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EXTERIOR FORMS IN n-DIMENSIONAL PREMANIFOLDS

Summary. The concept of an exterior differential form in n-dimensional premanifold has been introduced in the paper. The equivalence of a point - wise definition of an exterior differential form with a global one has been proved. The formula $d \circ d = 0$ which is fundamential for de Rham cohomology theory has been obtained as well.

0. PRELIMINARIES

We will consider a real or a complex (see [4],[5]) premanifold M such that its topology topM will be T_o- topology (see [4], [5], [2]). The domain of any function f will be denoted by D_f. For any functions f and g with values in a vector space V the function f+g will be always meant as the one having the domain $D_f \cap D_g$ such that $(f+g)(p) = f(p) + g(p)$ for $p \in D_f \cap D_g$. It may happen then $D_{f+g} = \phi$. If f is a real (complex) function, g is a function such that g(p) is a point of real (complex) vector space V_p for $p \in D_g$, then we have the function $f \cdot g$ with the domain $D_f \cap D_g$ and defined by usual formula $(f \cdot g)(p) = f(p) \cdot g(p)$ for $p \in D_f \cap D_g$. We will assume in the paper that M is n-dimensional (see [2]), i.e. every point p of M has a neighbourhood $U \in$ topM and a system $(e_1,\dots,e_n)$ of smooth vector fields in M defined on U such that: 1) $(e_1(p),\dots,e_n(p))$ is a base for the tangent space T_pM to M at the point p; 2) for any smooth vector field X defined in a neighbourhood D_X of p there exist $\alpha^1,\dots,\alpha^n \in M$ such that

$$X(q) = \alpha^i(q)e_i(q) \qquad \text{for} \qquad q \in U \cap D_X . \tag{0.0}$$

Such a system $(e_1,\dots,e_n)$ is said to be a local base for M on U. We remark that the set of all local vector fields together with the addition is not a group.

1. EXTERIOR FORMS

For any point p of M and any natural number k we have k-th exterior power $\Lambda^k(T_pM)^*$ of the dual space $(T_pM)^*$ to T_pM. The function ω with the domain $D_\omega \in$ topM such that $\omega(p)$ is an element of $\Lambda^k(T_pM)^*$ is said to be an exterior k-form in M. Assume that M is n-dimensional and $0 \le k \le n$. Then we have the duality $\langle\cdot|\cdot\rangle$ between $\Lambda^k T_pM$ and $\Lambda^k(T_pM)^*$ given by the formula $\langle v_1 \wedge \dots \wedge v_k | w_1 \wedge \dots \wedge w_k\rangle = \det[w_i(v_j);\ i,j \le k]$ for $v_1,\dots,v_k$ in T_pM and $w_1,\dots,w_k$ in $(T_pM)^*$. For any local vector fields $X_1,\dots,X_k$ and any point $p \in D_\omega \cap D_{X_1} \cap \dots \cap D_{X_k}$ we set

$$\omega'(X_1,\dots,X_k)(p) = \langle X_1(p)\wedge \dots \wedge X_k(p)|\omega(p)\rangle. \tag{1.1}$$

We have thus the function $\omega'(X_1,\dots,X_k)$ with

$$D_{\omega'}(X_1,\dots,X_k) = D_\omega \cap D_{X_1} \cap \dots \cap D_{X_k} . \tag{1.2}$$

We wil say that ω is smooth iff for any local smooth vector fields $X_1,\dots,X_k$ the function $\omega'(X_1,\dots,X_k)$ belongs to M.
The set of all smooth exterior k-forms in M will be denoted by $A^k(M)$.

Consider the set $A'^k(M)$ of all functions η satisfying the following conditions:

(i) D_η is the set of all k-tuples $(X_1,\dots,X_k)$ of smooth local vector fields in M and $\eta(X_1,\dots,X_k) \in M$ for $(X_1,\dots,X_k) \in D_\eta$;

(ii) for any smooth local vector fields $X_1,\dots,X_k,X$ in M and any $\alpha \in M$ we have

$$\eta(X_1,\dots,X_i + \alpha X,\dots,X_k) = \eta(X_1,\dots,X_k) +$$
$$+ \alpha \cdot \eta(X_1,\dots,X_{i-1},X,X_{i+1},\dots,X_k);$$

(iii) for any smooth local vector fields $X_1,\dots,X_k,Y_1,\dots,Y_k$ in M such that $D_{X_1} = D_{Y_1},\dots,D_{X_k} = D_{Y_k}$ we have

$$D_\eta(X_1,\dots,X_k) = D_\eta(Y_1,\dots,Y_k);$$

(iv) η is skew-symmetric.

1.1. For any $\eta \in A'^{k}(M)$ and any smooth local vector fields $X_1,\dots,X_k$ we have

$$D_{\eta(X_1,\dots,X_k)} = D_{X_1} \cap D_{X_k} \cap D_{\eta(0,\dots,0)}, \tag{1.3}$$

where 0 is zero vector field on the set of all points of M.
If $U \subset D_{\eta(X_1,\dots,X_k)}$ then

$$\eta(X_1|U,\dots,X_k|U) = \eta(X_1,\dots,X_k)|U. \tag{1.4}$$

Proof. Let 0_i be the zero vector field on D_{X_i}, $i = 1,\dots,k$.
From (iii) it follows that $D_{\eta(X_1,\dots,X_k)} = D_{\eta(0_1,\dots,0_k)}$.
On the other hand, we have $0_i = \alpha_i \cdot 0$ where $D_{\alpha_i} = D_{X_i}$ and $\alpha_i(p) = 1$ for $p \in D_{\alpha_i}$. Hence, by (ii), it follows that
$\eta(0_1,\dots,0_k) = \alpha_1\dots\alpha_k\ \eta\ (0,\dots,0)$. Thus, $D_{\eta(0_1,\dots,0_k)} =$

$= D_{\alpha_1} \cap \dots \cap D_{\alpha_k} \cap D_{\eta(0,\dots,0)} = D_{X_1} \cap \dots \cap D_{X_k} \cap D_{\eta(0,\dots,0)}$.

Similarly, (ii) yields $\eta(X_1|U,\dots,X_k|U) = \eta(\alpha \cdot X_1,\dots,\alpha \cdot X_k) =$

$= \alpha \cdot \eta(X_1,\dots,X_k) = \eta(X_1,\dots,X_k)|U$, where $D_\alpha = U$ and $\alpha(p) = 1$

for $p \in U$. Q.E.D.

The set $\eta(0,\dots,0)$ will be called the natural domain of η.

1.2. If M is ans n-dimensional premanifold, $\eta \in A'^{k}(M)$, $X_1,\dots,X_k$ are smooth local vector fields in M, $p \in D_{\eta(X_1,\dots,X_k)}$ and $X_h(p) = 0$ for some h, then $\eta(X_1,\dots,X_k)(p) = 0$.

Proof. Let $(e_1,\dots,e_k)$ be a local base for M on U, where U is a neighbourhood of the point p. We may assume that $U \subset D_{\eta(X_1,\dots,X_k)}$.

According to 1.1 we have $U \subset D_{X_1} \cap \dots \cap D_{X_k}$ and $X_h|U = \alpha^i e_i$ for some

$\alpha^1,\dots,\alpha^n \in M$. Hence it follows that $\eta(X_1,\dots,X_k)|U =$

$= \eta(X_1|U,\dots,X_{h-1}|U,\ \alpha^i e_i, X_{h+1}|U,\dots,X_k|U) =$

$= \alpha^i \eta(X_1,\dots,X_{h-1},e_i,X_{h+1},\dots,X_k)|U$ and $0 = X_h(p) = \alpha^i(p)e_i(p)$.

Therefore $\alpha^i(p) = 0$, $i = 1,\dots,k$. Q.E.D.

Let M be an n-dimensional premanifold and $\eta \in A'^k(M)$. Proposition 1.2 allowe us to accept the following definition of the k-form η^o:

$$\langle v_1 \wedge \dots \wedge v_k | \eta^o(p)\rangle = \eta(X_1,\dots,X_k)(p), \tag{1.5}$$

when $v_j = X_j(p)$, X_j is a smooth local vector field in M, $p \in D_{\eta(X_1,\dots,X_k)}$.

Indeed, if $Y_1,\dots,Y_k$ are also smooth vector fields with $p \in D_{\eta(Y_1,\dots,Y_k)}$

and $v_j = Y_j(p)$, then according to 1.2 we get

$(\eta(X_1,\dots,X_k) - \eta\ (Y_1,X_2,\dots,X_k))(p) = \eta\ (X_1 - Y_1,\ X_2,\dots,X_k)(p) = 0$

and subsequently $\eta(X_1,\dots,X_k)(p) = \eta\ (Y_1,Y_2,X_3,\dots,X_k)(p) = \dots =$

$= \eta\ (Y_1,\dots,Y_k)(p)$.

From (iv) it follows that there exists the only element $\eta^o(p)$ in $\Lambda^k(T_pM)^*$ such that (1.5) holds.

1.3. For any n-dimensional premanifold M we have a one-one correspondence

$$\omega \longmapsto \omega' : A^k(M) \longrightarrow A'^k(M).$$

Moreover, $(\alpha \cdot \omega)' = \alpha \cdot \omega'$ and $(\omega_1 + \omega_2)' = \omega_1' + \omega_2'$ for $\omega, \omega_1, \omega_2 \in A^k(M)$ and $\alpha \in M$.

Proof. Let us take any $\eta \in A'^k(M)$. Then for any smooth local vector fields $X_1,\dots,X_k$, according to (1.5) and (1.1), we have $\eta^{o'}(X_1,\dots,X_k) =$ $= \eta(X_1,\dots,X_k)$. Thus, $\eta^{o'} = \eta$. Now, assume that $\omega, \omega_1 \in A^k(M)$ and $\omega' = \omega_1'$.

According to (1.2) we have $D_\omega = D_{\omega',(0,\dots,0)}$ and, similarly, $D_{\omega_1} = D_{\omega_1',(0,\dots,0)}$.

Hence it follows that $D_\omega = D_{\omega_1}$ and, by (1.1),

$\langle e_{i_1}(p) \wedge \dots \wedge e_{i_k}(p) | \omega(p)\rangle = \langle e_{i_1}(p) \wedge \dots \wedge e_{i_k}(p) | \omega_1(p)\rangle$ for $p \in D_\omega$,

$1 \le i_1 < \dots < i_k \le n$, where $(e_1,\dots,e_n)$ is a local base for M on a neighbourhood of the point p. This yields $\omega = \omega_1$. Q.E.D.

Proposition 1.3 allows us not distinguish ω and ω'. We will often write in the sequel $\omega(X_1,\dots,X_k)$ instead of $\omega'(X_1,\dots,X_k)$.

2. LIE BRACKETS OF VECTOR FIELDS

Let X and Y be smooth local vector fields in a premanifold M. For any $p \in D_X \cap D_Y$ and any $\alpha \in M$ such that $p \in D_\alpha$ we set

$$[X,Y](p)(\alpha) = X(p)(\partial_Y\alpha) - Y(p)(\partial_X\alpha), \tag{2.1}$$

where for any $\alpha \in M$ and $q \in D_\alpha \cap D_X$ we write

$$(\partial_X\alpha)(q) = X(q)(\alpha). \tag{2.2}$$

We have thus define the smooth local vector field [X,Y], called Lie brackets of X and Y, in M with the domain $D_X \cap D_Y$.
From (2.1) and (2.2) it follows that for any smooth local vector fields X,Y,Z and any c in $\mathbb{R}$ when M is a real premanifold (in $\mathbb{C}$ when M is a complex one) we have

$$[X + cY, Z] = [X,Z] + c[Y,Z],$$

$$[X,Y] + [Y,X] = 0,$$

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0.$$

The above properties of Lie brackets allows us to consider the exterior diffferential for all elements of $A'^{k}(M)$, and next, under the hypothesis that M is an n-dimensional premanifold for all ω belonging to $A^{k}(M)$.

3. EXTERIOR DIFFERENTIAL

For any $\eta \in A'^{K}(M)$ and any smooth local vector fields $X_o,\dots,X_k$ we set

$$d\eta(X_o,\dots,X_k) = \sum_{i=0}^{k}(-1)^{i}\,\partial_{X_i}\omega(X_1,\dots,X_{i-1},X_{i+1},\dots,X_k) + \\ + \sum_{0\le i<j\le k}(-1)^{i+j}\omega([X_i,X_j],X_o,\dots,X_{i-1},X_{i+1},\dots,X_{j-1},X_{j+1},\dots,X_k). \tag{3.1}$$

3.1. For any $\eta \in A'^{k}(M)$ we have $d\eta \in A'^{k+1}(M)$, the natural domain of $d\eta$ coincides with the natural domain of η and

$$d \circ d = 0, \tag{3.2}$$

i.e. for any smooth local vector fields $X_o,\dots,X_{k+1}$ we have

$$dd\eta(X_o,\dots,X_{k+1})(q) = 0 \tag{3.3}$$

for q in the natural domain of η.

Proof. By an easy verification we state that $d\eta \in A'^{k+1}$ and $D_{d\eta(0,\dots,0)} = D_{\eta(0,\dots,0)}$. To prove (3.2) we take arbitrary smooth local vector fields $X_o,\dots X_{k+1}$ and set

$$U = D_{X_o} \cap \dots \cap D_{X_{k+1}} \cap D_{\eta(0,\dots,0)} .$$

From proposition 1.1 we get $dd_{\eta}(X_o,\dots,X_{k+1}) = dd_{\eta}(X_o|U,\dots,X_{k+1}|U)$. Let us remark that any set $U \in topM$ the set of all smooth local vector fields X such that $D_X = U$ is a group with respect to addition. Hence (see [3] and also [1]) we get (3.3). Q.E.D.

3.2. If M is an n-dimensional premanifold, then there exists exactly one operation d which to eah $\omega \in A^k(M)$ assigns $d\omega \in A^{k+1}(M)$ in such a way that for any smooth local vector fields $X_o,\dots,X_k$ we have

$$(d\omega)'(X_o,\dots,X_k) = d\omega'(X_o,\dots,X_k). \tag{3.4}$$

For each $\omega \in A^k(M)$ we have $D_{d\omega} = D_\omega$ and

$$dd\omega = 0 . \tag{3.5}$$

Proof. It sufficies to reffer to 1.3 and 3.1. Q.E.D.

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FORMY ZEWNĘTRZNE NA n-WYMIAROWEJ PREROZMAITOŚCI

S t r e s z c z e n i e

W pracy wprowadzono pojęcie formy zewnętrznej na n-wymiarowej prerozmaitości. Udowodniono równoważność punktowej i globalnej definicji formy zewnętrznej. Udowodniono też podstawowy dla teorii kohomologii de Rhama prerozmaitości skończenie wymiarowych wzór $d \circ d = 0$.

ВНЕШНИЕ ФОРМЫ НА n-МЕРНЫХ ПРЕМНОГООБРАЗИЯХ

Р е з ю м е

В работе вводится понятие внешней формы на n-мерном премнорообразии. Показывается равносильность точечного и глобального определений внешней формы. Доказывается также основная для теории когомологии де Рама конечномерных премногообразий формула $d \circ d = 0$