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THE THREE-DIMENSIONAL DIFFUSION EQUATION IN THE SPACE OF DISTRIBUTIONS

Summary. The solutions of the diffusion equation (5) that satisfy the conditions (6), (7), (8) have been considered in the paper.
The evidence for these solutions existence are enclosed in the papers [1] & [2].
They are of mathematical and physical character.

1. INTRODUCTION

The subject of this work is construction of the transport equation (5). The solutions of this equation satisfies the limit conditions (6), (7), (8). The proofs of the existence of the solutions of the equation are enclosed in [1], [2].
The results of this work are the generalization of the results enclosed in [1]. They have a mathematical and physical character.

The continuity equation (expressing the mass conservation law in physics) for the flux $\vec{\phi} = -\nabla p$ has the form

$$\forall V : \int_V \frac{\partial p}{\partial t}\, dV = \int_\Sigma \nabla p \cdot d\vec{\Sigma}\,, \quad x \in R^3, \; t \in R^+$$

$$p : R^3 \times R^+ \supset \Omega\,(x,\ t) \longrightarrow R^+ \tag{1}$$

Σ is the surface of the volume V (V is the open domain), $d\vec{\Sigma} = \vec{n}\, d\Sigma$, $\vec{n}$ is the external normal ((1) is true for V dependent also on t).

If we assume that $p \in C^2$ with respect to x and $p \in C^1$ with respect to t, then we have from the Gauss-Green theorem

$$\int_V \frac{\partial p}{\partial t}\,dV = \int_V \Delta p\,dV \tag{2}$$

Hence

$$\frac{\partial p}{\partial t} = \Delta p \tag{3}$$

The integrals in (1) have sense for p satisfying the Lipschitz condition with respect to x and t.
(The assumption can be weakened).

2. FUNDAMENTAL THEOREM ON THE DISTRIBUTION FORM OF THE EQUATION (1)

We shall prove that the distribution form of equation (1) is following:

$$\frac{\partial p}{\partial t} - \Delta p = \sum_{j=1}^{n}\left(\left.\frac{\partial p}{\partial t}\right|_{\phi_j^+(x)} - \left.\frac{\partial p}{\partial t}\right|_{\phi_j^-(x)}\right)\chi_{\phi_j}(t) -$$

$$-\sum_{j=1}^{n}\sum_{i=1}^{3}\left(\left.\frac{\partial p}{\partial x_i}\right|_{\psi_{j,i}^+(t)} - \left.\frac{\partial p}{\partial x_i}\right|_{\psi_{j,i}^-(t)}\right)\delta^i_{\psi_{j,i}}(t)$$

Assumptions: Let p satisfy the Lipschitz condition with respect to x and t. We assume then that the set S of the points of discontinuity of the derivatives $\frac{\partial p}{\partial x_i}$ is a finite family of smoth manifolds $S = S_j$, $j = 1,\dots,r$, with with dimensions 3, 2, 1 in the four-dimensional space $x \in R^3$, $t \in R^+$. (These manifolds from the physical point of view are the results of moving points and lines and surfaces in the three-dimensional space $x \in R^3$, S_j may be well independent of t). Let p be the function generating the distribution **p**. Let's assume that V_k are the volumes which do not contain the elements which are the subsets of S. We choose these V_k, $k = 1,2$, **which** are neighbouring with the common surface P to each other.

(The surface P is the common closure of the open domains V). Generally the surface P is the function of the time t, therefore the volume V_k is the function of the time too. There are the subsets S_j, $j = 1,\dots n$, belonging to S on their common surface P, $n \le r$.

Proof. Let $W = V_1 \cup V_2 \cup P$, $P \supset \bigcup_{j=1}^{n} S_j$

We write (2) as:

$$\forall W: \sum_{k=1}^{2} \int_{V_k} \frac{\partial p}{\partial t}\, dV + \int_W \sum_{j=1}^{n} \left(\frac{\partial p}{\partial t}\Big|_{\phi_j^+(x)} - \frac{\partial p}{\partial t}\Big|_{\phi_j^-(x)} \right) \chi_{\phi_j(x)} dV -$$

$$- \int_W \sum_{j=1}^{n} \left(\frac{\partial p}{\partial t}\Big|_{\phi_j^+(x)} - \frac{\partial p}{\partial t}\Big|_{\phi_j^-(x)} \right) \chi_{\phi_j(x)} dV = \sum_{k=1}^{2} \int_{V_k} \Delta p\, dV +$$

$$+ \int_W \sum_{j=1}^{n} \sum_{i=1}^{3} \left(\frac{\partial p}{\partial x_i}\Big|_{\psi_{j,i}^+(t)} - \frac{\partial p}{\partial x_i}\Big|_{\psi_{j,i}^-(t)} \right) \delta^i_{\psi_{j,i}(t)} dV - \tag{4}$$

$$- \int_W \sum_{j=1}^{n} \sum_{i=1}^{3} \left(\frac{\partial p}{\partial x_i}\Big|_{\psi_{j,i}^+(t)} - \frac{\partial p}{\partial x_i}\Big|_{\psi_{j,i}^-(t)} \right) \delta^i_{\psi_{j,i}(t)} dV$$

where δ^i, δ - distribution associate with x_i,

$$\chi_{\phi_j(x)} = \begin{cases} 1, & (x,t) \in S_j \\ 0, & (x,t) \notin S_j \end{cases}$$

$\phi_j(x)$ denotes this value of t for which $(x,t) \in S_j$ and
$\psi_{j,i}(t)$ denotes this value of x_i for which $(x,t) \in S_j$,
$j = 1,2,3$.

The sums of the first end the second integrals of the left side as well as on the right side of (4) are suitable derivatives in the distribution sense. Therefore we may write the equation of diffusion

$$\frac{\partial p}{\partial t} - \Delta p = \sum_{j=1}^{n} \left(\frac{\partial p}{\partial t}\bigg|_{\phi_j^+(x)} - \frac{\partial p}{\partial t}\bigg|_{\phi_j^-(x)} \right) \chi_{\phi_j(x)} dV -$$

$$- \sum_{j=1}^{n} \sum_{i=1}^{3} \left(\frac{\partial p}{\partial x_i}\bigg|_{\psi_{j,i}^+(t)} - \frac{\partial p}{\partial x_i}\bigg|_{\psi_{j,i}^-(t)} \right) \delta^i_{\psi_{j,i}(t)}$$

We are looking the solution of (5) in the domain $\Omega(x,t)$ with the limit conditions:

$$p(x,0) = \varphi(x), \ \varphi \text{ satisfying the Lipschitz condition} \tag{6}$$

$$p(x,t) = 0, \ (x,t) \in \partial\,\Omega, \tag{7}$$

$$\int_{\Omega(x,t)\cap\{t\}} p(x,t)dx = \text{const} \tag{8}$$

The set S is given, $\Omega(x,t) \in R^3 \times R^+$. (On the left side without misunderstanding we write p instead of p).

REFERENCES

[1] E. Bobula: Derivation of the diffusion equation in the space of distributions, Prace Mat., ZN Uniw. Jagiell., z. 22, 1981.

[2] E. Bobula: On Some Solution of Three-dimensional Diffusion Equation (to appear).

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TRÓJWYMIAROWE RÓWNANIE DYFUZJI W PRZESTRZENI DYSTRYBUCJI

S t r e s z c z e n i e

W pracy rozważano rozwiązanie równania dyfuzji, które spełniają warunki (6), (7), (8). Dowody istnienia tych rozwiązań są zawarte w pracach [1] i [2].
Wyniki tej pracy stanowią uogólnienia rezultatów zawartych w [1].
Mają one matematyczny i fizyczny charakter.

ТРЕХМЕРНОЕ УРАВНЕНИЕ ДИФФУЗИИ В ПРОСТРАНСТВЕ ОБОБЩЕННЫХ ФУНКЦИЙ

Р е з ю м е

В работе рассмотрим решения уравнения диффузии (5), которые удовлетворяют (6), (7), (8). Доказательства существования этих решений помещены в работах [1] и [2].
Результаты настоящей работы являются обобщениями результатов, помещенных в [1].
Они имеют математический и физический характер.