometric objects of a proper Klein space.

Results obtained are illustrated in Section IV on the examples of elementary Klein spaces such as vector space, unitary space, affine and Euclidean space. With the use of the notion of the object of transformations we formulate the definitions of tensors and tensor densities in a new approach.

A reference is always given when we quote a result of some other author. In other cases the results presented are obtained by the author or they are generally well known facts.

B. Szociński

Katowice, 1989.

Section I

INTRODUCTION

The aim of this section is to define basic notions and to introduce basic concepts, as well as presenting theorems, used in further parts of this paper.

§1. Operation of the group on the set

Let X be an arbitrary nonempty set, G - abstract group and let F be the operation

$$F: X \times G \rightarrow X \tag{1.1}$$

Definition 1.1. Any mapping (1.1) satisfying the following conditions:

$$\bigwedge_{x \in X} \quad \bigwedge_{g_1, g_2 \in G} \quad F\big(F(x, g_1), g_2\big) = F(x, g_2 \cdot g_1), \tag{1.2}$$

$$\bigwedge_{x \in X} F(x, e) = x \tag{1.3}$$

where e is a neutral element of G, and $g_1 \cdot g_2$ denotes the group multiplication, will be called a *(left) operation of the group G on the set X.*

The above condition are called respectively a *translation* (or *fundamental*) *condition* and *identity condition.* If the *effectivity condition* is fulfilled, i.e.

$$\bigwedge_{g \in G} \Big(\big(\bigwedge_{x \in X} F(x, g) = x \big) \Rightarrow g = e \Big) \tag{1.4}$$

then the operation F is called *effective*, and the group G operates on X *effectively.*

It is well known that the set of all bijective mappings of the nonempty set X onto itself with the operation of superposition forms a group. Such a group will be denoted by $\mathcal{G}(X)$ and called a *group of all transformations of the set X.*

Definition 1.2. Homomorphism

$$\varphi: G \rightarrow \mathcal{G}(X) \tag{1.5}$$

will be called a *representation of the abstract group G in the group of all transformations of the set X.*

Translation equation (1.2) and the identity condition (1.3) imply (cf. [12]) that the transformation

$$F_g: X \rightarrow X, \quad F_g(x) := F(x, g) \tag{1.6}$$

is a bijection of the set X onto itself. Hence, by (1.2), the operation (1.1) defines a representation

$$\hat{F}: G \rightarrow \mathcal{G}(X), \quad \hat{F}(g) := F_g \tag{1.7}$$

of this group in the group of all transformations of the set X. It is easily seen that the reverse is true: for a given representation (1.5) we may define an operation of the group G on the set X as follows:

$$F(x, g) := \varphi_g(x), \quad \text{where } \varphi_g = \varphi(g). \tag{1.8}$$

Definition 1.3. Homomorphism (1.7) will be called a *representation of the group G in the group of all transformations of the set X, defined* or *induced by the operation (1.1) of the group G on the set X.* Transformation (1.1) given by the formula (1.8) will be called *the operation of the group G on the set X defined* or *induced by the representation (1.5) of this group in the group of transformations $\mathcal{G}(X)$.*

As the immediate consequence of these definitions we may state a necessary and sufficient condition for the operation of the group on the set to be effective.

Corollary 1.1. The operation (1.1) of the group G on the set X is effective iff the representation (1.7) induced by this operation is a monomorphism.

Now, we will define a transitive operation.

Definition 1.4. The operation of the group G on the set X will be called *transitive* iff for every $x_1, x_2 \in X$ there exists a $g \in G$ such that

$$F(x_1, g) = x_2 \tag{1.9}$$

If the element g is unique, then the operation (1.1) will be called *directly transitive.* In such cases the group G operates on X *transitively* (or *directly transitively*).

§2. The category of abstract objects

Let us begin from the definitions of abstract object and Klein space (cf. [13], p. 12 and [5], [12], [15]).

Definition 2.1. Any triplet

$$(X,\ G,\ F) \tag{2.1}$$

consisting of an arbitrary nonempty set X, abstract group G and the operation F of this group on X will be called an *abstract object*. The set X will be called a *fibre* of this object, and its elements the *points* (or *particular objects*). The operation F will be called a *transformation formula* (or transformation law) of the object, and the representation $\hat{F}$ of the group G in the transformation group $\mathcal{G}(X)$ induced by the operation F - the *representation of the object (2.1)*.

Definition 2.2. If the operation F of the group G on the fibre X is effective (transitive) then the abstract object (2.1) is called *effective (transitive)*. Effective objects are called *Klein spaces*, whereas transitive - *homogeneous spaces*.

Example 2.1. Let us consider an arbitrary group G and the transformation

$$L:\ G\times G \to G, \quad L(x,g) := g\cdot x \tag{2.2}$$

(the left translation in the group G). It is easily seen that L is an effective and directly transitive operation of the group G on the set of its elements. Thus, the triplet

$$(G,\ G,\ L) \tag{2.3}$$

is the effective abstract object. Hence, it is an example of a Klein space.

Example 2.2. Let X be a topological space, and let G be the group of all homeomorphisms of X. The transformation

$$F:\ X\times X \to X, \quad F(x,g) := g(x)$$

is an effective operation of the group G on X. Therefore the triplet (2.1) is a Klein space. It is called a topological Klein space.

Example 2.3. $GL(n,\mathbb{K})$ will denote multiplicative group of nonsingular square matrixes of n-th degree with elements of a field $\mathbb{K}$. The set of all pairs:

$$GA(n,\mathbb{K}) := \{((a^j), [A^j_i]) : (a^j) \in \mathbb{K}^n \wedge [A^j_i] \in GL(n,\mathbb{K})\}$$

with operation "$*$" defined by the formula:

$$((b^j), [B^j_i]) * ((a^j), [A^j_i]) := ((b^j + B^j_i a^i), [B^j_i A^i_l])$$

(we use the Einstein's summation convention), forms the group called *affine group of n-th order over the field $\mathbb{K}$*. The transformation

$$f: \mathbb{K}^n \times GA(n,\mathbb{K}) \to \mathbb{K}^n,$$
$$f((x^i), ((a^j), [A^j_i])) := (a^j + A^j_i x^i) \tag{2.4}$$

is the effective and transitive operation of the affine group $GA(n,\mathbb{K})$ on the set $\mathbb{K}^n$. Abstract object

$$(\mathbb{K}^n,\ GA(n,\mathbb{K}),\ f) \tag{2.5}$$

is called *n-dimensional canonical affine Klein space over the field $\mathbb{K}$*.

Let us consider two abstract objects

$$(X_1,\ G_1,\ F_1) \tag{2.6}$$
$$(X_2,\ G_2,\ F_2) \tag{2.7}$$

and two transformations

$$(\psi, \varphi), \tag{2.8}$$

where ψ: $X_1 \to X_2$, and φ: $G_1 \to G_2$ is a homomorphism G_1 into G_2.

Definition 2.3. Any pair of transformations (2.8) satisfying *equivariance condition*

$$\bigwedge_{x_1 \in X_1} \bigwedge_{g_1 \in G_1} F_2(\psi(x_1), \varphi(g_1)) = \psi(F_1(x_1, g_1)). \tag{2.9}$$

will be called an *equivariant transformation* of abstract object (2.6) into abstract object (2.7).

Whenever such a pair (2.8) exists, the object (2.6) is *equivariant* with the object (2.7).

It is easy to verify (see [13], p. 18 and [9], [25]) that the class of all abstract objects, as well as the class of all Klein spaces with equivariant transformations as morphisms and with superposition of pairs of transformations (2.8) as composition form categories. These categories we will denote by

OA and PK, respectively. They will be called *the category of abstract objects and the category of Klein spaces.*

Now, let

$$(X_1, G, F_1) \tag{2.10}$$

$$(X_2, G, F_2) \tag{2.11}$$

be two abstract objects, and let

$$(\psi, id_G), \quad \psi: X_1 \to X_2 \tag{2.12}$$

be the equivariant transformation. The class of all abstract objects supported by the same group G, with equivariant transformations of the form (2.12) as the composition, form the category as well. We will denote it by OA(G) and call *the category of abstract objects supported by the group G.*

As the immediate corollary of the above definitions we may note that the categories OA(G) and PK are the subcategories of OA.

In [13] (cf. also [4], [12]) geometric object and Klein geometry are defined as follows:

Definition 2.4. Abstract object of category OA(G), i.e. object (2.1) supported by the same group as Klein space

$$(M, G, f) \tag{2.13}$$

will be called a *geometric object* of Klein space (2.13). The category OA(G) will be called *Klein geometry* of the group G or *G-geometry.*

In the following we will define the notions of invariants and comitants, very important for Klein geometries (cf. [13], p 21, also [4])).

Definition 2.5. The transformation $\psi: X_1 \to X_2$ of the fibre X_1 of object (2.10) into the fibre X_2 of object (2.11) will be called *invariant transformation* (or simply an *invariant*) iff the pair of transformations (2.21) is a morphism of the category OA(G), i.e. the condition

$$\bigwedge_{x_1 \in X_1} \bigwedge_{g \in G} F_2(\psi(x_1), g) = \psi(F_1(x_1, g)) \tag{2.14}$$

If ψ is surjection, the geometric object (2.11) will be called a *comitant* of the object (2.10).

The fundamental problems for each Klein geometry are to determine geometric objects and their invariants and comitants and to classify the objects, i.e. to determine classes of equivalent objects.

Since the classes of abstract objects, Klein spaces and geometric objects of a given Klein space form the categories, to define equivalence of objects we have to use the notion of isomorphism of respective categories (cf. [13], p. 21 and [16]).

Definition 2.6. Abstract objects (Klein spaces) are *abstractively equivalent* iff there exists a pair of transformations (2.8) being the isomorphism of the category OA (category PK).

B. Zaporowski proved in [25] the following theorem.

Theorem 2.1. Morphism (2.8) of the object (2.6) into object (2.7) is an isomorphism of the category OA iff ψ is a bijection and φ is a group isomorphism.

As the immediate consequence of this theorem and definition 2.6 we get:

Corollary 2.1. Abstract objects (Klein spaces) (2.6) and (2.7) are abstractively equivalent iff there exist a bijection $\psi\colon X_1 \to X_2$ and isomorphism $\varphi\colon G_1 \to G_2$ such that the equivariance condition (2.9) holds.

Definition 2.7. Objects (2.10) and (2.11) of category OA(G) are *geometrically equivalent* iff there exists a pair of transformations (2.12) being an isomorphism of this category.

It is easily seen that the following corollary is true.

Corollary 2.2. Objects (2.10) and (2.11) of category OA(G) are geometrically equivalent iff there exists a bijection $\psi\colon X_1 \to X_2$, being an invariant transformation, i.e. such that the equivariance condition (2.14) holds true.

As a consequence of the properties of the category isomorphism we get the following corollary.

Corollary 2.3. The relations of abstract and geometrical equivalence are equivalence relations, i.e. they are reflexive, symmetric and transitive.

From the definitions 2.6 and 2.7 we infer that every two objects (2.10) and (2.11) supported by the same group that are geometrically equivalent are abstractively equivalent. The natural problem arises, whether the abstract equivalence implies the geometrical equivalence of objects. The answer is negative, as demonstrates the example presented in [16]. In the same paper there are given some necessary and sufficient conditions for two abstractively equivalent objects of the category OA(G) to be geometrically equivalent.

In the sequel we will call abstractively (geometrically) equivalent objects simply equivalent, unless it may cause any misunderstandings. In particular, the equivalence of two objects supported by the same group means geometrical equivalence.

§3. Subobjects and partial objects

Let us consider an arbitrary abstract object

$$(X,\ G,\ F) \tag{3.1}$$

and the subgroup $\tilde{G} \subset G$. $\tilde{F}$ denotes the restriction of operation F to the set $X \times \tilde{G}$. It is easily seen that $\tilde{F}$ is an operation of the group $\tilde{G}$ on the set X. Therefore

$$(X,\ \tilde{G},\ \tilde{F}), \quad \tilde{F} := F|_{X\times\tilde{G}} \tag{3.2}$$

is an abstract object (cf. [13], p. 36 and [4]).

Definition 3.1. Abstract object (3.2) will be called a *subobject* of the object (3.1) supported by the subgroup $\tilde{G}$.

As the immediate consequence we get

Corollary 3.1. Every subobject of Klein space is a Klein space.

Such a method of defining subobjects can be generalized. Let $\tilde{G}$ be an arbitrary group and let

$$\varphi\colon\ \tilde{G} \rightarrow G \tag{3.3}$$

be a homomorphism. The operation

$$\tilde{F}\colon\ X\times\tilde{G} \rightarrow X, \quad \tilde{F}(x,\tilde{g}) := F\big(x,\varphi(\tilde{g})\big)$$

is the operation of the group $\tilde{G}$ on the set X (cf. [13], p. 37).

Thus

$$(X,\ \tilde{G},\ \tilde{F}), \quad \tilde{F}(x,\tilde{g}) := F(x,\varphi(\tilde{g})) \qquad (3.4)$$

is an abstract object.

Definition 3.2. Abstract object (3.4) will be called *induced* or *determined* by the object (3.1) and homomorphism (3.3).

In the particular case, if φ is an isomorphism, the following corollary holds true.

Corollary 3.2. Object (3.4) determined by the object (3.1) and isomorphism (3.3) is abstractively equivalent with the object (3.1). Moreover, (id_X, φ) is an isomorphism (of the category OA) of object (3.4) onto object (3.1).

As a consequence of definition 3.1 of subobject and definition 3.2 of induced object we get the following corollary.

Corollary 3.3. Object (3.4) induced by object (3.1) and imbedding (3.3) of the subgroup $\tilde{G}$ of the group G into G is a subobject (3.2) of the object (3.1), determined by the subgroup $\tilde{G}$.

Now, we will prove another corollary.

Corollary 3.4. The object induced by a Klein space (2.13) and a homomorphism (3.3) is a Klein space iff this homomorphism is a monomorphism.

Proof. On account of corollary 1.1, the representation $\hat{f}$ of Klein space (2.13) is a monomorphism. It is easily seen that the representation $\hat{\tilde{f}}$ of the object $(M,\ \tilde{G},\ \tilde{f})$

$$\tilde{f}(p,\tilde{g}) := f\big(p,\varphi(\tilde{g})\big)$$

induced by Klein space (2.13) satisfies the equality

$$\hat{\tilde{f}} = \hat{f}\circ\varphi.$$

Since $\hat{f}$ is a monomorphism, $\hat{\tilde{f}}$ is a monomorphism iff φ is a monomorphism. Hence, by the corollary 1.1 we get the thesis. □

To introduce the notion of partial object we will start from the definition of invariant subset (cf. [13], p. 35, also [4]).

Definition 3.3. A nonempty subset X_0 of the fibre X of object (3.1) will be called *invariant* (or *permissible*) iff

$$\bigwedge_{x\in X_o} \bigwedge_{g\in G} F(x, g)\in X_o$$

Definition 3.4. A subset of the fibre of object (3.1) defined by the formula:

$$W^F_{x_o} := \{F(x_o, g): g\in G\}$$

will be called a *transitive fibre* of this object, determined by $x_o\in X$.

Obviously, every transitive abstract object (3.1) has only one transitive fibre equal to whole fibre X. Any invariant subset X_o of the fibre of object (3.1) is either a transitive fibre or a union of a family of transitive fibres of this object.

It is easy to check that for arbitrary invariant subset X_o of the fibre of object (3.1) the restriction F_o of the transformation formula F of this object is an operation of the group G on the set X_o. Thus, the triplet

$$(X_o,\ G,\ F_o), \qquad F_o := F|_{X_o\times G} \tag{3.5}$$

is an abstract object.

Definition 3.5. Abstract object (3.5) will be called a *partial object* of the object (3.1) determined by invariant subset X_o.

The following simple corollary is a consequence of the above definition and the effectivity condition (1.4).

Corollary 3.5. If at least one partial object of object (3.1) is effective, then (3.1) is effective.

The method of construction of partial object can be generalized as well.

Definition 3.6. The bijection

$$\psi: Y \rightarrow X_o, \qquad X_o\subset X \tag{3.6}$$

of an arbitrary set Y onto subset X_o of the fibre X of object (3.1) will be called *invariant* iff X_o is an invariant subset of this object.

It can be proved (cf. [13], p. 37) that the transformation

$$F_1\colon Y\times G \to Y, \quad F_1(y,g) := \psi^{-1}\bigl(F(\psi(y),g)\bigr)$$

is an operation of the group G on the set Y. Hence, the triplet

$$(Y,\ G,\ F_1), \qquad F_1(y,g) := \psi^{-1}\bigl(F(\psi(y),g)\bigr) \tag{3.7}$$

is an abstract object.

Definition 3.7. Object (3.7) will be called *determined* (or *induced*) by the object (3.1) and the invariant bijection (3.6).

Two following corollaries are immediate consequences of the definition.

Corollary 3.6. Object (3.7) induced by object (3.1) and invariant bijection (3.6) is geometrically equivalent with partial object (3.5) of the object (3.1) determined by the invariant subset X_o. The pair (ψ, id_G) is an isomorphism of object (3.7) onto object (3.5).

Corollary 3.7. Object (3.7) induced by the object (3.1) and invariant bijection id_{X_o} is a partial object of the object (3.1) determined by invariant subset X_o.

It is always possible to define the object induced by the given object (3.1), group homomorphism (3.3) and invariant bijection (3.6). The simple example of such an object is the partial subobject

$$(X_o,\ \tilde{G},\ F|_{X_o\times\tilde{G}})$$

of the object (3.1) determined by the subgroup $\tilde{G}$ of the group G and invariant subset X_o of the fibre of the object (3.1).

§4. Objects of subsets of the fibre of object

Let

$$(X,\ G,\ F) \tag{4.1}$$

be a given abstract object. 2^X will denote the family of all subsets of the fibre of this object. The transformation

$$F^*\colon 2^X\times G \to 2^X$$

given by the formula

$$F^*(A, g) := F(A, g) = \{F(x, g): x \in A\} \tag{4.2}$$

is obviously an operation of the group G on the set 2^X, whereas the triplet

$$(2^X,\ G,\ F^*) \tag{4.3}$$

is an abstract object. For each invariant subset $\mathfrak{M}$ of the fibre of this object we can define a partial object

$$(\mathfrak{M},\ G,\ F^*|_{\mathfrak{M}\times G}), \quad \mathfrak{M}\subset 2^X. \tag{4.4}$$

Definition 4.1. Objects (4.4) and (4.3) will be called *object of subsets* and the *object of all subsets*, respectively, of the fibre of the object (4.1).

Let us consider two objects:

$$(X_1,\ G,\ F_1) \tag{4.5}$$

$$(X_2,\ G,\ F_2) \tag{4.6}$$

and the objects of all subsets of the fibres of these objects

$$(2^{X_1},\ G,\ F_1^*) \tag{4.7}$$

$$(2^{X_2},\ G,\ F_2^*) \tag{4.8}$$

In the sequel two following lemmas will prove useful.

Lemma 4.1. If objects (4.5) and (4.6) are equivalent, then the objects (4.7) and (4.8) of all subsets of the fibres of the objects (4.5) and (4.6), respectively, are also equivalent.

Proof. The assumption and the corollary 2.2 implies the existence of a bijection $\psi: X_1 \to X_2$ such that the equivariance condition (2.14) holds true. It is easily seen that the transformation

$$\psi_o: 2^{X_1} \to 2^{X_2} \quad ,$$

defined by the formula

$$\psi_o(A) := \psi(A) = \{\psi(x_1): x_1 \in A\},$$

is a bijection. The equality

$$F_2^*(\psi_o(A), g) = \psi_o(F_1^*(A, g)) \tag{4.9}$$

holds true for every $A\subset X_1$ and $g\in G$. Indeed, from (2.14) and the definitions of transformation formulas F_1^*, F_2^* and the bijection

ψ_0 we infer that for each subset A of the fibre X_1 of the object (4.5) and each $g \in G$

$$F_2^*(\psi_0(A), g) = F_2(\psi(A), g) = F_2(\{\psi(x_1): x_1 \in A\}, g) =$$
$$= \{F_2(\psi(x_1), g): x_1 \in A\} = \{\psi(F_1(x_1, g)): x_1 \in A\} =$$
$$= \psi(F_1(A, g)) = \psi_0(F_1^*(A, g))$$

Thus, the objects (4.7) and (4.8) are equivalent.□

Lemma 4.2. If the abstract object

$$(X_0, G, F_0) \qquad (4.10)$$

is a partial object of the object (4.1), then the object

$$(2^{X_0}, G, F_0^*) \qquad (4.11)$$

of all subsets of the fibre of the object (4.10) is a partial object of the object (4.3).

Proof. The fibre 2^{X_0} of the object (4.11) is obviously a subset of the fibre 2^X of the object (4.3). We will show that it is an invariant subset. For each subset A of the fibre X_0 of the object (4.10) and for each $g \in G$ we have

$$F_0^*(A, g) = F_0(A, g) = F(A, g) = F^*(A, g).$$

Thus

$$F_0^*(A, g) = F^*(A, g) \quad \text{for } A \in 2^{X_0},\ g \in G. \qquad (4.12)$$

Since $F_0^*(A, g) \in 2^{X_0}$, by (4.12) the set 2^{X_0} is an invariant subset of the fibre of object (4.3) and

$$F_0^* = F^*|_{2^{X_0} \times G}$$

what ends the proof.□

The objects of geometric figures of a given Klein space

$$(M, G, f) \qquad (4.13)$$

are the examples of the objects of subsets.

Definition 4.2. A subset A of the fibre M of the Klein space (4.13) will be called a *geometric figure* of this space. An object of subsets of the fibre of Klein sapce will be called an *object of geometric figures* of this space.

Example 4.1. As the object of geometric figures of n-dimensional affine space (cf. example 2.3) over the field $\mathbb{K}$ we may mention the object of k-dimensional hyperplanes ($1 \leq k < n$). In particular, the object of straight lines is such an object. The object of pencils of lines is an object of subsets of the fibre of the object of straight lines. The more sophisticated example of geometric object of affine space is tensor (cf. Section IV). It can be shown that it is also an object of subsets of the fibre of some geometric objects of affine space.

Taking all this into account we may state that the objects of subsets of the fibre of object play a particularly important role in the theory of Klein spaces.

§5. Remarks

The definition of Klein space given in §2 is to general. Beside geometric Klein spaces, i.e. the spaces being the subject of study of metageometry, it contains many other spaces, e.g. topological Klein space (cf. example 2.2). Thus, to the effectivity condition some other condition should be added, to assure that Klein space is geometric. Unfortunately, such conditions are not known, as yet. Since in all classical Klein spaces there exist m-repers (cf. definition III.2.3), the effectivity conditions could be replaced by a stronger one, postulating the existence of such repers. This condition does not solve the problem, though.

In the definition 2.4 we do not assume any relations between the fibre of the space and that of the object, neither we do between the transformations laws. Whether the abstract object is a geometric object of a given Klein space depends solely of the abstract group. Z. Moszner suggested that abstract objects, which are not properly related to the Klein space should not be considered as geometric objects of that space. In geometric studies such objects are simply useless. Therefore, we should either prove that there are proper relations between the objects of the category OA(G) and Klein space (M, G, f), or to accept as geometric only the objects of such subcategory that ensures the existence of such relations and, moreover, that contains all object traditionally viewed as geometric.

There are some reservations about the definition of Klein space. It can be shown that beside given Klein space (M, G, f) there exist effective objects of the category OA(G) which are not equivalent with it. As a result we have that the category OA(G) itself is a geometry of non-equivalent Klein spaces. E. Siwek and E. Kasparek shown (unpublished result) that in the category OA(G) there exist ever primitive and transitive Klein spaces, which are not equivalent. It seems that by a Klein geometry one should understand a pair consisting of a Klein space and a category of geometric objects.

Because of all the abovementioned reasons, in the sequel, by geometric objects of the space (M, G, f) we will understand the objects of some subcategory OG(f) of the category OA(G), and by a Klein geometry of this space - the pair

$$\big((M,\ G,\ f),\ OG(f)\big).$$

By a Klein space we will still undarstand any effective abstract object.

In the papers on Klein spaces that have been published, the equivalence of two Klein geometries was not defined. It will be done in §4 of the Section II.

Section II

THE NOTION OF KLEIN GEOMETRY

In this section we will define basic concepts of the theory of Klein geometries: geometric object, Klein geometry and the equivalence of two Klein geometries.

§1. <u>The definition of geometric object</u>

Let

$$(M,\ G,\ f) \tag{1.1}$$

be a Klein space. We will start from the definition of the standard geometric object of rank k of the space (1.1), and then the definition of any geometric object of this space.

<u>Definition 1.1.</u> Standard geometric object of rank k ($k\in\mathbf{N}$) of the Klein space (1.1) is an abstract object

$$(\Omega^{(k)}(M),\ G,\ f^{(k)}),\quad k\in\mathbf{N} \tag{1.2}$$

defined as follows

(a) for k=1 the object (1.2) is the object of all subsets of the fibre of Klein space (1.1), i.e.

$$\Omega^{(1)}(M) := 2^{M} \quad \text{and} \quad f^{(1)} := f^{*};$$

(b) object $(\Omega^{(m+1)}(M),\ G,\ f^{(m+1)})$ is the object of all subsets of the fibre of the object $(\Omega^{(m)}(M),\ G,\ f^{(m)})$, i.e.

$$\Omega^{(m+1)}(M) := 2^{\Omega^{(m)}(M)} \quad \text{and} \quad f^{(m+1)} := (f^{(m)})^{*}$$

<u>Definition 1.2.</u> The abstract object

$$(X,\ G,\ F) \tag{1.3}$$

equivalent with any partial object of a standard geometric object (1.2) will be called a *geometric object of the Klein space (1.1)*.

Hence, abstract object (1.3) is the geometric object of the Klein space (1.1) iff there exists a $k\in\mathbf{N}$ and the invariant subset $\Omega_{0}^{(k)}(M)$ of the fibre of the standard geometric object (1.2) of rank k such that the partial object

$$\left(\Omega_o^{(k)}(M),\ G,\ f_o^{(k)} \right); \quad f_o^{(k)} := f^{(k)}|_{\Omega_o^{(k)}(M)\times G} \tag{1.4}$$

is equivalent with the object (1.3). There exists then a bijection

$$\psi\colon\ X \rightarrow \Omega_o^{(k)}(M) \tag{1.5}$$

such that the equivariance condition

$$\bigwedge_{x\in X}\ \bigwedge_{g\in G}\quad f_o^{(k)}\big(\psi(x), g\big) = \psi\big(F(x, g)\big). \tag{1.6}$$

It follows that between the fibre X of an arbitrary geometric object (1.3) of Klein space (1.1) there exists a relation, determined by the invariant transformation (1.5). Transformation formulas f and F of the Klein space and its geometric object (1.3) are closely related through the equivariance condition (1.6).

<u>Lemma 1.1.</u> If an abstract object (1.3) is equivalent with a partial object (1.4) of the standard geometric object (1.2) of rank k of Klein space (1.1), then it is equivalent with some partial object of every standard geometric object of rank m ($m > k$) of this space.

Proof. Let $\mathfrak{M}$ denote the family of all singletons of the fibre of the object (1.4), i.e.

$$\mathfrak{M} = \{ \{A\}:\ A\in\Omega_o^{(k)}(M) \}.$$

It is easily seen that $\mathfrak{M}$ is an invariant subset of the fibre of standard geometric object of rank k+1. Thus, we can define a partial object

$$(\mathfrak{M},\ G,\ f_o^{(k+1)}),\qquad f_o^{(k+1)} := f^{(k+1)}|_{\mathfrak{M}\times G} \tag{1.7}$$

of this object. It can be proved, by direct calculation, that the bijection

$$\psi\colon\ \Omega_o^{(k)}(M) \rightarrow \mathfrak{M},\qquad \psi(A) := \{A\}$$

satisfies the equivariance condition

$$f_o^{(k+1)}\big(\psi(A), g\big) = \psi\big(f_o^{(k)}(A, g)\big).$$

Hence, the objects (1.4) and (1.7) are equivalent. The assumption and the transitivity of equivalence relation imply that the object (1.3) is equivalent with the object (1.7). Thus, the thesis is true for m=k+1. One can easily prove by induction that it is also true for every positive integer $m > k$. □

Definition 1.3. A positive integer k will be called a *rank of the geometric object (1.3) of the Klein space (1.1)* iff this object is equivalent with some partial object (1.4) of the object (1.2) and is not equivalent with any partial object of the standard geometric object of rank $m < k$ of this space.

Let us note that each object (1.2) is a geometric object of Klein space (1.1) in the sense of definition 1.2, and k is a rank of this object in the sense of definition 1.3. Each partial object (1.4) of object (1.2) is a geometric object of this space as well. The rank of this object is no greater that k.

The following two corollaries follows immediately from the definitions.

Corollary 1.1. Each abstract object (1.3) equivalent with a geometric object of rank k of Klein space (1.1) is a geometric object of this space of the same rank.

Corollary 1.2. Each object of geometric figures of Klein space (1.1) is a geometric object of this space of rank 1.

It is easily seen that the family

$$\mathfrak{M} = \{\langle p\rangle : p \in M\}$$

of singletons of the fibre of Klein space (1.1) is an invariant subset of the fibre of standard geometric object of rank 1 of this space, whereas the partial object

$$(\mathfrak{M}, G, f_{\circ}^{(1)}), \quad f_{\circ}^{(1)} := f^{(1)}|_{\mathfrak{M}\times G}$$

is equivalent with the given Klein space (1.1). As the immediate consequence we have the following corollary.

Corollary 1.3. Each Klein space (1.1) is a geometric object of rank 1 of this space.

Klein space considered as a geometric object of itself is usually called a *point object*.

From lemma 1.1 and corollary 1.3 we infer that for every positive integer k, a standard geometric object (1.2) of Klein space (1.1) has a partial object equivalent with this space. Such an object is effective, as equivalent with effective object, i.e.

with the space (1.1). Thus, from corollary I.3.5 we get the corollary.

Corollary 1.4. Standard geometric object of any rank is an effective object.

Using objects (1.2) we can define further abstract objects. Let

$$\Omega^{(\infty)}(M) := \bigcup_{k=1}^{\infty} \Omega^{(k)}(M)$$

and let $f^{(\infty)}\colon \Omega^{(\infty)}(M) \to \Omega^{(\infty)}(M)$ be a transformation given by the formula

$$f^{(\infty)}(A, g) := f^{(k)}(A, g) \qquad \text{for } A \in \Omega^{(k)}(M).$$

It is easy to verify that

$$(\Omega^{(\infty)}(M),\ G,\ f^{(\infty)}) \tag{1.8}$$

is an effective abstract object. Hence, it is a Klein space. Let

$$(\Omega^{(\infty,n)}(M),\ G,\ f^{(\infty,n)}),\quad n \in N \tag{1.9}$$

denote the standard geometric object of rank n of this space. The above method of construction of abstract objects can be iterated to define successive Klein spaces of the form (1.8) and their standard geometric objects.

From the well known results of set theory follows that the power of the fibre of object (1.9) is greater than the power of the fibre of any of the objects (1.2), so the object (1.9) is not a geometric object of Klein space (1.1) in the sense of definition 1.2. Further considerations presented in this paper let us assume that the objects (1.8) and (1.9) are excessive. Geometric interpretation of these objects in not known and they are not found in geometry. Therefore, we do not consider them geometric objects of the space (1.1), although their fibres and transformation formulas are related with the fibre M and transformation formula f of Klein space (1.1).

§2. The category of geometric objects

The class of all geometric objects of a given Klein space

$$(M,\ G,\ f) \tag{2.1}$$

with equivariant transformations of the form

$$(\psi,\ id_G),\quad \psi\colon X_1 \to X_2$$

(X_1 and X_2 are the fibres of geometric objects) as morphisms and the superposition of such transformations as a composition forms a category. This category will be denoted as OG(f) and called a *category of geometric objects of Klein space (2.1).*

It is not difficult to prove that OG(f) is indeed a category. One has only to verify the axioms of the category. It is equally simple to check that OG(f) is a subcategory of OA(G) of abstract objects supported by the same group G.

We will start from the following theorem.

Theorem 2.1. Categories of geometric objects of two geometrically equivalent Klein spaces are identical.

Proof. Let the object

$$(\bar{M},\ G,\ \bar{f}) \tag{2.2}$$

be a Klein space geometrically equivalent with the space (2.1) and let

$$(\Omega^{(k)}(M),\ G,\ f^{(k)}),\quad k \in \mathbf{N} \tag{2.3}$$

$$(\Omega^{(k)}(\bar{M}),\ G,\ \bar{f}^{(k)}),\quad k \in \mathbf{N} \tag{2.4}$$

be standard geometric objects of rank k of the space (2.1) and (2.2), respectively. Using lemma I.4.1 one can easily prove, by induction, that these objects are equivalent. Hence, there exists a bijection

$$\psi\colon \Omega^{(k)}(M) \to \Omega^{(k)}(\bar{M})$$

such that the equivariance condition

$$\bigwedge_{A \in \Omega^{(k)}(M)} \quad \bigwedge_{g \in G} \quad \bar{f}^{(k)}\big(\psi(A), g\big) = \psi\big(f^{(k)}(A, g)\big) \tag{2.5}$$

holds true.

Now, let

$$(X,\ G,\ F) \tag{2.6}$$

be a geometric object of rank k of the space (2.1). According to definition 1.2 of geometric object, object (2.6) is equivalent with some partial object (1.4) of the object (2.3). It follows from the equivariance condition (2.5) that the set

$$\Omega_0^{(k)}(\bar{M}) := \psi\left(\Omega_0^{(k)}(M)\right)$$

is an invariant subset of the fibre of object (2.4) and that the partial object (1.4) of object (2.3) is equivalent with partial object

$$\left(\Omega_0^{(k)}(\bar{M}),\ G,\ \bar{f}^{(k)}|_{\Omega_0^{(k)}(\bar{M})\times G}\right) \tag{2.7}$$

of the object (2.4). Since the equivalence relation is transitive, objects (2.6) and (2.7) are equivalent and, therefore, (2.6) is a geometric object of the space (2.2).

We have shown that every object of the category OG(f) is an object of the category of geometric objects OG($\bar{f}$) of Klein space (2.2). Similarly, we can prove the converse, i.e. that every object of the category OG($\bar{f}$) is an object of the category OG(f). Hence, classes of objects of these categories are equal. Therefore, considered categories are identical.□

The converse theorem is not true. To show it, let us consider again standard geometric objects (2.3) of Klein space (2.1). These are effective objects (cf. corollary 1.4), and, therefore, they are Klein spaces. Since the fibres of any two of these spaces are the sets of different powers, these spaces cannot be equivalent. We will prove also the following lemma.

<u>Lemma 2.1.</u> For any positive integer k the categories OG($f^{(k)}$) of geometric objects of Klein space (2.3) and OG(f) of geometric objects of Klein space (2.1) are identical.

Proof. It is enough to show that the classes of objects of category OG($f^{(k)}$) and OG(f) are equal. From definition 1.2 of geometric object it follows that each object of the category OG($f^{(k)}$) is an object of the category OG(f). Lemma 1.1 implies that any object of the category OG(f) is an object of the category OG($f^{(k)}$). Hence, classes of objects of two considered categories are identical.□

From the above considerations we infer that OG(f) is a category of geometric objects not only of Klein space (2.1), but also of infinitely many spaces (2.3), which are not equivalent. Because of the reasons explained in §5 of Section 1 we will define Klein geometry of a given Klein space as follows.

Definition 2.1. A pair

$$\big((M,\ G,\ f),\ OG(f)\big), \tag{2.8}$$

of Klein space (2.1) and a category of geometric objects of this space will be called *Klein geometry of Klein space (2.1)*.

Now, let $\tilde{G}$ be a subgroup of G, supporting Klein space (2.1). According to corollary I.3.1, the subobject

$$(M,\ \tilde{G},\ \tilde{f}), \qquad \tilde{f} := f\,\big|_{M\times\tilde{G}} \tag{2.9}$$

of the space (2.1) is also a Klein space. Let us consider the category $OG(\tilde{f})$ of geometric objects of Klein space (2.9) and its geometry

$$\big((M,\ \tilde{G},\ \tilde{f}),\ OG(\tilde{f})\big). \tag{2.10}$$

Although $OG(\tilde{f})$ is not a subcategory of OG(f), the following definition is usually accepted.

Definition 2.2. Klein geometry (2.10) of Klein space (2.9), being a subobject of the space (2.1) will be called a *subgeometry* of Klein geometry (2.8) of the space (2.1).

It appears that subobject of any geometric object of the space (2.1) determined by the subgroup $\tilde{G}$ of the group G is a geometric object of the space (2.9). This fact will be shown in the sequel.

§3. **Some properties of geometric objects**

Let us consider an arbitrary Klein space

$$(M,\ G,\ f) \tag{3.1}$$

and its geometric object

$$(X,\ G,\ F). \tag{3.2}$$

Theorem 3.1. Each partial object of geometric object (3.2) of Klein space (3.1) is a geometric object of this space and its rank is not greater than the rank of the object (3.2).

Proof. Let (3.2) be a geometric object of rank k of the space (3.1). In virtue of definition of geometric object there exist a partial object (1.4) of the object (1.2) and a bijection

(1.5) such that the condition (1.6) holds. Let us consider an arbitrary partial object

$$(X_1,\ G,\ F_1),\quad F_1 = F|_{X_1 \times G} \tag{3.3}$$

of the object (3.2) and a subset $\Omega_1^{(k)}(M)$ of the fibre of the object (1.4), defined by the formula

$$\Omega_1^{(k)}(M) := \psi(X_1). \tag{3.4}$$

Since X_1 is an invariant subset of the fibre of object (3.2), by (1.6) the set (3.4) is an invariant subset of the fibre of object (1.4). Hence, we may define a partial object

$$(\Omega_1^{(k)}(M),\ G,\ f_1^{(k)}),\qquad f_1^{(k)} = f^{(k)}|_{\Omega_1^{(k)}(M)\times G} \tag{3.5}$$

of the object (1.2). Let $\psi_1 := \psi|_{X_1}$. By (1.6) we have, then

$$\bigwedge_{x\in X_1}\ \bigwedge_{g\in G}\quad f_1^{(k)}\big(\psi_1(x), g\big) = \psi_1\big(F_1(x, g)\big).$$

Thus, objects (3.3) and (3.5) are equivalent and, therefore, (3.3) is a geometric object of the space (3.1). It is easily seen that, due to definition 1.3, rank of this object is no greater than k.□

From the above theorem and corollaries I.3.6 and 1.1 we get, as the immediate consequence, the following corollary.

<u>Corollary 3.1.</u> Object induced by geometric object (3.2) of Klein space (3.1) and an invariant bijection is a geometric object of this space.

Using lemmas I.4.1 and I.4.2 we can prove the following theorem.

<u>Theorem 3.2.</u> Each object of subsets of the fibre of geometric object (3.2) of Klein space (3.1) is a geometric object of this space.

Proof. Let $k\in\mathbf{N}$ be a rank of geometric object (3.2). As we know, it has to be equivalent with some partial object (1.4) of a standard geometric object (1.2) of rank k. By lemma I.4.1 the object

$$(2^X,\ G,\ F^*) \tag{3.6}$$

of all subsets of the fibre of object (3.2) is equivalent with the object

$$\left(2^{\Omega_o^{(k)}(M)},\ G,\ f_o^{(k)*}\right) \tag{3.7}$$

of all subsets of the fibre of object (1.4). It follows from lemma I.4.2 that (3.7) is a partial object of object of all subsets of the fibre of object (1.2), i.e. standard geometric object of rank k+1. Thus, the object (3.6) is equivalent with the partial object (3.7) of standard geometric object of rank k+1 and, therefore, it is a geometric object of Klein space (3.1). By theorem 3.1, any object of subsets of the fibre of object (3.2) is a geometric object, as a partial object of geometric object (3.6). □

The next theorem and its consequences will play an important role in further considerations. Beside Klein space (3.1) let us consider another space

$$(\bar{M},\ \tilde{G},\ \bar{f}) \tag{3.8}$$

and a homomorphism

$$\varphi\colon\ \tilde{G} \to G. \tag{3.9}$$

Let

$$(M,\ \tilde{G},\ \tilde{f}), \qquad \tilde{f}(p, \tilde{g}) := f\left(p, \varphi(\tilde{g})\right) \tag{3.10}$$

and

$$(X,\ \tilde{G},\ \tilde{F}), \qquad \tilde{F}(x, \tilde{g}) := F\left(x, \varphi(\tilde{g})\right) \tag{3.11}$$

be the objects determined by Klein space (3.1) and homomorphism (3.9) and by geometric object (3.2) and homomorphism (3.9), respectively.

<u>Theorem 3.3.</u> If the object (3.10) induced by Klein space (3.1) and homomorphism (3.9) is a geometric object of Klein space (3.8), then the object (3.11) induced by geometric object (3.2) of Klein space (3.1) and homomorphism (3.9) is also a geometric object of Klein space (3.8).

Proof. Let

$$(\Omega^{(k)}(M),\ \tilde{G},\ \tilde{f}^{(k)}), \qquad k \in \mathbf{N} \tag{3.12}$$

be a geometric object determined by the following conditions (cf. definition 1.1):

(a) for k=1 object (3.12) is the object of all subsets of the fibre of geometric object (3.10) of the space (3.8);

(b) for k=m+1 object (3.12) is the object of all subsets of the fibre of the object

$$(\Omega^{(m)}(M),\ \tilde{G},\ \tilde{f}^{(m)}).$$

Using theorem 3.2 we can easily prove, by induction, that for each $k\in N$ object (3.12) is a geometric object of the space (3.8). Also by induction we can prove that the following relation

$$\bigwedge_{A\in\Omega^{(k)}(M)} \quad \bigwedge_{\bar{g}\in\tilde{G}} \quad \tilde{f}^{(k)}(A,\bar{g}) = f^{(k)}(A,\varphi(\bar{g})). \tag{3.13}$$

Indeed, for every subset $A\in\Omega^{(1)}(M)$ and $\bar{g}\in\tilde{G}$ we have

$$\tilde{f}^{(1)}(A,\bar{g}) = \tilde{f}(A,\bar{g}) = f(A,\varphi(\bar{g})) = f^{(1)}(A,\varphi(\bar{g})),$$

what means that (3.13) holds true for k=1. Now, let us assume that (3.13) holds true for k=m. Then, for every set $A\in\Omega^{(m+1)}(M)$ and every $\bar{g}\in\tilde{G}$ we have

$$\tilde{f}^{(m+1)}(A,\bar{g}) = \tilde{f}^{(m)}(A,\bar{g}) = f^{(m)}(A,\varphi(\bar{g})) = f^{(m+1)}(A,\varphi(\bar{g}))$$

Thus, (3.13) holds true for k=m+1 as well.

We assumed that the object (3.2) is a geometric object of the space (3.1). Therefore, there exist a partial object (1.4) of the object (1.2) and a bijection (1.5), such that the condition (1.6) holds, implying that

$$\bigwedge_{x\in X} \quad \bigwedge_{\bar{g}\in\tilde{G}} \quad f_o^{(k)}(\psi(x),\varphi(\bar{g})) = \psi(F(x,\varphi(\bar{g}))). \tag{3.14}$$

The set $\Omega_o^{(k)}(M)$ is an invariant subset of the fibre of object (1.2). From this fact and from the relation (3.13) we infer that it is an invariant subset of the fibre of the object (3.12), as well. Hence, we can define a partial object

$$(\Omega_o^{(k)}(M),\ \tilde{G},\ \tilde{f}_o^{(k)}), \qquad \tilde{f}_o^{(k)} = \tilde{f}^{(k)}|_{\Omega_o^{(k)}(M)\times G} \tag{3.15}$$

of the object (3.12), and by (3.13) we have

$$\bigwedge_{A\in\Omega_o^{(k)}(M)} \quad \bigwedge_{\bar{g}\in\tilde{G}} \quad \tilde{f}_o^{(k)}(A,\bar{g}) = f_o^{(k)}(A,\varphi(\bar{g})),$$

and, therefore,

$$\bigwedge_{x\in X} \quad \bigwedge_{\bar{g}\in\tilde{G}} \quad \tilde{f}_o^{(k)}(\psi(x),\bar{g}) = f_o^{(k)}(\psi(x),\varphi(\bar{g})). \tag{3.16}$$

Thus, by (3.11) and (3.14) we get

$$\bigwedge_{x\in X}\ \bigwedge_{\tilde{g}\in\tilde{G}}\ \tilde{f}_{o}^{(k)}(\psi(x),\tilde{g})=\psi(\tilde{F}(x,\tilde{g})),$$

which means that the objects (3.11) and (3.15) are equivalent. Object (3.15) as a partial object of geometric object (3.12) is a geometric object of the space (3.8). Therefore, (3.11) is also a geometric object of this space. □

As a particular case of the above theorem (cf. corollary I.3.4 and corollary 1.3) we get the following corollary.

<u>Corollary 3.2.</u> If (3.2) is an object of category OG(f) of geometric objects of Klein space (3.1), then the object (3.11) determined by object (3.2) and monomorphism (3.9) is an object of category OG($\tilde{f}$) of geometric objects of Klein space (3.10) determined by Klein space (3.1) and monomorphism (3.9).

If $\tilde{G}$ is a subgroup of G and monomorphism (3.9) an inclusion map of $\tilde{G}$ into G (cf. corollary I.3.3), corollary 3.2 may be formulated as follows.

<u>Corollary 3.3.</u> If Klein space (3.10) is a subobject of Klein space (3.1), then subobject (3.11) of an arbitrary geometric object (3.2) of Klein space (3.1), determined by subgroup $\tilde{G}$ of the group G is a geometric object of Klein space (3.10).

§4. <u>Equivalence of Klein geometries</u>

The notion of equivalence of two Klein geometries

$$((M,\ G,\ f),\ OG(f)) \tag{4.1}$$

and

$$((\bar{M},\ \bar{G},\ \bar{f}),\ OG(\bar{f})) \tag{4.2}$$

and, respectively, two Klein spaces

$$(M,\ G,\ f) \tag{4.3}$$

and

$$(\bar{M},\ \bar{G},\ \bar{f}) \tag{4.4}$$

we will define using a covariant functor of category OG(f) onto category OG($\bar{f}$). In the sequel the notion of the functor plays an important role, hence we will remind its definition.

A function T which to each object A of a category $\mathcal{C}$ assigns an object T(A) of a category $\tilde{\mathcal{C}}$, and to each morphism ω: $A \to A_1$ assigns a morphism $T(\omega)$: $T(A) \to T(A_1)$ will be called a *covariant functor* iff the following conditions hold:

FUN 1. For each object A of category $\mathcal{C}$

$$T(id_A) = id_{T(A)}.$$

FUN 2. If ω: $A \to A_1$ and ω_1: $A_1 \to A_2$ are morphisms of category $\mathcal{C}$, then

$$T(\omega_1 \circ \omega) = T(\omega_1) \circ T(\omega).$$

If

$$\varphi: \tilde{G} \to G \tag{4.5}$$

is an isomorphism of a group $\tilde{G}$ onto G, then T_φ will denote a function which:

1° to each object

$$(X, \ G, \ F) \tag{4.6}$$

of the category OG(f) assigns an object induced by the object (4.6) and isomorphism (4.5), i.e.

$$T_\varphi\big((X, G, F)\big) := (X, \tilde{G}, \tilde{F}), \quad \text{where } \tilde{F}(x, \tilde{g}) := F\big(x, \varphi(\tilde{g})\big). \tag{4.7}$$

2° to each morphism (ψ, id_G) of the object (4.6) of category OG(f) into object

$$(X_1, \ G, \ F_1) \tag{4.8}$$

of this category assigns a pair $(\psi, id_{\tilde{G}})$, i.e.

$$T_\varphi\big((\psi, id_G)\big) := (\psi, id_{\tilde{G}}). \tag{4.9}$$

Let us note that, by corollary I.3.4, object

$$(M, \ \tilde{G}, \ \tilde{f}), \qquad \tilde{f}(p, \tilde{g}) := f\big(p, \varphi(\tilde{g})\big) \tag{4.10}$$

is a Klein space. We will prove the following lemma.

Lemma 4.1. If Klein space (4.4) is geometrically equivalent with Klein space (4.10) induced by the space (4.3) and isomorphism (4.5), then T_φ, defined by the formulas (4.7) and (4.9) is a bijective functor of the category OG(f) onto category OG($\tilde{f}$).

Proof. Let (4.6) be an arbitrary object of category OG(f).

By corollary 3.2, object (4.7) is of the category $OG(\tilde{f})$ of geometric objects of the space (4.10) induced by the space (4.3) and isomorphism (4.5). From the assumption and theorem 2.1, the categories $OG(\tilde{f})$ and $OG(\bar{f})$ are identical. Thus, T_* assigns objects of the category $OG(\bar{f})$ to the objects of the category $OG(f)$. Moreover, if (ψ, id_G) is a morphism (of category $OG(f)$) of the object (4.6) into the object (4.8), then

$$\bigwedge_{x \in X} \bigwedge_{g \in G} F_1\big(\psi(x), g\big) = \psi\big(F(x, g)\big).$$

Thus,

$$\bigwedge_{x \in X} \bigwedge_{\tilde{g} \in \tilde{G}} F_1\big(\psi(x), \varphi(\tilde{g})\big) = \psi\big(F(x, \varphi(\tilde{g}))\big),$$

and, therefore,

$$\bigwedge_{x \in X} \bigwedge_{\tilde{g} \in \tilde{G}} \tilde{F}_1\big(\psi(x), \tilde{g}\big) = \psi\big(\tilde{F}(x, \tilde{g})\big).$$

Thus, pair $(\psi, id_{\tilde{G}})$ is a morphism (of category $OG(\bar{f})$) of the object (4.7) into the object $T_*\big((X_1, G, F_1)\big)$. Hence, T_* assigns objects (morphisms) of category $OG(\bar{f})$ to objects (morphisms) of category $OG(f)$.

Since identity morphisms of the objects (4.6) and (4.7) are pairs (id_X, id_G) and $(id_X, id_{\tilde{G}})$, by (4.9) T_* satisfies the condition FUN 1.

Now, let (ψ, id_G) be a morphism of the object (4.6) into object (4.8), and let (ψ_1, id_G) be a morphism of the object (4.8) into object (X_2, G, F_2). Then

$$T_*\big((\psi_1, id_G) \circ (\psi, id_G)\big) = T_*\big((\psi_1 \circ \psi, id_G)\big) = (\psi_1 \circ \psi, id_{\tilde{G}})$$

and

$$T_*\big((\psi_1, id_G)\big) \circ T_*\big((\psi, id_G)\big) = (\psi_1, id_{\tilde{G}}) \circ (\psi, id_{\tilde{G}}) = (\psi_1 \circ \psi, id_{\tilde{G}}).$$

Thus, T_* satisfies condition FUN 2 as well.

We have proved that T_* is a covariant functor of category $OG(f)$ into category $OG(\bar{f})$.

Klein spaces (4.4) and (4.10) are geometrically equivalent, by assumption. Hence, there exists a bijection ψ: $\bar{M} \to M$ satisfying equivariance condition

$$\bigwedge_{\bar{p} \in \bar{M}} \bigwedge_{\tilde{g} \in \tilde{G}} f\big(\psi(\bar{p}), \varphi(\tilde{g})\big) = \psi\big(\bar{f}(\bar{p}, \tilde{g})\big).$$

This condition cam be rewritten in the form

$$\bigwedge_{p\in M} \bigwedge_{g\in G} \bar{f}\left(\psi^{-1}(p), \varphi^{-1}(g)\right) = \psi^{-1}\left(f(p, g)\right)$$

meaninig that the space (4.3) is geometrically equivalent with the space induced by space (4.4) and isomorphism φ^{-1}. This and the above considerations imply that we can define a covariant functor $T_{\varphi^{-1}}$ of category $OG(\bar{f})$ into category $OG(f)$. It is easily seen that the functors T_φ and $T_{\varphi^{-1}}$ satisfiy the conditions

$$T_\varphi \circ T_{\varphi^{-1}} = id_{OG(\bar{f})}, \quad T_{\varphi^{-1}} \circ T_\varphi = id_{OG(f)}.$$

Thus, T_φ is a bijective functor.□

<u>Definition 4.1.</u> A bijective functor T of category of geometric objects $OG(f)$ of Klein space (4.3) onto category of geometric objects $OG(\bar{f})$ of Klein space (4.4) will be called *simple* iff it satisfies the following conditions:

(a) there exists a group isomorphism φ: $\tilde{G} \to G$ such that for each object (4.6) and each morphism (ψ, id_G) of category $OG(f)$ the following equalities:

$$T\left((X, G, F)\right) = T_\varphi\left((X, G, F)\right), \quad T\left((\psi, id_G)\right) = T_\varphi\left((\psi, id_G)\right)$$

hold;

(b) Klein space (4.4) is geometrically equivalent with the space $T\left((M, G, f)\right)$.

The following two lemmas state some properties of simple functor. The first one is the immediate consequence of corollary I.3.2.

<u>Lemma 4.2.</u> If T is a simple functor of category $OG(f)$ onto category $OG(\bar{f})$, then each object (4.6) of category $OG(f)$ is abstractly equivalent with the object $T\left((X, G, F)\right)$.

<u>Lemma 4.3.</u> Klein spaces (4.4) and (4.3) are equivalent (abstractively) iff there exists a simple functor of category $OG(f)$ onto category $OG(\bar{f})$.

Proof. First, let us assume that the spaces (4.4) and (4.3) are equivalent. There exists, then, a pair

$$(\psi, \varphi), \quad \psi: \bar{M} \to M, \quad \varphi: \tilde{G} \to G,$$

where φ is a group isomorphism, such that the equivariance condition

$$\bigwedge_{\bar{p}\in\bar{M}} \bigwedge_{\bar{g}\in\tilde{G}} f\big(\psi(\bar{p}), \varphi(\bar{g})\big) = \psi\big(\bar{f}(\bar{p}, \bar{g})\big) \qquad (4.11)$$

holds. By (4.10) we have

$$\bigwedge_{\bar{p}\in\bar{M}} \bigwedge_{\bar{g}\in\tilde{G}} \tilde{f}\big(\psi(\bar{p}), \bar{g}\big) = \psi\big(\bar{f}(\bar{p}, \bar{g})\big). \qquad (4.12)$$

With the isomorphism φ we can define a functor T_φ. By (4.12), Klein space (4.4) is geometrically equivalent with the space

$$(M, \tilde{G}, \tilde{f}) = T_\varphi\big((M, G, f)\big).$$

Thus, T_φ is a simple functor.

Now, if there exists a simple functor of category OG(f) onto category OG($\bar{f}$), then, by axiom (b) of definition 4.1, there exists a bijection ψ: $\bar{M} \to M$, such that condition (4.12) holds true. Thus, (4.12) and (4.10) imply that equivariance condition (4.11) holds true as well, what proves the equivalence of Klein spaces (4.4) and (4.3).□

Stated above properties of simple functor motivate the following definition of equivalence of Klein geometries.

Definition 4.2. Klein geometry (4.1) of Klein space (4.3) will be called *equivalent* with Klein geometry (4.2) of Klein space (4.4) iff there exists a simple functor of category OG(f) onto category OG($\bar{f}$).

It is easily seen that the relation of equivalence of Klein geometries is an equivalence relation, i.e. it is reflexive, symmetric and transitive.

The following theorem is an immediate consequence of the above definition and lemma 4.3.

Theorem 4.1. Klein geometries (4.1) and (4.2) of Klein spaces (4.3) and (4.4), respectively, are equivalent iff the spaces are equivalent.

It would seem to be more natural to define the equivalence of Klein geometries as follows: Klein geometries (4.1) and (4.2) will be called equivalent if there exists a bijective covariant functor T of category OG(f) onto category OG($\bar{f}$) satisfying the condition

$$T\big((M,\ G,\ f)\big) = (\bar{M},\ \bar{G},\ \bar{f}).$$

Such a definition should also imply theorem 4.1, though. For the present, proof of such theorem is not known in this case.

To study properties of Klein geometry (4.1) of a given Klein space (4.3) we usually consider the simplest (canonical) space in the class of spaces equivalent with the given one. We will do it in Section IV, to present elementary Klein spaces.

Section III

PROPERTIES OF GEOMETRIC OBJECTS

We have shown, in §3 of the previous section, methods of construction of new geometric objects of a Klein space: partial objects and objects of subsets of the fibre of object. Now, we will present further methods of construction of geometric objects and some of their consequences. We will discuss some properties of a scalar, G-product of objects, object of transformations, factor object and disjoint union of objects.

§1. Scalars

In category of geometric objects of Klein space

$$(M, G, f) \tag{1.1}$$

the object called *scalar* plays an important role.

Definition 1.1. Abstract object

$$(S, G, I), \quad \text{where } I(s, g) := s \tag{1.2}$$

for all $s \in S$ and $g \in G$ will be called a *scalar*.

The following two corollaries are the immediate consequences of the definition.

Corollary 1.1. Two scalars: (1.2) and

$$(\bar{S}, G, \bar{I}), \quad \bar{I}(\bar{s}, g) := \bar{s} \tag{1.3}$$

are equivalent iff their fibres S and $\bar{S}$ are equinumerous.

Proof. Indeed, if objects (1.2) and (1.3) are equivalent, then there exists a pair (ψ, id_G), where $\psi: S \to \bar{S}$ is a bijection, such that the equivariance equation

$$\bar{I}(\psi(s), g) = \psi(I(s, g)) \quad \text{for all } s \in S \text{ and } g \in G \tag{1.4}$$

holds. Thus, S and $\bar{S}$ are equinumerous.

Conversely, if the fibres S and $\bar{S}$ are equinumerous, then there exists a bijection $\psi: S \to \bar{S}$. Every such bijection satisfies condition (1.4) since both sides of the equation are equal to $\psi(s)$. Hence, objects (1.2) and (1.3) are equivalent.□

Corollary 1.2. Each object geometrically equivalent with a scalar is a scalar.

Proof. Let (X, G, F) be an object geometrically equivalent with a scalar (1.2). There exists then a bijection $\psi: S \to X$ such that

$$\bigwedge_{s\in S} \bigwedge_{g\in G} F(\psi(s), g) = \psi(I(s, g)) = \psi(s).$$

Taking $x=\psi(s)$ we get

$$F(x, g) = x \quad \text{for all } x\in X \text{ and } g\in G.$$

Thus, object (X, G, F) is a scalar.□

Let us note that not every scalar (1.2) is a geometric object of the space (1.1). Indeed, if - for example - the fibre of the object (1.2) is equinumerous with the fibre $\Omega^{(m,n)}(M)$ of the object (II.1.9), then it is easily seen that the scalar (1.2) in not a geometric object of the space (1.1).

Definition 1.2. Scalar (1.2) being a geometric object of Klein space (1.1) will be called a *geometric scalar* of this space.

From theorem II.3.1 and corollary 1.1 we get another corollary.

Corollary 1.3. A scalar (supported by a group G) whose fibre is equinumerous with an arbitrary, nonempty subset of the fibre of a geometric scalar of Klein space (1.1) is a geometric scalar of this space.

Proof. Let (1.2) be a geometric scalar. Every nonempty subset S_0 of its fibre is an invariant subset. Hence, we can define a partial object

$$(S_0, G, I|_{S_0\times G}),$$

which is obviously a scalar. By the theorem II.3.1 it is a geometric scalar. By corollary 1.1, it is equivalent with every scalar whose fibre is equinumerous with S_0. Thus, it is a geometric object.□

The theorem below states that whether a scalar is geometric depends solely of the power of its fibre. It also gives a method to determine geometric scalars.

Theorem 1.1. Scalar (1.2) is a geometric object of Klein space (1.1) iff its fibre is equinumerous with some family of invariant subsets of some standard geometric object of this space.

Proof. If scalar (1.2) is a geometric object of Klein space (1.1), then, by the definition of geometric object, it is equivalent with a partial object

$$(\Omega_o^{(k)}(M),\ G,\ f_o^{(k)}), \qquad f_o^{(k)} = f^{(k)}|_{\Omega_o^{(k)}(M)\times G} \tag{1.5}$$

of a standard geometric object

$$\left(\Omega^{(k)}(M),\ G,\ f^{(k)}\right). \tag{1.6}$$

Therefore, there exists a bijection

$$\psi\colon\ S \to \Omega_o^{(k)}(M), \tag{1.7}$$

satisfying the equivariance condition. By corollary 1.2, object (1.5) is a scalar. Hence

$$f^{(k)}(A,g) = A \quad \text{for all } A\in\Omega_o^{(k)}(M),\ g\in G. \tag{1.8}$$

If $k>1$, then the elements A of the fibre of the object (1.5) are the subsets of the fibre of standard geometric object of rank $k-1$. By (1.8) they are invariant subsets. Thus, by (1.7), the fibre S is equinumerous with the family $\Omega_o^{(k)}(M)$ of invariant subsets of the fibre of standard geometric object of rank $k-1$. If $k=1$, then the elements A of the fibre of object (1.5) are invariant subsets of the fibre of the space (1.1), which is a standard geometric object of rank 1 (cf. corollary II.1.3). It follows that there exists a family of invariant subsets

$$\mathcal{A} = \left\{\{p\}\colon\ p\in A\right\}$$

of the fibre of standard geometric object of rank $k=1$, which is equinumerous with the fibre of the scalar (1.2).

Conversely, if the fibre S of scalar (1.2) is equinumerous with a family $\mathfrak{M}$ of invariant subsets of the fibre of the object (1.6), then it is easy to note that the partial object

$$\left(\mathfrak{M},\ G,\ f^{(k+1)}|_{\mathfrak{M}\times G}\right) \tag{1.9}$$

of a standard geometric object of rank k+1 is a geometric scalar of the space (1.1). Thus, by corollary 1.1, objects (1.2) and (1.9) are equivalent and, therefore, scalar (1.2) is a geometric object of the space (1.1).□

Let us define the sets

$$\Omega^{(l)}(\mathbf{N}), \qquad l=0,1,2,\dots \tag{1.10}$$

where **N** is a set of all positive integers, by the formulas

$$\Omega^{(0)}(\mathbf{N}) := \mathbf{N}, \qquad \Omega^{(1)}(\mathbf{N}) := 2^{\mathbf{N}}$$
$$\Omega^{(m+1)}(\mathbf{N}) := 2^{\Omega^{(m)}(\mathbf{N})}. \tag{1.11}$$

We will prove the following lemma.

<u>Lemma 1.1.</u> If the fibre of Klein space (1.1) is an infinite set, then the scalar (1.2) with the fibre equinumerous with one of the sets (1.10) is a geometric scalar of this space.

Proof. In virtue of corollary 1.1 it is sufficient to show that the scalars

$$\left(\Omega^{(l)}(\mathbf{N}),\ G,\ I^{(l)}\right), \quad I^{(l)}(x,g) = x, \quad l=0,1,2,\dots \tag{1.12}$$

are geometric objects of the space (1.1). We will do it by induction. Let M_n denote the family of all n-element subsets of the fibre of the space (1.1). It is esily seen that for every positive integer n, M_n is an invariant subset of the fibre of standard geometric object of rank 1 of the space (1.1). The family $\mathfrak{M}$ of all such invariant subsets is countable. Thus, by theorem 1.1, scalar (1.12) with l=0 is a geometric object of the space (1.1).

Now, let us assume that the object (1.12) is a geometric scalar for l=m. In virtue of theorem II.3.2, the object of subsets

$$\left(2^{\Omega^{(m)}(\mathbf{N})},\ G,\ I^{(m)*}\right)$$

is a geometric object of the space (1.1). It is easy to note that it is the scalar (1.12) for l=m+1. Hence, each one of the objects (1.12) is a geometric scalar.□

<u>Lemma 1.2.</u> If the fibre of Klein space (1.1) is a finite set or is equinumerous with one of the stes (1.10), then the scalar

$$(M,\ G,\ I_1), \qquad I_1(p, g) = p \tag{1.13}$$

is a geometric object of this space.

Proof. If the fibre M of the space (1.1) is a set equinumerous with one of the sets (1.10), this lemma is an immediate consequence of the previous one.

Now, let us consider the case when the fibre M of the space (1.1) is a finite set containing m elements. Similarly to the proof of lemma 1.1, M_n ($1 \leq n \leq m$) will denote the family of all n-element subsets of M. Each one of these families is an invariant subset of the fibre of standard geometric object of rank 1. This fact and theorem 1.1 imply that the object (1.13) is a geometric scalar of the space (1.1).□

Geometric scalars play a significant role. As we will see in §5, whether a non-transitive object of category OA(G) is an object of its subcategory OG(f) depends largely of these scalars. Geometricity of the scalar (1.13) implies (cf. corollary 4.3 and the proof of lemma 3.1) that transitive objects of category OA(G) are simultaneously the objects of subcategory OG(f). From lemma 1.1 and corollary 1.3 it follows that the set of geometric scalars of a given space is relatively large. However, we do not know if the scalar (1.13) is a geometric object for an arbitrary Klein space (1.1).

Z. Moszner noted (unpublished result) that lemmas 1.1 and 1.2 can be generalized as follows:

<u>Lemma 1.3.</u> If there exists a cardinal α such that:

(a) there exist at least α distinct and less or equal than α cardinals,

(b) $$\alpha \leq \overline{\overline{M}},$$

(c) for some set Z of power α and some positive integer m the fibre of scalar

$$(\Omega,\ G,\ I) \tag{1.14}$$

satisfies the condition

$$\alpha \leq \overline{\overline{\Omega}} \leq \overline{\overline{\Omega^{(m)}(Z)}},$$

then the object (1.14) is a geometric scalar of the space (1.1).

We will present the sketch of the proof. As in the proof of lemma 1.1 we can show that the scalars with the fibre $\Omega^{(m)}(Z)$ are geometric. For m=0 we replace the family M_n with family M_β of all subsets of power β, for β belonging to the set of cardinals satisfying condition (a). Scalar (1.14) may be viewed as a partial object of the scalar with fibre $\Omega^{(m)}(Z)$. Thus, in virtue of theorem II.3.1, (1.14) is a geometric scalar of the space (1.1).□

Taking $\alpha = \overline{\overline{N}}$ and $\Omega = \Omega^{(m)}(N)$ we obtain lemma 1.1. If the set M is finite, by taking $\alpha = \overline{\overline{M}}$ we obtain the first part of lemma 1.2. If the set M is equinumerous with one of the sets (1.10), then by taking $\alpha = \overline{\overline{N}}$ and $\Omega = \Omega^{(m)}(N)$ we get the second part of this lemma.

Lemma 1.3 is more general than lemmas 1.1 and 1.2, since without assuming the continuum hypothesis we can obtain some results for cardinals between powers of the sets $\Omega^{(m)}(Z)$.

Z. Moszner noted that the problem whether in an arbitrary Klein space (1.1) scalar (1.13) is a geometric object, together with lemma 1.3, suggest a problem, interesting from the point of wiev of set theory:

if for every cardinal β there exists such a cardinal α satisfying condition (a) from lemma 1.3) and such positive integer n, that

$$\alpha \leqslant \beta \leqslant \alpha(n), \quad \text{where } \alpha(1) = \alpha, \ \alpha(n+1) = 2^{\alpha(n)} \quad ?$$

A. Tyszka proved (unpublished result), that the property formulated in the question above is undecidable (independent) on the basis of ZFC axioms. Therefore the problem of geometricity of the scalar (1.13) is either undecidable on the basis of this axiomatic, or positively decidable.

§2. G-products of objects

Let us consider m abstract objects

$$(X_i, \ G, \ F_i), \quad i=1,2,\dots,m \tag{2.1}$$

supported by the same group G, and the transformation

$$F:\ (X_1\times X_2\times\ldots\times X_m)\times G \to X_1\times X_2\times\ldots\times X_m,$$

defined by the formula

$$F\big((x_1, x_2, \ldots, x_m), g\big) := \big(F_1(x_1, g), F_2(x_2, g), \ldots, F_m(x_m, g)\big). \qquad (2.2)$$

F turns out to be an operation of the group G on cartesian product $X_1\times X_2\times\ldots\times X_m$ of the fibres of objects (2.1). Hence, we can define a new abstract object (cf. [14], p. 17, also [5])

$$(X_1\times X_2\times\ldots\times X_m,\ G,\ F). \qquad (2.3)$$

Definition 2.1. Abstract object (2.3) with transformation formula F defined by (2.2) will be called a *G-product of objects (2.1)* or, simply, a *product object*.

Due to its applications, particularly important is a G-product of m examples of a point object. We will start from a lemma, and next we will prove a theorem concerning geometricity of a G-product of geometric objects of a given Klein space.

Now, let us consider a standard geometric object (1.6) of Klein space (1.1) and its two arbitrary partial objects

$$\begin{aligned}&\big(\Omega_i^{(k)}(M),\ G,\ f_i^{(k)}\big),\\ &f_i^{(k)} = f^{(k)}\big|_{\Omega_i^{(k)}\times G}\end{aligned} \qquad i=1,2 \qquad (2.4)$$

and the cartesian product

$$\Omega_1^{(k)}(M)\times\Omega_2^{(k)}(M) \qquad (2.5)$$

of the fibres of these objects. We will prove:

Lemma 2.1. Cartesian product (2.5) of the fibres of two arbitrary partial objects (2.4) of standard geometric object of rank k of Klein space (1.1) is an invariant subset of the fibre of standard geometric object of rank k+2

$$\big(\Omega^{(k+2)}(M),\ G,\ f^{(k+2)}\big) \qquad (2.6)$$

of this space. For any $A_i\in\Omega_i^{(k)}(M)$, i=1,2 and $g\in G$ the condition

$$f^{(k+2)}\big((A_1, A_2), g\big) = \big(f^{(k)}(A_1, g), f^{(k)}(A_2, g)\big) \qquad (2.7)$$

is fulfilled.

Proof. First, let us note that for arbitrary $A_i\in\Omega_i^{(k)}(M)$, i=1,2, ordered pair

$$(A_1, A_2) = \big\{\{A_1\}, \{A_1, A_2\}\big\}$$

belongs to the fibre of object (2.6). By the definition of transformation formulas of standard geometric objects we have

$$\begin{aligned} f^{(k+2)}\big((A_1, A_2), g\big) &= f^{(k+1)}\big(\{(A_1), (A_1, A_2)\}, g\big) = \\ &= \big\{f^{(k+1)}\big((A_1), g\big), f^{(k+1)}\big((A_1, A_2), g\big)\big\} = \\ &= \big\{f^{(k)}\big((A_1), g\big), f^{(k)}\big((A_1, A_2), g\big)\big\} = \\ &= \big\{(f^{(k)}(A_1, g)), (f^{(k)}(A_1, g), f^{(k)}(A_2, g))\big\} = \\ &= \big(f^{(k)}(A_1, g), f^{(k)}(A_2, g)\big), \end{aligned}$$

what proves the equation (2.7). Since $\Omega_i^{(k)}(M)$, $i=1,2$, are invariant, it follows from (2.7) that the cartesian product (2.5) is invariant subset of the fibre of the object (2.6), what ends the proof.□

<u>Theorem 2.1.</u> G-product (2.3) of geometric objects (2.1) of Klein space (1.1) is a geometric object of this space.

Proof. We will prove the thesis by induction. First, let us consider two geometric objects

$$(X_i, \ G, \ F_i), \quad i=1,2 \tag{2.8}$$

of Klein space (1.1). Let k_1 and k_2 be the ranks of these objects, respectively, and let $k := \max(k_1, k_2)$. By definition II.1.2 of geometric object and lemma II.1.1 there exist partial objects (2.4) of standard geometric object of rank k and bijections

$$\psi_i\colon X_i \to \Omega_i^{(k)}(M), \qquad i=1,2 \tag{2.9}$$

such that for every $x_i \in X_i$ and $g \in G$ the conditions

$$f_i^{(k)}\big(\psi_i(x_i), g\big) = \psi_i\big(F_i(x_i, g)\big), \quad i=1,2 \tag{2.10}$$

hold. In virtue of lemma 2.1 the set

$$\Omega_o^{(k+2)}(M) := \Omega_1^{(k)}(M) \times \Omega_2^{(k)}(M) \tag{2.11}$$

is an invariant subset of the fibre of the object (2.6) and the equation

$$f_o^{(k+2)}\big((A_1, A_2), g\big) = \big(f_1^{(k)}(A_1, g), f_2^{(k)}(A_2, g)\big), \tag{2.12}$$

where

$$f_o^{(k+2)} := f^{(k+2)}\big|_{\Omega_o^{(k+2)}(M)\times G} \, ,$$

holds true for every $A_i \in \Omega_i^{(k)}(M)$ and $g \in G$.
We will prove that the partial object

$$\left(\Omega_o^{(k+2)}(M),\ G,\ f_o^{(k+2)}\right) \qquad (2.13)$$

of the object (2.6) is equivalent with G-product

$$(X_1\times X_2,\ G,\ F),\quad F\big((x_1,x_2),g\big) = \big(F_1(x_1,g),F_2(x_2,g)\big) \qquad (2.14)$$

of objects (2.8). For, let us consider the transformation

$$\psi:\ X_1\times X_2 \rightarrow \Omega_o^{(k+2)}(M),$$

defined by the formula

$$\psi\big((x_1,x_2)\big) := \big(\psi_1(x_1),\psi_2(x_2)\big), \qquad (2.15)$$

where ψ_1 and ψ_2 are transformations (2.9). Transformation (2.15) is obviously a bijection. Using relations (2.15), (2.12), (2.10) and (2.14) we get

$$\begin{aligned} f_o^{(k+2)}\big(\psi\big((x_1,x_2)\big),g\big) &= f_o^{(k+2)}\big((\psi_1(x_1),\psi_2(x_2)),g\big) = \\ &= \big(f_1^{(k)}(\psi_1(x_1),g), f_2^{(k)}(\psi_2(x_2),g)\big) = \\ &= \big(\psi_1(F_1(x_1,g)),\psi_2(F_2(x_2,g))\big) = \\ &= \psi\big((F_1(x_1,g),F_2(x_2,g))\big) = \psi\big(F\big((x_1,x_2),g\big)\big), \end{aligned}$$

what proves the equivalence of objects (2.13) and (2.14). It follows that (2.14) is a geometric object of Klein space (1.1). So, we have proved the thesis of the theorem for m=2.

Now, let us assume that G-product of objects (2.1) $(i=1,2,\dots,l)$:

$$(X_1\times X_2\times\dots\times X_l,\ G,\ \bar{F}),$$

where

$$\bar{F}\big((x_1,x_2,\dots,x_l),g\big) = \big(F_1(x_1,g),F_2(x_2,g),\dots,F_l(x_l,g)\big)$$

and object

$$(X_{l+1},\ G,\ F_{l+1})$$

are geometric objects of the space (1.1). Due to the first part of the proof, G-product of these objects

$$\big((X_1\times X_2\times\dots\times X_l)\times X_{l+1},\ G,\ \bar{\bar{F}}\big) \qquad (2.16)$$

where

$$\bar{\bar{F}}\big(((x_1,x_2,\dots,x_l),x_{l+1}),g\big) = \big(\bar{F}((x_1,x_2,\dots,x_l),g),F_{l+1}(x_{l+1},g)\big).$$

is a geometric object. We will show that it is equivalent with the object

$$(X_1 \times X_2 \times \ldots \times X_l \times X_{l+1},\ G,\ \bar{F}). \tag{2.17}$$

where

$$\bar{F}\big((x_1, \ldots, x_{l+1}), g\big) = \big(F_1(x_1, g), \ldots, F_{l+1}(x_{l+1}, g)\big).$$

Let

$$\psi_o:\ (X_1 \times \ldots \times X_l) \times X_{l+1} \to X_1 \times \ldots \times X_l \times X_{l+1}$$

be a bijection defined by the formula

$$\psi_o\big((x_1, \ldots, x_l), x_{l+1}\big) := (x_1, \ldots, x_l, x_{l+1}).$$

Easy calculation shows that the pair (ψ_o, id_G) is an isomorphism (in category OG(f)) of the object (2.16) onto the object (2.17). Thus, these objects are equivalent. Object (2.17) is a geometric object of the space (1.1) as an object equivalent with geometric object (2.16) of this space. In virtue of induction principle, for any positive integer m G-product (2.3) of m geometric objects (2.1) of the space (1.1) is a geometric object of this space.□

We will prove one more important lemma. First, we will introduce the concepts of non-effectivity subgroup and reper of order m (cf. [14], pp. 24 and 49, also [8]).

Definition 2.2. Subgroup

$$\{g \in G:\ \bigwedge_{y \in Y} F(y, g) = y\}$$

of the group G will be called a *non-effectivity group* of a nonempty subset Y of abstract object (X, G, F).

Definition 2.3. Every finite sequence of m distinct points $p_1, p_2, \ldots, p_m$ belonging to the fibre of Klein space (1.1) such that the non-effectivity group of the set $\{p_1, p_2, \ldots, p_m\}$ is trivial will be called a *reper of order m* or simply *m-reper* in this space.

It appears that in some Klein spaces m-repers may not exist (cf. [14], p. 49).

Lemma 2.2. If there exist m-repers in Klein space (1.1), then the abstract object (cf. example I.2.1)

$$(G,\ G,\ L), \quad L(x, g) = g \cdot x \tag{2.18}$$

is a geometric object of this space.

Proof. Let us define G-product of m-spaces (1.1)

$$(M^m,\ G,\ f^m)$$
$$f^m((p_1, \dots, p_m), g) = (f(p_1, g), \dots, f(p_m, g)). \tag{2.19}$$

By corollary II.1.3 and theorem 2.1 it is a geometric object of this space. It can be proved (cf. [14], p. 57, also [8]) that the set M^m_o of all m-repers is an invariant subset of the fibre of the object (2.19). Thus, we can define partial object

$$(M^m_o,\ G,\ f^m|_{M^m_o \times G}). \tag{2.20}$$

Let $\mathfrak{M}$ be an arbitrary transitive fibre of the object (2.20) (if the object is transitive itself, we define $\mathfrak{M} = M^m_o$). In virtue of theorem II.3.1 partial object

$$(\mathfrak{M},\ G,\ f^m|_{\mathfrak{M}\times G}) \tag{2.21}$$

of the object (2.19) is a geometric object of the space (1.1). It can be shown (cf. [14], p. 58), that objects (2.18) and (2.21) are equivalent. Therefore, (2.18) is a geometric object.□

§3. Objects of transformations

Let

$$(X_1,\ G,\ F_1) \tag{3.1}$$

and

$$(X_2,\ G,\ F_2) \tag{3.2}$$

be two abstract objects supported by the same group G, and let $\mathcal{F}(X_1, X_2)$ be a set of all transformations γ: $X_1 \to X_2$ defined on the fibre of object (3.1) with values in the fibre of object (3.2). Let us consider also representations $\hat{F}_1$ and $\hat{F}_2$ of objects (3.1) and (3.2), respectively, (cf. definition I.2.1), and the transformation

$$F:\ \mathcal{F}(X_1, X_2)\times G \to \mathcal{F}(X_1, X_2),$$

defined by the formula

$$F(\gamma, g) := F_{2g}\circ\gamma\circ F_{1g^{-1}}, \tag{3.3}$$
where $\quad F_{1g^{-1}} = \hat{F}_1(g^{-1}), \quad F_{2g} = \hat{F}_2(g).$

It is easily seen that transformation F is an operation of the group G on the set of transformation $\mathcal{F}(X_1, X_2)$. Hence, we can define new abstract object

$$(\mathcal{F}(X_1, X_2),\ G,\ F). \tag{3.4}$$

<u>Definition 3.1.</u> Abstract object (3.4) with transformation formula F defined by (3.3) and its every partial object will be called *objects of transformations of the fibre of object (3.1) into the fibre of object (3.2)* or simply *transformation objects.*

First, we will prove the theorem concerning geometricity of transformation object.

<u>Theorem 3.1.</u> If abstract objects (3.1) and (3.2) are geometric objects of Klein space (1.1), then each object of transformations of the fibre of object (3.1) into the fibre of object (3.2) is a geometric object of this space.

Proof. In virtue of theorem II.3.1 concerning geometricity of partial objects, it is enough to prove the thesis for the object (3.4) of all transformations. Let us consider G-product of objects (3.1) and (3.2)

$$(X_1 \times X_2,\ G,\ \bar{F}),\qquad \bar{F}\big((x_1, x_2), g\big) = \big(F_1(x_1, g), F_2(x_2, g)\big),$$

and then the object of all subsets of the fibre of this G-product:

$$\big(2^{X_1 \times X_2},\ G,\ \bar{F}^*\big),\qquad \bar{F}^*(A, g) = \bar{F}(A, g). \tag{3.5}$$

It follows from assumption and theorems 2.1 and II.3.2, that (3.5) is a geometric object of Klein space (1.1). Since each transformation $\gamma: X_1 \to X_2$ can be represented in the form

$$\gamma = \big\{(x_1, \gamma(x_1)):\ x_1 \in X_1\big\},$$

$\mathcal{F}(X_1, X_2)$ is a subset of the fibre of object (3.5). For any $\gamma \in \mathcal{F}(X_1, X_2)$ and $g \in G$ we have

$$\begin{aligned}\bar{F}^*(\gamma, g) &= \bar{F}\big(\big\{(x_1, \gamma(x_1)):\ x_1 \in X_1\big\}, g\big) = \\ &= \big\{\bar{F}\big((x_1, \gamma(x_1)), g\big):\ x_1 \in X_1\big\} = \\ &= \big\{\big(F_1(x_1, g), F_2(\gamma(x_1), g)\big):\ x_1 \in X_1\big\}.\end{aligned}$$

Denoting

$$y_1 := F_1(x_1, g)\qquad (\text{then } x_1 = F_1(y_1, g^{-1}))$$

we get

$$\bar{F}^*(\gamma, g) = \{(y_1, F_2(\gamma(F_1(y_1, g^{-1})), g)) : y_1 \in X_1\}$$

and, therefore,

$$\bar{F}^*(\gamma, g) = \{(x_1, F_{2g} \circ \gamma \circ F_{1g^{-1}}(x_1)) : x_1 \in X_1\} \qquad (3.6)$$

for each $\gamma \in \mathcal{F}(X_1, X_2)$ and $g \in G$. Hence $\mathcal{F}(X_1, X_2)$ is an invariant subset of the fibre of geometric object (3.5). By theorem II.3.1, partial object

$$(\mathcal{F}(X_1, X_2),\ G,\ \bar{F}^*_\circ), \qquad \bar{F}^*_\circ = \bar{F}^*|_{\mathcal{F}(X_1, X_2) \times G} \qquad (3.7)$$

of the object (3.5) is a geometric object of the space (1.1). Moreover, by (3.6) we have

$$\bar{F}^*_\circ(\gamma, g) = F_{2g} \circ \gamma \circ F_{1g^{-1}},$$

what means, by (3.3), that transformation formulas $\bar{F}^*_\circ$ and F are identical. Therefore, object of transformations (3.4) and geometric object (3.7) are identical.□

Till today, objects of transformations were not considered in papers on theory of Klein spaces. Further parts of this paper will convince us about their usefulness.

For an arbitrary abstract object

$$(X,\ G,\ F) \qquad (3.8)$$

we can define the object of all transformations

$$(\mathcal{F}(X, X),\ G,\ \Phi_\circ), \qquad \Phi_\circ(\gamma, g) = F_g \circ \gamma \circ F_{g^{-1}} \qquad (3.9)$$

of the fibre of object (3.8) into itself. If $\gamma: X \to X$ is a bijection, i.e. it belongs to the group $\mathcal{G}(X)$ of all transformations of the set X, then, since F_g is also a bijection, we have

$$\bigwedge_{\gamma \in \mathcal{G}(X)} \bigwedge_{g \in G} \quad F_g \circ \gamma \circ F_{g^{-1}} \in \mathcal{G}(X).$$

Thus, $\mathcal{G}(X)$ is an invariant subset of the fibre of object (3.9). Moreover,

$$\bigwedge_{x, g \in G} \quad \Phi_\circ(F_x, g) = F_g \circ F_x \circ F_{g^{-1}} = F_{g \cdot x \cdot g^{-1}} \in \hat{F}(G),$$

and, therefore, image $\hat{F}(G)$ of the group G by representation of object (3.8) is also an invariant subset of the fibre of object (3.9). Partial objects

$$(\mathcal{G}(X),\ G,\ \Phi_\circ|_{\mathcal{G}(X) \times G}) \qquad (3.10)$$

and

$$\left(\hat{f}(G),\ G,\ \Phi_a|_{\hat{f}(G)\times G}\right) \tag{3.11}$$

of object (3.9) are the examples of transformation objects of the fibre of object (3.8) into itself. As the immediate consequence of theorem 3.1 we have the following corollary.

Corollary 3.1. For each geometric object (3.8) of Klein space (1.1) transformation objects (3.9)-(3.11) are geometric objects of this space. In particular, object

$$\left(\hat{f}(G),\ G,\ \Phi\right), \qquad \Phi(f_x, g) = f_g \circ f_x \circ f_{g^{-1}} \tag{3.12}$$

is a geometric object of the space (1.1).

It is easy to note that the triplet

$$(G,\ G,\ J), \qquad J(x,g) := g\cdot x\cdot g^{-1} \tag{3.13}$$

is an abstract object. Using the above corollary we will prove

Corollary 3.2. For an arbitrary Klein space (1.1) abstract object (3.13) is a geometric object of this space.

Proof. It is easy to check that the pair $(\hat{f}, id_G)$ is an equivariant transformation of the object (3.13) into object (3.12). Indeed, for all $x, g\in G$ we have

$$\Phi\left(\hat{f}(x), g\right) = f_g \circ f_x \circ f_{g^{-1}} = f_{g\cdot x\cdot g^{-1}} = \hat{f}(g\cdot x\cdot g^{-1}) = \hat{f}\left(J(x,g)\right).$$

Moreover, the representation $\hat{f}\colon G \to \hat{f}(G)$ of Klein space is a bijection (cf. corollary I.1.1). Hence, objects (3.13) and (3.12) are equivalent. Thus, in virtue of corollary 3.1, (3.13) is a geometric object.□

Let $C(G)$ denote the centre of the group G. Then

$$\bigwedge_{x\in C(G)}\ \bigwedge_{g\in G} \quad J(x,g) = g\cdot x\cdot g^{-1} = x$$

Hence, $C(G)$ is an invariant subset of the fibre of object (3.13). Thus, we get

Corollary 3.3. Partial object

$$\left(C(G),\ G,\ J|_{C(G)\times G}\right) \tag{3.14}$$

of the object (3.13) is a geometric scalar of the space (1.1).

In the previous part of this section we have shown (lemma 2.2) that if in Klein space there exist m-repers, then (2.18) is a geometric object of this space. The object (2.18) can be also a geometric object in the case when m-repers do not exist in the space (1.1), as will be shown by the following lemma.

<u>Lemma 3.1.</u> If the fibre of Klein space (1.1) is a finite or equinumerous with one of the sets

$$\Omega^{(l)}(N) \quad \text{for } l=0,1,2,\ldots$$

then abstract object (2.18) is a geometric object of the space (1.1).

Proof. By assumption and lemma 1.2, scalar (1.13) is a geometric object. Thus, in virtue of theorem 3.1, object of transformations

$$\left(\hat{f}(G),\ G,\ \Phi_1\right), \qquad \Phi_1(f_x, g) = f_g \circ f_x \circ I_{1g^{-1}} = f_g \circ f_x \tag{3.15}$$

of the fibre of scalar (1.13) into the fibre of space (1.1) is also a geometric object. It is easy to verify that the objects (2.18) and (3.15) are equivalent. Indeed, for all $x, g \in G$ we have

$$\Phi_1\left(\hat{f}(x), g\right) = f_g \circ f_x = f_{g \cdot x} = \hat{f}(g \cdot x) = \hat{f}\left(L(x, g)\right),$$

where $\hat{f}\colon G \to \hat{f}(G)$ is a bijection. Object (2.18) is equivalent with geometric object (3.15) and, therefore, it is a geometric object, as well. □

§4. <u>Factor objects</u>

To define a factor object we will start from the notion of congruence (cf. [13], p. 39, also [4]) in the fibre of abstract object

$$(X,\ G,\ F) \tag{4.1}$$

<u>Definition 4.1.</u> Every equivalence relation r defined on the fibre X and consistent with the object (4.1), i.e. satisfying the condition

$$\bigwedge_{x_1, x_2 \in X} \ \bigwedge_{g \in G} \quad x_1 \, r \, x_2 \Rightarrow F(x_1, g) \, r \, F(x_2, g) \tag{4.2}$$

will be called a *congruence in the fibre X of object (4.1).*

Let X/r denote factor set with respect to the congruence r, and let [x] denote abstraction class determined by $x \in X$. It is easy to check that the transformation

$$F^r\colon (X/r) \times G \to X/r, \qquad F^r\left([x], g\right) := \left[F(x, g)\right] \tag{4.3}$$

is an operation of the group G on the set X/r (see [13], p. 40). Thus, the triplet

$$(X/r,\ G,\ F^r) \tag{4.4}$$

is an abstract object supported by the group G.

Definition 4.2. Abstract object (4.4) with operation F^r defined by formula (4.3) will be called a *factor object of the object (4.1) with respect to congruence r.*

The following two lemmas state certain important properties of factor objects. Their proofs are to be found in [13], pp. 45 and 43.

Lemma 4.1. Each comitant of geometric object (4.1) of Klein space (1.1) is equivalent with some factor object (4.4) of the object (4.1) with respect to congruence *r* defined in the fibre X of object (4.1).

Lemma 4.2. Every transitive abstract object (4.1) is equivalent with factor object

$$(G/r,\ G,\ L^r) \tag{4.5}$$

of the object (2.18) with respect to some congruence defined in the fibre G of the object (2.18).

It is a well known fact (see e.g. [2], pp. 68-69) that the fibre G/r of object (4.5) is a factor set G/H of the group G by some subgroup H. Hence, the elements of the set G/r are left cosets of the group G with respect to subgroup H.

Now, let us consider the object

$$\left(2^X,\ G,\ F^*\right) \tag{4.6}$$

of all subsets of the fibre of object (4.1). Let us note that the fibre of factor object (4.4) is an invariant subset of the fibre of object (4.6). Indeed, due to condition (4.2), for all $[x]\in X/r$ and $g\in G$ we have

$$\begin{aligned} F^*\left([x],g\right) &= F\left(\{y\in X:\ x\ r\ y\},g\right) = \\ &= \left\{F(y,g)\in X:\ F(x,g)\ r\ F(y,g)\right\} = \left[F(x,g)\right]\in X/r. \end{aligned}$$

Also, it follows that the factor object (4.4) is identical with partial object

$$\{X/r,\ G,\ F^*|_{(X/r)\times G}\}$$

of object (4.6). Hence, factor objects of the object (4.1) are the objects of subsets of the fibre of object (4.1). Thus, as the immediate consequence of theorem II.3.2 we get

Theorem 4.1. If abstract object (4.1) is a geometric object of Klein space (1.1), then each factor object (4.4) of this object is also a geometric object of this space.

The above theorem and lemma 4.1 imply the following

Corollary 4.1. Each comitant of geometric object of Klein space (1.1) is a geometric object of this space.

At last, in virtue of lemmas 2.2, 3.1, 4.2 and theorem 4.1 we get the corollary.

Corollary 4.2. If there exist m-repers in Klein space (1.1) or the fibre M of this space is finite or equinumerous with one of the sets

$$\Omega^{(l)}(N) \qquad \text{for } l=0,1,2,\ldots \qquad (4.7)$$

(defined by the formulas (1.11)), then every transitive abstract object (4.1) is a geometric object of the space (1.1).

From geometrical point of view, assumptions of the above corollary are not too restricting (cf. Section I, §5), since they are satisfied by all Klein spaces discussed in geometry. According to corollary 4.2, for a geometric Klein space (1.1) every transitive object of category OA(G) is an object of category OG(f).

Since abstract object (2.18) is transitive (cf. example I.2.1), by theorem 4.1 and lemma 4.2 we get the following simple corollary.

Corollary 4.3. Necessary and sufficient condition for that every object of category OA(G) is an object of category OG(f) of geometric objects of Klein space (1.1) is that abstract object (2.18) is a geometric object of this space.

Unfortunately, it is not known if there exist Klein spaces (1.1) such that abstract object (2.18) is not a geometric object.

Using corollary 4.2 we will prove one more lemma.

Lemma 4.3. If there exist m-repers in Klein space (1.1) or the fibre M of this space is finite or equinumerous with one of the sets (4.7), then for each epimorphism $\varphi: G \twoheadrightarrow H$ of the group G onto the group H the triplet

$$(H, G, F_1), \quad F_1(h, g) := \varphi(g) * h, \tag{4.8}$$

where * denotes group operation in H, is a geometric object of the space (1.1)

Proof. First, we will show that F_1 is an operation of the group G on the set of elements of H. For all $h \in H$ and $g_1, g_2 \in G$ we have

$$F_1(h, g_2 \cdot g_1) = \varphi(g_2 \cdot g_1) * h = \varphi(g_2) * \varphi(g_1) * h =$$
$$= \varphi(g_2) * F_1(h, g_1) = F_1(F_1(h, g_1), g_2).$$

Thus, F_1 satisfies the translation equation. Moreover, for $h \in H$ we have

$$F_1(h, e) = \varphi(e) * h = e_H * h = h,$$

where e and e_H are the neutral elements of G and H, respectively. Hence, F_1 satisfies the identity condition as well and, therefore, it is an operation of G on H, and the triplet (4.8) is an abstract object. It is easy to note that (4.8) is a transitive object (since φ is a surjection). Thus, in virtue of corollary 4.2, abstract object (4.8) is a geometric object of Klein space (1.1). □

§5. Disjoint union of objects

Let

$$(X_s, G, F_s), \quad s \in S \tag{5.1}$$

be a family of abstract objects and let

$$\bigvee_{s \in S} X_s := \bigcup_{s \in S} X_s \times \{s\}$$

be a disjoint union of the family of all fibres of objects (5.1). Let us also consider a transformation

$$\bar{F}: \left(\bigvee_{s \in S} X_s\right) \times G \to \bigvee_{s \in S} X_s$$

defined by the formula

$$\bar{F}\big((x_s, s), g\big) := \big(F_s(x_s, g), s\big) \quad \text{for } x_s \in X_s \text{ and } s \in S. \tag{5.3}$$

It is easy to note that $\bar{F}$ is an operation of the group G on the set (5.2). We will call it a *disjoin union of operations* F_s and denote

$$\bigvee_{s\in S} F_s := \bar{F}. \qquad (5.4)$$

Definition 5.1. Abstract object

$$\left(\bigvee_{s\in S} X_s,\ G,\ \bigvee_{s\in S} F_s\right) \qquad (5.5)$$

with transformation formula (5.4) defined by (5.3) will be called a *disjoint union of abstract objects (5.1)*.

Expressively speaking, object (5.5) is constructed of objects (5.1) by "glueing" together their fibres X_s, preserving operations F_s of the group G on these fibres.

Let us also consider a scalar

$$(S,\ G,\ I), \qquad I(s,g) = s. \qquad (5.6)$$

We will start from the following theorem.

Theorem 5.1. If abstract objects (5.1) and (5.6) are geometric objects of Klein space (1.1) and the set of ranks of all these object is bounded from above, then disjoint union (5.5) of objects (5.1) is a geometric object of this space.

Proof. Let k_s ($s\in S$) denote the rank of the object (5.1) and k_* - the rank of the scalar (5.6). By assumption, there exists a positive integer k such that

$$\bigwedge_{s\in S} \quad k_s \leqslant k \quad \text{and } k_* \leqslant k.$$

In virtue of assumption and lemma II.1.1 there exist partial objects

$$\left(\Omega_s^{(k)}(M),\ G,\ f_s^{(k)}\right), \qquad f_s^{(k)} = f^{(k)}|_{\Omega_s^{(k)}(M)\times G} \qquad (5.7)$$

and

$$\left(\Omega_*^{(k)}(M),\ G,\ f_*^{(k)}\right), \qquad f_*^{(k)} = f^{(k)}|_{\Omega_*^{(k)}(M)\times G} \qquad (5.8)$$

of the standard geometric object of rank k of the space (1.1), and bijections

$$\psi_s\colon X_s \to \Omega_s^{(k)}(M),\ s\in S, \qquad \psi_*\colon\ S \to \Omega_*^{(k)}(M)$$

such that for all $x_s\in X_s$, $s\in S$ and $g\in G$ the following equalities

$$f_s^{(k)}\left(\psi_s(x_s), g\right) = \psi_s\left(F_s(x_s, g)\right) \qquad (5.9)$$

and

$$f_*^{(k)}\big(\psi_*(s), g\big) = \psi_*\big(I(s, g)\big) = \psi_*(s) \qquad (5.10)$$

hold true. By (5.10), object (5.8) ia a scalar. Thus, by lemma III.2.1, for all $s\in S$ the set

$$\Omega_*^{(k)}(M)\times\{\psi_*(s)\}$$

is an invariant subset of the fibre of standard geometric object of rank k+2 and the equality

$$f^{(k+2)}\big((A_*, \psi_*(s)), g\big) = \big(f^{(k)}(A_*, g), \psi_*(s)\big), \qquad (5.11)$$

where $A_*\in\Omega_*^{(k)}(M)$, holds true. It follows that the set

$$\Omega_o^{(k+2)}(M) := \bigcup_{s\in S} \Omega_*^{(k)}(M)\times\{\psi_*(s)\}$$

is also an invariant subset of the fibre of standard geometric object of rank k+2. Hence, we can define a partial object

$$\big(\Omega_o^{(k+2)}(M),\ G,\ f_o^{(k+2)}\big) \qquad (5.12)$$

where

$$f_o^{(k+2)} = f^{(k+2)}\big|_{\Omega_o^{(k+2)}(M)\times G}$$

By (5.11) we get

$$f_o^{(k+2)}\big((A_*, \psi_*(s)), g\big) = \big(f_*^{(k)}(A_*, g), \psi_*(s)\big) \qquad (5.13)$$

for every $A_*\in\Omega_*^{(k)}(M)$ and $s\in S$. It is easy to check that the transformation

$$\psi\colon \bigvee_{s\in S} X_s \to \Omega_o^{(k+2)}(M)$$

defined by the formula

$$\psi\big((x_s, s)\big) := \big(\psi_s(x_s), \psi_*(s)\big)$$

is a bijection. Thus, due to (5.13), (5.9) and (5.3) we get succesively

$$f_o^{(k+2)}\big(\psi\big((x_s, s)\big), g\big) = f_o^{(k+2)}\big((\psi_s(x_s), \psi_*(s)), g\big) =$$
$$= \big(f_*^{(k)}(\psi_s(x_s), g), \psi_*(s)\big) = \big(\psi_s(F_s(x_s, g)), \psi_*(s)\big) =$$
$$= \psi\big((F_s(x_s, g), s)\big) = \psi\big(F((x_s, s), g)\big),$$

what proves the equivalence of objects (5.5) and (5.12). Thus, disjoint union (5.5) of objects (5.1) is a geometric object of Klein space (1.1).□

Now, let

$$(X, G, F) \tag{5.14}$$

be an arbitrary, non-transitive abstract object supported by the group G and let $(X_s)_{s\in S}$ be the family of all transitive fibres of this object. Then the triplets

$$(X_s, G, F_s), \quad F_s := F|_{X_s \times G}, \quad s \in S \tag{5.15}$$

are transitive partial objects of the object (5.14). This, with the definition 5.1, imply the following corollary.

Corollary 5.1. Each non-transitive abstract object (5.14) is equivalent with the disjoint union (5.5) of all its transitive partial objects (5.15).

Now, we will formulate necessary and sufficient conditions for a non-transitive abstract object (5.14) to be a geometric object of the space (1.1).

Theorem 5.2. It is necessary and sufficient for a non-transitive abstract object (5.14) to be a geometric object of Klein space (1.1), to satisfy the following three conditions:

(a) each transitive partial object (5.15) of the object (5.14) is a geometric object of the space (1.1);

(b) the set of ranks of all transitive partial objects (5.15) of the object (5.14) is bounded from above;

(c) there exists a geometric scalar of Klein space (1.1) with the fibre equinumerous with the set of all transitive fibres of object (5.14).

Proof. Let us assume that the abstract object (5.14) is a geometric object of rank k of the space (1.1). In virtue of theorem II.3.1, every transitive partial object (5.15) of object (5.14) is a geometric object of this space, and its rank k_s is no geater than the rank k of object (5.14). Thus, conditions (a) and (b) are satisfied. By definition of geometric object, the object (5.14) is equivalent with some partial object

$$(\Omega_o^{(k)}(M), G, f_o^{(k)}), \quad f_o^{(k)} = f^{(k)}|_{\Omega_o^{(k)}(M)\times G} \tag{5.16}$$

of a standard geometric object of rank k of the space (1.1). Let $\mathfrak{M}$ denote the set of all transitive fibres of partial object

(5.16). It is easily seen that $\mathfrak{M}$ is an invariant subset of the fibre of standard geometric object of rank k+1, and the partial object

$$\left(\mathfrak{M},\ G,\ f_o^{(k+1)}\right),\qquad f_o^{(k+1)} = f^{(k+1)}\big|_{\mathfrak{M}\times G} \tag{5.17}$$

is a geometric scalar of the space (1.1). Due to equivalence of objects (5.14) and (5.16), there exists a bijection

$$\psi:\ X \rightarrow \Omega_o^{(k)}(M),$$

satisfying the equivariance condition. It is a well known fact (cf. [25], [23]) that such bijections transforms transitive fibres of object (5.14) into transitive fibres of object (5.16). Thus, the set of all transitive fibres of object (5.14) is equinumerous with the fibre $\mathfrak{M}$ of geometric scalar (5.17), what ends the proof of condition (c).

Conversely, let us assume that a non transitive abstract object (5.14) satisfies conditions (a), (b) and (c). By corollary 5.1, object (5.14) is equivalent with the disjoint union (5.15) of all its transitive partial objects and, due to condition (c) and corollary 1.1, scalar (5.6) is a geometric object of the space (1.1). Hence, by conditions (a) and (b), all assumptions of theorem 5.1 are satisfied and, therefore, a disjoint union (5.5) is a geometric object of Klein space (1.1). Since the object (5.14) is equivalent with this disjoint union, it is a geometric object, what ends the proof of the theorem. □

We know, from considerations conducted in §4, that in Klein space satisfying the assumptions of corollary 4.2 every transitive abstract object supported by the group G is a geometric object of this space. Hence, in virtue of above theorem, the following corollary is true.

Corollary 5.2. Let us assume that there exist m-repers in Klein space (1.1) or that the fibre M of this space is finite or equinumerous with one of the sets (4.7). Then a non-transitive abstract object (5.14) is a geometric object of this space iff conditions (b) and (c) are satisfied.

Section IV

ELEMENTARY KLEIN SPACES

General properties of Klein spaces and its geometric object will be now illustrated by examples of elementary Klein spaces, such as vector space, unitary space, affine space and Euclidean space.

§1. Abstract linear objects

First, we will define a linear object and linear Klein space (cf. [13], p. 47 and [23]).

Definition 1.1. Abstract object

$$(V,\ G,\ F) \tag{1.1}$$

will be called *linear* over the field $\mathbb{K}$ iff its fibre V is a linear space over $\mathbb{K}$ and transformation formula F satisfies the condition

$$F(\lambda_1 v_1+\lambda_2 v_2, g) = \lambda_1 F(v_1, g) + \lambda_2 F(v_2, g).$$

Linear object over $\mathbb{K}$ (1.1) will be called *n-dimensional* iff $\dim_{\mathbb{K}} V = n$. Effective linear objects over $\mathbb{K}$ will be called *linear Klein spaces* over the field $\mathbb{K}$.

Let U^k and W^m be a k-dimensional and m-dimensional, respectively, linear spaces over the same field $\mathbb{K}$. Let us consider two linear object over $\mathbb{K}$

$$(U^k,\ G,\ F_1) \tag{1.2}$$

and

$$(W^m,\ G,\ F_0) \tag{1.3}$$

supported by the same group G, and the object

$$\left(\mathcal{F}(U^k, W^m),\ G,\ \bar{F}\right), \qquad \bar{F}(\gamma, g) = F_{0g}\circ\gamma\circ F_{1g^{-1}}, \tag{1.4}$$

of all transformations of the fibre of object (1.2) into the fibre of object (1.3). Let $\mathcal{L}(U^k, W^m)$ denote the set of all linear transformations of the space U^k into the space W^m. According to definition 1.1 of linear object, bijections F_{1g} and F_{0g} are the linear transformations. Hence, the implication

$$\gamma\in\mathcal{L}(U^k, W^m) \Rightarrow F_{0g}\circ\gamma\circ F_{1g^{-1}}\in\mathcal{L}(U^k, W^m)$$

holds true for all $g\in G$. Thus, $\mathcal{L}(U^k, W^m)$ is an invariant subset of the fibre of object (1.4). Therefore, we can define a partial object

$$\left(\mathcal{L}(U^k, W^m),\ G,\ F\right),\quad F(\gamma, g) = F_{0g}\circ\gamma\circ F_{1g^{-1}} \tag{1.5}$$

of object (1.4). It is a well known fact (cf. [19]), that the fibre of object (1.5) with operations defined as follows:

$$(\gamma_1 + \gamma_2)(u) := \gamma_1(u) + \gamma_2(u), \qquad (\lambda\gamma)(u) := \lambda\gamma(u)$$

is $(k\cdot m)$-dimensional linear space over $\mathbb{K}$. We can check by simple calculation that for all $\lambda_1, \lambda_2\in\mathbb{K}$, $\gamma_1, \gamma_2\in\mathcal{L}(U^k, W^m)$ and $g\in G$ we have

$$F(\lambda_1\gamma_1+\lambda_2\gamma_2, g) = \lambda_1 F(\gamma_1, g) + \lambda_2 F(\gamma_2, g). \tag{1.6}$$

It follows that (1.5) is a linear object over $\mathbb{K}$. Thus, we get the following corollary.

Corollary 1.1. If (1.2) and (1.3) are two linear objects over $\mathbb{K}$ and their dimension is k and m, respectively, then the object (1.5) of all linear transformations of the fibre of object (1.2) into the fibre of object (1.3) is a $(k\cdot m)$-dimensional linear object over the field $\mathbb{K}$.

By theorem III.3.1 we get another corollary.

Corollary 1.2. If linear objects (1.2) and (1.3) are geometric objects of n-dimensional linear Klein space over the field $\mathbb{K}$

$$(V^n,\ G,\ f), \tag{1.7}$$

then linear object (1.5) is also a geometric object of this space.

We can generalize the above considerations. Let us examine (instead one object (1.2)) k linear objects over $\mathbb{K}$

$$\left(V_i^{n_i},\ G,\ F_i\right),\qquad n_i = \dim_{\mathbb{K}} V_i^{n_i}\quad i=1,2,\ldots,k \tag{1.8}$$

and G-product of these objects

$$\left(V_1^{n_1}\times\ldots\times V_k^{n_k},\ G,\ \bar{F}\right). \tag{1.9}$$

Let

$$\left(\mathcal{F}(V_1^{n_1}\times\ldots\times V_k^{n_k}, W^m),\ G,\ \bar{F}\right) \tag{1.10}$$

be the object of all transformations of the fibre of G-product (1.9) into the fibre of object (1.3). It is easily seen that the set

$$\mathcal{L}(V_1^{n_1}, \ldots, V_k^{n_k}; W^m)$$

of all k-linear transformations (i.e. linear with respect to each variable separately, with remaining variables fixed)

$$\gamma:\ V_1^{n_1}\times\ldots\times V_k^{n_k} \rightarrow W^m$$

is an invariant subset of the fibre of object (1.10). Let us consider a partial object

$$\left(\mathcal{L}(V_1^{n_1}, \ldots, V_k^{n_k}; W^m),\ G,\ F\right) \tag{1.11}$$

of this object. We know (cf. [19]), that the fibre of object (1.11), with operations defined as usual, is an $(n_1\cdot\ldots\cdot n_k\cdot m)$-dimensional linear space over $\mathbb{K}$. It is easy to check by direct calculation that for all $\lambda_1, \lambda_2\in \mathbb{K}$, $g\in G$ and every $\gamma_1, \gamma_2\in\mathcal{L}(V_1^{n_1}, \ldots, V_k^{n_k}; W^m)$ the equality (1.6) remains true. Therefore, (1.11) is a linear object over the field $\mathbb{K}$. Thus, we get another corollary, being a generalization of corollary 1.1.

Corollary 1.3. If (1.8) and (1.3) are the linear objects over $\mathbb{K}$, then the object (1.11) of all k-linear transformations of the fibre of object (1.9) into the fibre of object (1.3) is an $(n_1\cdot\ldots\cdot n_k\cdot m)$-dimensional linear object over the field $\mathbb{K}$.

Subsequent corollary follows immediately from theorems III.2.1 and III.3.1.

Corollary 1.4. If linear objects (1.8) and (1.3) are geometric objects of linear Klein space (1.7), then linear object (1.11) is also a geometric object of this space.

To study the properties of Klein geometry of a given Klein space, it is convenient to choose the simplest (canonical) space of the class of equivalent spaces and conduct studies in it. We will do it in the following parts of this section.

Let $GL(n, \mathbb{K})$ denote the multiplicative group of all non-singular square matrixes of n-th order with elements belonging to the field K, and $\widetilde{GL}(n,K)$ - an arbitrary subgroup of $GL(n,K)$. Abstract object (cf. [13], p. 23, [5], [23])

$$(K^n,\ GL(n,K),\ f),\qquad f\left((x^i),[A^i_j]\right) := \left(A^i_j\, x^j\right) \tag{1.12}$$

(we use the Einstein's sumation convention) and each its subobject

$$(K^n,\ \widetilde{GL}(n,K),\ \tilde{f}),\qquad \tilde{f} = f\,|_{K^n\times\widetilde{GL}(n,K)} \tag{1.13}$$

are the examples of n-dimensional linear Klein space over the field K. In the sequel they will be called *canonical* linear Klein spaces. It follows from considerations presented in [23] that every n-dimensional linear Klein space over K is equivalent with one of the spaces (1.12), (1.13). Object (1.13) is also called an n-dimensional canonical *vector* Klein space. Due to accepted definition, vector Klein space is a linear Klein space, but the converse generally is not true. In the sequel we will discuss the most important geometric objects of vector Klein space.

§2. Covariant and contravariant vectors

Let us consider an n-dimensional (canonical) vector Klein space over the field K

$$(K^n,\ GL(n,K),\ f),\qquad f\left((x^i),[A^i_j]\right) = \left(A^i_j\, x^j\right). \tag{2.1}$$

Due to corollary II.1.3, space (2.1) is its own geometric object called, in general case, a point object. For a vector Klein space, though, we will bring the following definition (cf. [13], p. 23).

Definition 2.1. Point object of a vector Klein space (2.1) will be called a *contravariant vector*.

As we know, the centre of a general linear group $GL(n,K)$ is a group of scalar matrixes. We will denote this group by $S(n,K)$. By corollary III.3.3, the object

$$\left(S(n,K),\ GL(n,K),\ \bar{I}\right),\qquad \bar{I}(X,A) = X \quad \text{for } X\in S(n,K),\ A\in GL(n,K)$$

is a geometric scalar of Klein space (2.1). Since the set of all scalar matrixes is equinumerous with the field K, by corollaries III.1.1 and II.1.1 we get the following corollary.

Corollary 2.1. Abstract object

$$(\mathbb{K},\ GL(n,\mathbb{K}),\ I),\quad I(\lambda, A) := \lambda \tag{2.2}$$

is a geometric scalar of n-dimensional canonical vector Klein space over the field $\mathbb{K}$.

It is easily seen that the following corollary is also true.

Corollary 2.2. Geometric scalar (2.2) of the space (2.1) is a one-dimensional linear object over the field $\mathbb{K}$.

The next corollary follows easily from the previous one and corollaries 1.1 and 1.2.

Corollary 2.3. Object

$$\begin{gathered}(\mathcal{L}(K^n, K),\ GL(n,K),\ F)\\ F(\omega, A) = I_A \circ \omega \circ f_{A^{-1}} = \omega \circ f_{A^{-1}}\end{gathered} \tag{2.3}$$

of all linear mappings $\omega\colon K^n \to K$ of the fibre of contravariant vector into the fibre of scalar (2.2) is a geometric, n-dimensional linear object of Klein space (2.1).

Definition 2.2. Geometric object (2.3) will be called a *covector* or a *covariant vector* of vector space (2.1).

Now, let us consider abstract linear object over K (cf. [13], p. 23, also [23])

$$(K^n,\ GL(n,K),\ \bar{f}),\quad \bar{f}\left((x_i), [A^i_j]\right) := (\bar{A}^j_i\, x_j), \tag{2.4}$$

where $\bar{A}^j_i$ are the elements of matrix A^{-1}, inverse of the matrix $A = [A^i_j]$, and the mapping of the fibre of object (2.3) into the fibre of object (2.4)

$$\psi\colon \mathcal{L}(K^n, K) \to K^n,\quad \psi(\omega) := (\omega_i), \tag{2.5}$$

where $\omega_i := \omega(e_i)$, and $e_i = (\delta^j_i)$, $i, j = 1, 2, \dots, n$ is a base of the fibre K^n of contravariant vector (2.1) (δ^j_i denotes the Kronecker's symbol). It is proven (see e.g. [19]) that such defined mapping is a linear isomorphism. We will show that the pair $(\psi, id_{GL(n,K)})$ is an isomorphism (of category $OG(f)$) of object (2.3) onto object (2.4). Due to definition of mapping (2.5) and the transformation formula F of object (2.3), for each $\omega \in \mathcal{L}(K^n, K)$ and $A \in GL(n,K)$ we have

$$\psi(F(\omega, A)) = \psi(\omega \circ f_{A^{-1}}) = (\omega \circ f_{A^{-1}}(e_i)) = (\omega(f(e_i, A^{-1}))).$$

Let

$$f(e_i, A^{-1}) = v_i = (v_i^j) \in K^n.$$

Then $f(v_i, A) = e_i$ and

$$f(v_i, A) = f\left((v_i^j), [A_j^k]\right) = (A_j^k v_i^j) = e_i = (\delta_i^k).$$

Thus,

$$A_j^k v_i^j = \delta_i^k.$$

Therefore, $[v_i^j]$ is an inverse matrix of A, i.e.

$$v_i^j = \bar{A}_i^j.$$

Hence,

$$\psi(F(\omega, A)) = (\omega(v_i)) = (\omega(v_i^j e_j)) = (\omega(\bar{A}_i^j e_j)) =$$
$$= (\bar{A}_i^j \omega(e_j)) = \bar{f}(\psi(\omega), [A_i^j]),$$

and, therefore,

$$\bar{f}(\psi(\omega), A) = \psi(F(\omega, A)).$$

Since ψ is a bijection, pair $(\psi, id_{GL(n,K)})$ is an isomorphism of category OG(f). The results obtained above we will formulate as a lemma.

Lemma 2.1. Transformation (2.5) is a linear isomorphism, and pair $(\psi, id_{GL(n,K)})$ - an isomorphism of object (2.3) onto object (2.4).

As the immediate consequence of this lemma we get:

Corollary 2.4. Object (2.4) is a geometric object of a vector Klein space (2.1), equivalent with covariant vector (2.3).

Definition 2.3. Geometric object (2.4) will be called a *canonical covariant vector* of vector Klein space (2.1).

It appears (cf. [16], [23]) that objects (2.1) and (2.4) are abstractively equivalent, but not geometrically equivalent. In geometric interpretation it means that these objects, considered as Klein spaces are equivalent, but treated as geometric objects of vector Klein space are not. It can be shown that the covector of covariant vector (2.4) is an object equivalent with contravariant vector.

§3. Tensors

Let us consider a contravariant vector and a covariant vector (2.4) of vector Klein space (2.1). Define the cartesian product

$$E^{(p,q)} := \underbrace{K^n\times\ldots\times K^n}_{q}\times\underbrace{K^n\times\ldots\times K^n}_{p}, \tag{3.1}$$

where the first q factors are the fibres of object (2.1), and the next p - the fibres of object (2.4). One of the numbers p, q can be equal to zero. Let

$$\begin{gathered}(E^{(p,q)},\ GL(n,K),\ \bar{F}),\\ \bar{F}\big((v_1,\ldots,u_p),A\big) = \big(f(v_1,A),\ldots,\bar{f}(u_p,A)\big)\end{gathered} \tag{3.2}$$

be a product object defined of objects (2.1) and (2.4). Since covariant and contravariant tensors are geometric, n-dimensional linear objects over K, in virtue of corollaries 1.3 and 1.4 we have

Corollary 3.1. Abstract object

$$\begin{gathered}(\mathcal{L}(E^{(p,q)},K),\ GL(n,K),\ F)\\ F(\sigma,A) = I_A\circ\sigma\circ\bar{F}_{A^{-1}} = \sigma\circ\bar{F}_{A^{-1}}\end{gathered} \tag{3.3}$$

of all (p+q)-linear mappings $\sigma\colon E^{(p,q)}\to K$ of the fibre of object (3.2) into the fibre of scalar (2.2) is an n^{p+q}-dimensional, linear over K, geometric object of the space (2.1).

Definition 3.1. Geometric object (3.3) of vector Klein space (2.1) will be called a *tensor of valence (p, q)* or *tensor contravariant of degree p and covariant of degree q*.

Now, let us consider the cartesian product $K^{n^{p+q}}$ of the field K. Any element of this product we will denote by the symbol

$$(a^{i_1\ldots i_p}_{j_1\ldots j_q}),\ \text{where } i_1,\ldots,i_p,j_1,\ldots,j_q = 1,2,\ldots,n.$$

Let us define an abstract linear object over K (cf. [13], p. 25 and [5])

$$\big(K^{n^{p+q}},\ GL(n,K),\ \tilde{F}\big) \tag{3.4}$$

with the transformation formula defined as follows:

$$\tilde{F}\big((a^{i_1\ldots i_p}_{j_1\ldots j_q}),[A^j_i]\big) := \big(A^{k_1}_{i_1}\ldots A^{k_p}_{i_p}\bar{A}^{j_1}_{l_1}\ldots\bar{A}^{j_q}_{l_q}\,a^{i_1\ldots i_p}_{j_1\ldots j_q}\big), \tag{3.5}$$

where $\bar{A}^j_i$ are the elements of a matrix A^{-1}, inverse for $A=[A^j_i]$.

Finally, let

$$\psi\colon \mathcal{L}\left(E^{(p,q)}, K\right) \to K^{n^{p+q}} \tag{3.6}$$

be a transformation of the fibre of object (3.3) into the fibre (3.4), defined by the formula

$$\psi(\sigma) := \left(\sigma(e_{i_1}, \ldots, e_{i_q}, e^{k_1}, \ldots, e^{k_p})\right), \tag{3.7}$$

where

$$e_i = \left(\delta_i^j\right), \quad i=1,2,\ldots,n \tag{3.8}$$

and

$$e^k = \left(\delta_i^k\right), \quad k=1,2,\ldots,n \tag{3.9}$$

are the bases of contravariant vector (2.1) and covariant vector (2.4), respectively, and δ_i^j denotes Kronecker's symbol.

We will prove the following lemma.

Lemma 3.1. Transformation (3.6) defined by the formula (3.7) is a linear isomorphism, and the pair $\left(\psi, id_{GL(n,K)}\right)$ is an isomorphism of the object (3.3) onto object (3.4).

Proof. Let us note that transformation (3.6) is a linear isomorphism (see e.g. [19]). In the previous part of this section we have shown (see the proof of lemma 2.1) that the implication

$$f(e_i, A^{-1}) = v_i = (v_i^j) \Rightarrow v_i^j = \bar{A}_i^j. \tag{3.10}$$

Similarly, we can show that

$$\bar{f}(e^k, A^{-1}) = u^k = (u_i^k) \Rightarrow u_i^k = A_i^k. \tag{3.11}$$

For the sake of simplicity, we will carry the proof of second part of the lemma in the case p = q = 1. Using the above implications and relations (3.3), (3.7) and (3.2), for each $\sigma\in\mathcal{L}\left(E^{(p,q)}, K\right)$ and $A\in GL(n,K)$ we have

$$\begin{aligned}\psi\left(F(\sigma, A)\right) &= \psi\left(\sigma\circ\bar{F}_{A^{-1}}\right) = \left(\sigma\circ\bar{F}_{A^{-1}}(e_i, e^k)\right) = \\ &= \left(\sigma(f(e_i, A^{-1}), \bar{f}(e^k, A^{-1}))\right) = \left(\sigma(v_i, u^k)\right) = \\ &= \left(\sigma(v_i^j\, e_j, u_l^k\, e^l)\right) = \left(\sigma(\bar{A}_i^j\, e_j, A_l^k\, e^l)\right) = \\ &= \left(A_l^k\, \bar{A}_i^j\, \sigma(e_j, e^l)\right) = \bar{F}\left(\psi(\sigma), A\right).\end{aligned}$$

Thus, the pair $(\psi, id_{GL(n,K)})$ is an isomorphism of the object (3.3) onto object (3.4). For arbitrary p and q the proof is similar. □

As an immediate consequence of the above lemma we get the following corollary.

Corollary 3.2. Abstract object (3.4) is a geometric object of vector Klein space (2.1), equivalent with tensor (3.3).

Definition 3.2. Geometric object (3.4) of Klein space (2.1) will be called a *canonical tensor of valence (p, q)*.

Using denotations:

$$A = [A_i^{i'}], \qquad A^{-1} = [A_{i'}^{i}]$$

and

$$(a^{i'_1 \dots i'_p}_{j'_1 \dots j'_q}) = \bar{F}((a^{i_1 \dots i_p}_{j_1 \dots j_q}), [A^j_i])$$

we can express the transformation formula of the object (3.4) in in the form well known in tensor calculus:

$$a^{i'_1 \dots i'_p}_{j'_1 \dots j'_q} = A^{i'_1}_{i_1} \dots A^{i'_p}_{i_p} A^{j_1}_{j'_1} \dots A^{j_q}_{j'_q} a^{i_1 \dots i_p}_{j_1 \dots j_q} .$$

Contravariant and covariant vectors are obviously tensors of valences (1,0) and (0,1), respectively. Some of the properties of tensors over the field of real numbers are discussed in [13]. They can be easily transferred to the tensors over arbitrary field $\mathbb{K}$.

§4. Densities

Let be given an n-dimensional vector Klein space over the field of real numbers R

$$(R^n,\ GL(n, R),\ f), \qquad f((x^i), [A_i^j]) = (A_i^j x^i), \tag{4.1}$$

and a homomorphism φ_o: $R_o \to R_o$ of a multiplicative group $R_o = R\setminus\{0\}$ of reals into itself.

In the space (4.1) there exist n-repers. These are (cf. [14], p. 52) all bases of the fibre R^n. Moreover, the mapping $\varphi := \varphi_o \circ \det$ is an epimorphism of the group GL(n, R) onto the group $\varphi_o(R_o)$. It follows (cf. lemma III.4.3) that the triplet

$$(\varphi_o(R_o),\ GL(n, R),\ F_o), \quad F_o(x, A) := \varphi_o(\det A)\cdot x \tag{4.2}$$

is a geometric object of the space (4.1). Object (4.2) will be called a *generalized density*. It appears (cf. [1]) that the only measurable homomorphisms $\varphi_0 : R_0 \to R_0$ are the functions of the form

$$\varphi_0(t) = |t|^\alpha \tag{4.3}$$

and

$$\varphi_0(t) = (\mathrm{sgn}\, t)|t|^\alpha \tag{4.4}$$

where α is an arbitrary real number. Hence, we can define (cf. [13], p. 25, also [5]) abstract objects

$$(R,\ GL(n,R),\ F_1), \qquad F_1(x,A) := |\det A|^\alpha \cdot x \tag{4.5}$$

and

$$(R,\ GL(n,R),\ F_2), \qquad F_2(x,A) := \mathrm{sgn}(\det A)\cdot|\det A|^\alpha \cdot x. \tag{4.6}$$

Definition 4.1. Abstract object (4.5) and (4.6) will be called a *W-density* (or *Weyl density*) *of weight* $(-\alpha)$ and *G-density* (or *ordinary density*) *of weight* $(-\alpha)$.

It is easily seen that the following corollary holds true.

Corollary 4.1. W-densities and G-densities of an arbitrary weight are linear object over the field R.

Let us note that W-density of weight 0 is a scalar (with $\mathbb{K}=R$), and G-density of weight 0 is an abstract object of the form

$$(R,\ GL(n,R),\ F_3), \qquad F_3(x,A) = \mathrm{sgn}(\det A)\cdot x \tag{4.7}$$

called a *biscalar*.

First, we will prove the following theorem.

Theorem 4.1. W-density of weight $(-\alpha)$ is a geometric object of vector Klein space (4.1).

Proof. If $\alpha=0$, then the object (4.5) is a scalar and, therefore, a geometric object (cf. corollary 2.1). Hence, we can assume that $\alpha\neq0$. Let us note that the mapping

$$\varphi:\ GL(n,R) \to R_+ = (0,+\infty), \qquad \varphi := \varphi_0\circ\det,$$

where φ_0 is defined by the formula (4.3), is an epimorphism of the group GL(n, R) onto multiplicative group R_+. Thus, due to lemma III.4.3, abstract object

$$(R_+,\ GL(n,R),\ F_4), \qquad F_4(x,A) := |\det A|^\alpha \cdot x, \quad \alpha\neq0 \tag{4.8}$$

is a geometric object of the space (4.1). Moreover, the object

$$\left(R_{-},\ GL(n,R),\ F_5\right),\qquad F_5(x,A) := |\det A|^{\alpha}\cdot x,\quad \alpha\neq 0, \qquad (4.9)$$

where $R_{-} := (-\infty, 0)$, is equivalent with the object (4.8). Indeed,

$$\psi\colon\ R_{+} \to R_{-},\qquad \psi(x) := -x$$

is a bijective function, and since

$$F_5\left(\psi(x), A\right) = |\det A|^{\alpha}\cdot\psi(x) = -|\det A|^{\alpha}\cdot x = \psi(|\det A|^{\alpha}\cdot x) = \psi\left(F_4(x,A)\right)$$

the pair $(\psi, id_{GL(n,R)})$ is a morphism of object (4.8) onto object (4.9). It follows that (4.9) is a geometric object of the space (4.1). Since the scalar (2.2) (K=R) is geometric, in virtue of corollary III.1.3, each scalar with a finite fibre is geometric. In particular, the scalar with one-element fibre

$$\left(\{0\},\ GL(n,R),\ I_0\right),\qquad I_0(0,A) = 0 \in R \qquad (4.10)$$

is geometric. Geometric objects (4.8)-(4.10) forms a family of all transitive partial objects of the object (4.5) for $\alpha\neq 0$. It follows then, by theorem III.5.2, that W-density of weight $(-\alpha)$ $(\alpha\neq 0)$ is a geometric object of Klein space (4.1), what ends the proof.□

Lemma 4.1. Biscalar (4.7) is a geometric object of vector Klein space (4.1).

Proof. Mapping

$$\varphi\colon\ GL(n,R) \to \{-1, 1\},\qquad \varphi(A) := \operatorname{sgn}(\det A)$$

is an epimorphism of the group GL(n, R) onto multiplicative group $\{-1, 1\}$. Due to lemma III.4.3, the object

$$\left(\{-1, 1\},\ GL(n,R),\ F_6\right),\qquad F_6(x,A) = \operatorname{sgn}(\det A)\cdot x \qquad (4.11)$$

is a geometric object. It is easy to note that, for an arbitrarily fixed $a\in R_{+}$, it is equivalent with the object

$$\left(\{-a, a\},\ GL(n,R),\ F_7\right),\qquad F_7(x,A) = \operatorname{sgn}(\det A)\cdot x. \qquad (4.12)$$

Therefore, (4.12) is a geometric object. Objects (4.10)-(4.12) forms the family of all transitive partial objects of biscalar (4.7). Geometricity of biscalar (4.7) follows immediately from this fact and theorem III.5.2.□

Theorem 4.2. G-density of weight $(-\alpha)$ is a geometric object of vector Klein space (4.1).

Proof. It follows from lemma 4.1 that the theorem is true for $\alpha=0$. Let us assume then, that $\alpha\neq 0$. Mapping

$$\varphi\colon GL(n,\mathbf{R}) \to \mathbf{R}_o, \qquad \varphi := \varphi_o \circ \det,$$

where φ_o denote the function (4.4), is a group epimorphism. Thus, by lemma III.4.3, the object

$$(\mathbf{R}_o,\ GL(n,\mathbf{R}),\ F_e), \qquad F_e(x,A) = \varphi(A)\cdot x \tag{4.13}$$

is geometric. G-density of weight $(-\alpha)$ $(\alpha\neq 0)$ has only two transitive partial objects, i.e. (4.13) and (4.10). In virtue of theorem III.5.2, it is a geometric object of the space (4.1).□

§5. Tensor densities

Let us consider again the n-dimensional vector Klein space (4.1) over the field **R** of real numbers, and the object

$$(\mathbf{R},\ GL(n,\mathbf{R}),\ \Phi), \qquad \Phi(x,A) := \varphi(A)\cdot x, \tag{5.1}$$

where $\varphi\colon GL(n,\mathbf{R}) \to \mathbf{R}$ is a transformation defined by the formula

$$\varphi(A) = |\det A|^{\alpha}, \qquad \alpha\in\mathbf{R}, \tag{5.2}$$

or the formula

$$\varphi(A) = \mathrm{sgn}(\det A)|\det A|^{\alpha}, \qquad \alpha\in\mathbf{R}. \tag{5.3}$$

Depending on whether φ is defined by (5.2) or (5.3), geometric object (5.1) is either W-density or G-density, respectively, of weight $(-\alpha)$. Let us also consider product object (3.2) ($\mathbf{K}=\mathbf{R}$)

$$\begin{gathered}(E^{(p,q)},\ GL(n,\mathbf{R}),\ \bar{F})\\ \bar{F}((v_1,\dots,u_p),A) = (f(v_1A),\dots,\bar{f}(u_pA))\end{gathered} \tag{5.4}$$

and object

$$\begin{gathered}(\mathcal{L}(E^{(p,q)},\mathbf{R}),\ GL(n,\mathbf{R}),\ F),\\ F(\sigma,A) = \Phi_A\circ\sigma\circ\bar{F}_{A^{-1}}\end{gathered} \tag{5.5}$$

of all (p+q)-linear mappings $\sigma\colon E^{(p,q)} \to \mathbf{R}$ of the fibre of object (5.4) into the fibre of object (5.1).

Corollaries 1.3 and 1.4 imply the following:

Corollary 5.1. Object (5.1) is a linear over **R** geometric object of vector Klein space (4.1).

Definition 5.1. Geometric object (5.5) of vector Klein space

(4.1) will be called a *tensor density of valence* (p,q) *and weight* $(-\alpha)$ (or, more precisely, *tensor W-density* or *G-density of valence* (p,q) *and weight* $(-\alpha)$, depending of whether φ is defined by formula (5.2) or (5.3)).

Let us consider a linear object over R (cf. [13], p. 25, also [5])

$$\left(R^{n^{p+q}},\ GL(n,R),\ \tilde{F}\right), \tag{5.6}$$

with the transformation formula $\tilde{F}$ defined as follows

$$\tilde{F}\left((a^{i_1 \dots i_p}_{j_1 \dots j_q}), A\right) := \left(\varphi(A)\ A^{k_1}_{i_1} \dots A^{k_p}_{i_p}\ \bar{A}^{j_1}_{l_1} \dots \bar{A}^{j_q}_{l_q}\ a^{i_1 \dots i_p}_{j_1 \dots j_q}\right), \tag{5.7}$$

where φ is defined by (5.2) or (5.3) and

$$A = [A^i_j], \qquad A^{-1} = [\bar{A}^i_j].$$

Let (3.8) and (3.9) be the bases of, respectively, fibres of contravariant and covariant vectors over R, and let

$$\psi\colon\ \mathcal{L}(E^{(p,q)}, R) \to R^{n^{p+q}} \tag{5.8}$$

be a mapping of the fibre of object (5.5) into the fibre of object (5.6), defined by the formula (3.7).

<u>Lemma 5.1.</u> Pair $(\psi, id_{GL(n,R)})$, where ψ denotes the transformation (5.8) defined by (3.7), is an isomorphism of object (5.5) onto the object (5.6).

Proof. As in §3, we will carry the proof only for the particular case p=q=1. For all other p and q the proof is quite similar. Using (5.5), (3.7), (5.4), (3.10), (3.11) and (5.1) we get, for each $\sigma \in \mathcal{L}(E^{(p,q)}, R)$ and $A \in GL(n,R)$,

$$\begin{aligned}\psi(F(\sigma, A)) &= \psi(\Phi_A \circ \sigma \circ \bar{F}_{A^{-1}}) = \left(\Phi_A \circ \sigma \circ \bar{F}_{A^{-1}}(e_i, e^k)\right) = \\ &= \left(\Phi_A\left(\sigma(f(e_i, A^{-1}), \bar{f}(e^k, A^{-1}))\right)\right) = \left(\Phi_A\left(\sigma(v_i, u^k)\right)\right) = \\ &= \left(\Phi_A\left(\sigma(v^j_i\, e_j, u^k_l\, e^l)\right)\right) = \left(\varphi(A)\ A^k_l\ \bar{A}^j_i\ \sigma(e_j, e^l)\right)\end{aligned}$$

and, therefore,

$$\psi(F(\sigma, A)) = \tilde{F}(\psi(\sigma), A).$$

Since ψ is a linear isomorphism (cf. lemma 3.1), it is a bijection. Hence, pair $(\psi, id_{GL(n,R)})$ is an isomorphism of category OG(f).□

<u>Corollary 5.2.</u> Abstract object (5.6) is a geometric object

of Klein space (4.1), equivalent with tensor density (5.5).

Definition 5.2. Geometric object (5.6) of vector Klein space (4.1) will be called a *canonical tensor density of valence (p, q) and weight (-α).*

Tensor densities and their properties are presented in [13].

§6. Geometric objects of elementary Klein spaces

Let us consider an n-dimensional vector Klein space over the field **K**

$$(\mathbf{K}^n,\ GL(n, \mathbf{K}),\ f), \qquad f\left((x^i), [A^j_i]\right) = (A^j_i\, x^i) \tag{6.1}$$

and an n-dimensional affine Klein space over the same field (cf. example I.2.3)

$$(\mathbf{K}^n,\ GA(n, \mathbf{K}),\ \bar f), \quad \bar f\left((x^i), \left((a^j), [A^j_i]\right)\right) = (a^j + A^j_i\, x^i). \tag{6.2}$$

Let $H(n, \mathbf{K})$ denote an arbitrary subgroup of linear group $GL(n, \mathbf{K})$. An important example of such a group is orthogonal group over **K**, defined by the formula

$$O(n, \mathbf{K}) := \{A \in GL(n, \mathbf{K}):\ A \cdot A^T = A^T \cdot A = E\},$$

where E denotes unit matrix and A^T - transposed matrix of A. To a subgroup $H(n, \mathbf{K})$ corresponds a subgroup (cf. [13], p. 29 [5], [23])

$$GH(n, \mathbf{K}) := \{(a, A):\ a \in \mathbf{K}^n \wedge A \in H(n, \mathbf{K})\}$$

of the affine group $GA(n, \mathbf{K})$. Subgroup

$$E(n, \mathbf{K}) := \{(a, A):\ a \in \mathbf{K}^n \wedge A \in O(n, \mathbf{K})\}$$

of the affine group, corresponding to the orthogonal group $O(n, \mathbf{K})$, will be called *Euclidean group of degree n over K.*

Let us consider subobjects

$$(\mathbf{K}^n,\ H(n, \mathbf{K}),\ f_o), \qquad f_o := f\,|_{\mathbf{K}^n \times H(n, \mathbf{K})} \tag{6.3}$$

and

$$(\mathbf{K}^n,\ GH(n, \mathbf{K}),\ \bar f_o), \qquad \bar f_o := \bar f\,|_{\mathbf{K}^n \times GH(n, \mathbf{K})} \tag{6.4}$$

of spaces (6.1) and (6.2), determined by subgroups $H(n, \mathbf{K})$ and $GH(n, \mathbf{K})$, respectively. Due to corollaries I.3.1, they are also Klein spaces. As we know, the space (6.3) is called an n-dimensional linear Klein space over K.

Definition 6.1. Abstract object (6.4) will be called an *n-dimensional subaffine Klein space over K.*

Space (6.3) supported by the orthogonal group $O(n, K)$ is an important example of Klein space. It is called an *n-dimensional unitary Klein space over K.* Subaffine space (6.4) supported by the group $E(n, K)$ is called an *n-dimensional Euclidean Klein space over K.* More examples of subaffine spaces over R can be found in [13].

Definition 6.2. Klein space (M, G, f) equivalent with one of the spaces (6.1)-(6.4) will be called an *n-dimensional elementary Klein space over K.*

To study the properties of elementary Klein spaces we usually consider canonical elementary spaces (6.1)-(6.4). The following lemma is a base for further considerations.

Lemma 6.1. Abstract object

$$(K^n,\ GA(n, K),\ \bar{F}),\quad \bar{F}\big((v^i), ((a^j), [A^j_i])\big) := (A^j_i v^i) \qquad (6.5)$$

is a geometric object of affine Klein space (6.2).

Proof. Let us consider product object of the space (6.2)

$$(K^n \times K^n,\ GA(n, K),\ \bar{f}^2), \qquad (6.6)$$

where

$$\bar{f}^2\big(((x^i), (y^i)), ((a^j), [A^j_i])\big) = \big((a^j + A^j_i x^i), (a^j + A^j_i y^i)\big),$$

and the transformation

$$\psi\colon K^n \times K^n \to K^n,\quad \psi\big(((x^i), (y^i))\big) := (y^i - x^i)$$

of the fibre of object (6.6) into the fibre of object (6.5). Due to theorem III.2.1, object (6.6) is a geometric object of the affine space (6.2), whereas ψ, as is easily seen (cf. [13], p. 32, [5]), is an invariant and surjective transformation. Thus, object (6.5) is a comitant of geometric object (6.6). Hence, by corollary III.4.1, (6.5) is a geometric object of Klein space (6.2).□

Definition 6.3. Geometric object (6.5) of affine Klein space (6.2) will be called a *contravariant vector* of this space.

It is easy to note that the transformation

$$\varphi\colon GA(n,K) \to GL(n,K), \qquad \varphi\big((a,A)\big) := A \tag{6.7}$$

is a homomorphism of affine group into general linear group (cf. [13], p. 28). Moreover, object (6.5) is induced by vector Klein space (6.1) and homomorphism (6.7). Hence, by lemma 6.1 and theorem II.3.3, the following important corollary is true.

Corollary 6.1. Each object induced by a geometric object of vector Klein space (6.1) and homomorphism (6.7) is a geometric object of affine Klein space (6.2).

For example, the object

$$(K^n, GA(n,K), \bar{F}_1), \qquad \bar{F}_1\big((u_j), ((a^j), [A^j_i])\big) = (\bar{A}^j_i u_j)$$

induced by covariant vector (2.4) of vector Klein space (6.1) and homomorphism (6.7), is a geometric object of affine space (6.2). We will call it a *covariant vector of affine Klein space (6.2)*.

Now, we will introduce the following general definition.

Definition 6.4. Object induced by tensor (3.5) (by W-density (4.5), G-density (4.6), tensor density (5.6) for K=R) and homomorphism (6.7) will be called a *tensor of valence (p, q) (W-density, G-density, tensor density) of affine Klein space (6.2)*.

Tensors (and also densities and tensor densities for K=R) can be defined for an arbitrary linear Klein space (6.3) and arbitrary subaffine Klein space (6.4), using the following corollary, being an immediate consequence of corollary II.3.3.

Corollary 6.2. Subobject of an arbitrary geometric object of vector Klein space (6.1) (affine Klein space (6.2)), determined by subgroup H(n, K) of the group GL(n, K) (subgroup GH(n, K) of the group GA(n, K)) is a geometric object of linear Klein space (6.3) (subaffine Klein space (6.4)).

It follows that for an arbitrary elementary Klein space, beside the objects of geometric figures (cf. definition I.4.2 and corollary II.1.2) there are other geometric objects: tensors, and, in the case K=R, also densities and tensor densities. These are all geometric objects of elementary Klein spaces with practical applications. So, indroducing the definition II.1.2 of geometric object is fully reasonable.

CONCLUSIONS

We noted, in §5, Section I, that one should expect certain correlations between the fibres and transformation formulas of a given Klein space and its geometric object. These correlations exist for all objects of category OG(f) (cf. §1, Section II). Hence, in virtue of corollary III.4.2, the existence of repers of finite order in Klein space guarantees such correlations for every transitive object of category OA(G). So, if we replaced the effectivity condition in definition of Klein space by the stronger axiom of existence of m-repers (cf. §5, Section I), then the category OA(G) could be called a category of geometric objects and a pair

$$((M,G,f), OA(G))$$

a Klein geometry of this space. The notion of equivaleuce of geometries can be introduced with the use of simple functor (cf. §4, Section II) for such defined Klein geometries. It seems, though, that even in this case, definition II.1.2, accepted in this paper, is more properly designed, as prove the properties of geometric objects of elementary Klein spaces, discussed in Section IV.

This paper, although it forms a certain whole, does not exhaust the subject. Beside, undoubtedly important, elementary spaces, in geometry there are also discussed classical Klein spaces, such as projective, elliptic, hyperbolic, Grassman and Stiefel space (cf. [14], §7, Section I). Presenting the properties of these spaces and their geometric objects exceeds the limits of this paper, though.

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PODSTAWOWE POJĘCIA GEOMETRII KLEINA

S t r e s z c z e n i e

Chociaż od sławnego Programu z Erlangen Feliksa Kleina upłynęło już ponad sto lat, nie jest on do tej pory w pełni wykorzystany. Główna przyczyna tego tkwi między innymi w tym, że nie został on dostatecznie precyzyjnie przedstawiony. Oryginalną definicję geometrii, podaną przez F. Kleina (p. [6]), można przedstawić następująco: Geometrią zbioru M, względem grupy przekształceń G(M) tego zbioru, lub krótko G(M)-geometrią, nazywamy zbiór wszystkich własności figur geometrycznych, które nie ulegają zmianie przy przekształceniach grupy G(M). Własności takie nazywamy niezmiennikami lub własnościami geometrycznymi. Po pojawieniu się konieczności uprawiania geometrii opartych na zbiorach przekształceń, nie koniecznie tworzących grupy, przestrzenie z grupą przekształceń zaczęto nazywać przestrzeniami Kleina.

R. Sulanke przez przestrzeń Kleina rozumie (p. [21], [22]) tranzytywną, lewostronną grupę Liego przekształceń, tzn. trójkę (M,G,f), gdzie M jest rozmaitością, G - grupą Liego, zaś f - tranzytywnym, lewostronnym działaniem grupy G na M. G-geometrią natomiast nazywa pewną kategorię związaną z grupą Liego G. Wydaje się, że określenie geometrii jako pewnej kategorii jest zgodne z oryginalną definicją Kleina. Niezmienniki, o których mówi definicja Kleina, są po prostu morfizmami odpowiedniej kategorii.

E.J. Jasińska i M. Kucharzewski w pracy [4] G-geometrią nazwali efektywny obiekt abstrakcyjny (M,G,f). W dalszych swych publikacjach M. Kucharzewski (p. [12], [13]), opierając się na pewnych ideach, zawartych w pracach R. Sulanke (p. [21], [22]), pojęcia przestrzeni Kleina, obiektu geometrycznego i geometrii określił tak, jak to przedstawiono w §2 rozdziału I niniejszej pracy. Definicje tych po-

jęć budzą jednak pewne zastrzeżenia (por. §5, rozdz. I). Podstawowym mankamentem w definicji obiektu geometrycznego jest brak związku między włóknem obiektu a włóknem przestrzeni oraz brak zależności między prawami transofrmacji obiektu i przestrzeni. Ponadto określenie geometrii jako kategorii obiektów geometrycznych prowadzi do tego, że niektóre nierównoważne przestrzenie Kleina posiadają tę samą geometrię.

Celem tej pracy jest uściślenie niektórych pojęć teorii przestrzeni Kleina i podanie pewnych ich własności. Rozdział I ma charakter wstępny. Omówiono w nim podstawowe pojęcia, niezbędne do zrozumienia dalszej części pracy.

W rozdziale II podano nowe definicje obiektu geometrycznego i geometrii Kleina. Wprowadzono w nim również niezdefiniowane dotychczas pojęcie równoważności dwóch geometrii Kleina oraz wykazano warunek konieczny i dostateczny na to, aby dwie geometrie były równoważne.

Rozdział III poświęcony jest metodom konstrukcji obiektów geometrycznych. Określono w nim dwa nowe obiekty, a mianowicie obiekt odwzorowań oraz sumę rozłączną obiektów. Wykazano również, że obiekty odwzorowań, obiekty ilorazowe, a także G-produkty i sumy rozłączne obiektów geometrycznych danej prz-strzeni Kleina są obiektami geometrycznymi tej przestrzeni. Podano także pewne warunki konieczne i dostateczne na to, aby obiekty kategorii obiektów abstrakcyjnych, opartych na tej samej grupie, były obiektami geometrycznymi odpowiedniej przestrzeni Kleina.

Uzyskane rezultaty zilustrowano w rozdziale IV na przykładach elementarnych przestrzeni Kleina, takich jak przestrzeń wektorowa, unitarna, afiniczna i euklidesowa. Wykorzystując pojęcie obiektu odwzorowań podano definicje tensorów i gęstości tensorowych w nowym ujęciu.

ОСНОВНЫЕ ПОНЯТИЯ ГЕОМЕТРИИ КЛЕЙНА

Р е з ю м е

Хотя знаменитая "Эрлангенская программа" была изложена Клейном уже сто лет тому назад, до этих пор она не использована, поэтому что она не достаточно точная.

Оригинальное определение геометрии изложено Клейном (см. [6]) можно представить следующим образом: Геометрия множества M относительно группы преобразований G(M) этого множества, или коротко G(M)-геометрия, это множество всех свойств геометрических фигур, которые не изменяются при преобразованиях группы G(M). Эти свойства называем инвариантами или геометрическими свойствами.

Когда возникла необходимость рассматривания геометрий опирающихся на множествах преобразований не обязательно групп, пространства с группой преобразований начато называть пространствами Клейна.

Сулянке Р. пространством Клейна называет (см. [21], [22]) транзитивную, левостороннюю группу Ли преобразований, т.е. тройку (M,G,f) где M-многообразие, G - группа Ли, f - транзитивное, левостороннее действие группы G на M. G-геометрия в свою очередь это некоторая категория связана с группой Ли. Определение геометрии как некоторой категории сходно с оригинальным определением Клейна, потому что инварианты это морфизмы соответствующей категории.

Ясинская Е., Кухажевский М. в работе [4] G-геометрией называют эффективный, абстрактный объект (M,G,f). В других работах Кухажевский М. (см. [12], [13]) используя некоторые идеи Сулянке Р. (см. [21], [22]), понятия пространства Клейна, геометрического объекта и геометрии определил

так, как представлено в §2 главы I этой работы. Эти определения возбуждают однако некоторые сомнения (см. §5 гл. II. Основным недостаткам определения геометрического объекта это отсутствие связи между расслоением объекта и расслоением пространства, а тоже отсутствие зависимости между законами трансформации объекта и пространства. Кроме того определение геометрии как категории геометрических объектов приводит к тому, что некоторые неэквивалентные пространства Клейна имеют одинаковую геометрию.

Целью этой работы является уточнение некоторых понятий теории пространств Клейна и подача некоторых их свойств.

В главе I изложены основные понятия необходимые для понятия дальнейшей части работы.

Глава II содержит новые определения геометрического объекта и геометрии Клейна. Представлено може в ней неопределено до сих пор понятие эквивалентности геометрии Клейна, а также доказано необходимое и достаточное условие эквивалентности двух геометрий.

В главе III рассмотрены методы построения геометрических объектов. Определено в ней два новые объекты т.е. объект отображений и прямую сумму объектов. Доказано тоже, что объекты отображений, фактор-объекты, G-произведения и прямые суммы геометрических объектов пространства Клейна это геометрические объекты этого пространства. Представлено тоже необходимые и достаточные условия того, что объекты категории абстрактных объектов являются геометрическими объектами соответствующего пространства Клейна.

Получены результаты проиллюстрировано в главе IV примерами элементарных пространств Клейна таких как векторное, унитарное, аффинное, евклидого пространства. Используя понятие объекта отображений представлено новые определения тензоров и тензорных плотности.