ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1992
Seria: MATEMATYKA-FIZYKA z. 67 Nr kol. 1115

Janina MACURA

VARIATIONAL METHOD IN THE CLASS S_c

Summary. In this paper the family S_c of all functions analytic, univalent in the unit disk and satysfying conditions: $f(0)=0$, $f'(c)=1$ for fixed c from the unit disk is considered.
Adapting the variational method, used by A. Chang, M.M. Schiffer and G. Schober in [2], formulas for the first and the second variation for the family S_c were led out. These formulas make possible the consideration of the problem of finding $\max_{g\in S_c} \mathrm{Re}\{L(g)\}$, where L is a linear and continuous functional defined on the space of all functions analytic in the unit disk. The first variation gave the Schiffer equation, the second one was used to prove that the right side of this equation is nonpositive on the unit circle. Some conditions satosfied bythe function realizing maximum of the functional Re{K} were led out.
Using the boundary variation, like P. Duren in [3] for the family S' it is proved that the image of the unit disk through the function realizing the maximum of the Re{L} is the complement of a single analytic arc.

Let H(U) denote the space of all functions analytic in the unit disk $U = \{z: |z| < 1\}$ with the topology of uniform convergence on compact subsets of the disk U, H'(U) - the space of all linear and continuous functionals defined on H(U). Let S_c be the family of all univalent functions f from H(U) satisfying conditions: $f(0) = 0$ and $f'(c) = 1$ for fixed c from the disk U. This family was introduced by W. Janowski in [4].

At first we shall prove the following theorem:

Theorem 1. Let f be an arbitrary function from the family S_c. Then for real ε lying sufficiently near 0 to S_c belongs the function $\tilde{f}$ of the form

$$\tilde f(z) = f(z) + \varepsilon \tilde f_1(z) + \varepsilon^2 \tilde f_2(z) + o(\varepsilon^2) , \qquad (1)$$

where

$$\tilde f_1(z) = e^{i\alpha}\frac{f^2(\zeta)}{\varphi^2(\zeta)}\left\{\frac{\varphi^2(\zeta)}{f(\zeta)}\,\frac{\{[d-2f(\zeta)]d+f(\zeta)f(z)\}f(z)}{[d-f(\zeta)]^2[f(z)-f(\zeta)]} - \right.$$

$$\left. - \frac{1}{2}\varphi(z)\frac{z+\zeta}{z-\zeta} + \frac{1}{2} f(z)\frac{d}{dz}\left[\varphi(z)\frac{z+\zeta}{z-\zeta}\right]_{z=c}\right\} +$$

$$+ e^{-i\alpha}\frac{\overline{f^2(\zeta)}}{\overline{\varphi^2(\zeta)}}\left\{\frac{1}{2}\varphi(z)\frac{1+\bar\zeta z}{1-\bar\zeta z} - \frac{1}{2} f(z)\frac{d}{dz}\left[\varphi(z)\frac{1+\bar\zeta z}{1-\bar\zeta z}\right]_{z=c}\right\},$$

$$\tilde f_2(z) =$$

$$= e^{2i\alpha}\frac{f^4(\zeta)}{\varphi^4(\zeta)}\left\{\frac{1}{2}\,\frac{\zeta\varphi^2(\zeta)\varphi'(\zeta)}{f(\zeta)[d-f(\zeta)]^2}\,\frac{[f(z)-d]^2 f(z)}{[f(z)-f(\zeta)]^2} + \right.$$

$$+ \frac{\varphi^2(\zeta)}{f(\zeta)}\left[\frac{1}{2}\,\frac{\zeta^2\varphi'^2(\zeta)}{\varphi^2(\zeta)} - \frac{1}{6}\zeta^2\{f(\zeta),\zeta\} - \frac{1}{2}\,\frac{\zeta\varphi'(\zeta)}{f(\zeta)} + \frac{\zeta\varphi'(\zeta)}{d-f(\zeta)} + \right.$$

$$\left. + \frac{\varphi^2(\zeta)}{[d-f(\zeta)]^2}\right]\frac{\{[d-2f(\zeta)]d+f(\zeta)f(z)\}f(z)}{[d-f(\zeta)]^2[f(z)-f(\zeta)]} -$$

$$- \frac{1}{2}\,\frac{\varphi^2(\zeta)}{[d-f(\zeta)]^2}\left[\frac{f(z)\varphi(z)}{f(z)-f(\zeta)} - \frac{\{[d-2f(\zeta)]d+f(\zeta)f(z)\}\varphi(z)}{[f(z)-f(\zeta)]^2}\right]\cdot\frac{z+\zeta}{z-\zeta} +$$

$$+ \frac{1}{8} z \frac{d}{dz}\left[\varphi(z)\frac{(z+\zeta)^2}{(z-\zeta)^2}\right] +$$

$$+ \frac{1}{2}\,\frac{\varphi^2(\zeta)}{f(\zeta)}\,\frac{d}{dz}\left[\varphi(z)\frac{z+\zeta}{z-\zeta}\right]_{z=c}\frac{\{[d-2f(\zeta)]d+f(\zeta)f(z)\}f(z)}{[d-f(\zeta)]^2[f(z)-f(\zeta)]} -$$

$$- \frac{1}{4}\,\frac{d}{dz}\left[\varphi(z)\frac{z+\zeta}{z-\zeta}\right]_{z=c}\varphi(z)\frac{z+\zeta}{z-\zeta} + \left[\frac{1}{4}\left[\frac{d}{dz}\left[\varphi(z)\frac{z+\zeta}{z-\zeta}\right]_{z=c}\right]^2 +\right.$$

$$+\frac{\varphi^2(\zeta)c}{[d-f(\zeta)]^3}\frac{c+\zeta}{c-\zeta} - \frac{1}{8}\frac{d}{dz}\left[\varphi(z)\frac{(z+\zeta)^2}{(z-\zeta)^2}\right]_{z=c} -$$

$$-\frac{1}{8} c\frac{d^2}{dz^2}\left[\varphi(z)\frac{(z+\zeta)^2}{(z-\zeta)^2}\right]_{z=c}\Bigg] f(z)\Bigg\} +$$

$$+\frac{|f(\zeta)|^4}{|\varphi(\zeta)|^4}\Bigg\{\frac{|\zeta|^2}{(1-|\zeta|^2)^2}\frac{\varphi^2(\zeta)}{f(\zeta)}\frac{\{[d-2f(\zeta)]d+f(\zeta)f(z)\}f(z)}{[d-f(\zeta)]^2[f(z)-f(\zeta)]}+$$

$$+\frac{1}{2}\frac{\varphi^2(\zeta)}{[d-f(\zeta)]^2}\left[\frac{f(z)\varphi(z)}{f(z)-f(\zeta)} - \frac{\{[d-2f(\zeta)]d+f(\zeta)f(z)\}\varphi(z)}{[f(z)-f(\zeta)]^2}\right]\frac{1+\bar\zeta z}{1-\bar\zeta z} -$$

$$-\frac{1}{4} z\frac{d}{dz}\left[\varphi(z)\frac{(z+\zeta)(1+\bar\zeta z)}{(z-\zeta)(1-\bar\zeta z)}\right] + \frac{1}{4}\frac{d}{dz}\left[\varphi(z)\frac{z+\zeta}{z-\zeta}\right]_{z=c}\varphi(z)\frac{1+\bar\zeta z}{1-\bar\zeta z} -$$

$$-\frac{1}{2}\frac{\varphi^2(\zeta)}{f(\zeta)}\frac{d}{dz}\left[\varphi(z)\frac{1+\bar\zeta z}{1-\bar\zeta z}\right]_{z=c}\frac{\{[d-2f(\zeta)]d+f(\zeta)f(z)\}f(z)}{[d-f(\zeta)]^2[f(z)-f(\zeta)]}+$$

$$+\frac{1}{4}\frac{d}{dz}\left[\varphi(z)\frac{1+\bar\zeta z}{1-\bar\zeta z}\right]_{z=c}\varphi(z)\frac{z+\zeta}{z-\zeta} + \Bigg[-\frac{\varphi^2(\zeta)c}{[d-f(\zeta)]^3}\frac{1+\bar\zeta c}{1-\bar\zeta c} -$$

$$-\frac{1}{2}\frac{d}{dz}\left[\varphi(z)\frac{z+\zeta}{z-\zeta}\right]_{z=c}\frac{d}{dz}\left[\varphi(z)\frac{1+\bar\zeta z}{1-\bar\zeta z}\right]_{z=c}+$$

$$+\frac{1}{4}\frac{d}{dz}\left[\varphi(z)\frac{(z+\zeta)(1+\bar\zeta z)}{(z-\zeta)(1-\bar\zeta z)}\right]_{z=c}+$$

$$+\frac{1}{4} c\frac{d^2}{dz^2}\left[\varphi(z)\frac{(z+\zeta)(1+\bar\zeta z)}{(z-\zeta)(1-\bar\zeta z)}\right]_{z=c}\Bigg] f(z)\Bigg\}+$$

$$+ e^{-2i\alpha} \frac{\overline{f^4(\zeta)}}{\overline{\varphi^4(\zeta)}} \left\{ \frac{1}{8} z \frac{d}{dz} \left[\varphi(z) \frac{(1+\bar{\zeta} z)^2}{(1-\bar{\zeta} z)^2} \right] - \right.$$

$$- \frac{1}{4} \frac{d}{dz} \left[\varphi(z) \frac{1+\bar{\zeta} z}{1-\bar{\zeta} z} \right]_{z=c} \varphi(z) \frac{1+\bar{\zeta} z}{1-\bar{\zeta} z} + \left[\frac{1}{4} \left[\frac{d}{dz} \left[\varphi(z) \frac{1+\bar{\zeta} z}{1-\bar{\zeta} z} \right]_{z=c} \right]^2 - \right.$$

$$\left. \left. - \frac{1}{8} \frac{d}{dz} \left[\varphi(z) \frac{(1+\bar{\zeta} z)^2}{(1-\bar{\zeta} z)^2} \right]_{z=c} - \frac{1}{8} c \frac{d^2}{dz^2} \left[\varphi(z) \frac{(1+\bar{\zeta} z)^2}{(1-\bar{\zeta} z)^2} \right]_{z=c} \right] f(z) \right\},$$

$\frac{o(\varepsilon^2)}{\varepsilon^2} \longrightarrow 0$, while $\varepsilon \longrightarrow 0$, uniformly on compact subsets of U, ζ is an arbitrary but fixed point of the disk U, α–an arbitrary real number, $\varphi(z)=zf'(z)$, $d=f(c)$, $\{f(\zeta),\zeta\}$ denotes the Schwarzian derivative.

PROOF. Let f be an arbitrary function from S_c. We shall assume additionally that f is analytic also on the boundary ∂U and let $D=f(U)$ $(0\in D)$ and $\Gamma=\partial D$. The curve Γ is, because of our assumption about f, the analytic Jordan curve. Using the function

$$w^*(w) = w + \varepsilon v_1(w) + \varepsilon^2 v_2(w) + \ldots, \tag{2}$$

where ε is an arbitrary real number and functions v_k $(k=1,2,\ldots)$ are defined and analytic in a neighbourhood of Γ, we deform the curve Γ. If ε lies sufficiently near 0, we obtain a new Jordan curve Γ^* which bounds domain D^* containing 0 . There exists, by Riemann mapping theorem, the function f^*,analytic and univalent in the disk U and satisfying condition $f^*(0)=0$. Function f^* depends on ε. M.M.Schiffer has proved in [5] that it is an analytic dependence and one can write

$$f^*(z) = f(z) + \varepsilon f_1(z) + \varepsilon^2 f_2(z) + o(\varepsilon^2), \tag{3}$$

where f_1 and f_2 are functions from H(U), $\frac{o(\varepsilon^2)}{\varepsilon^2} \longrightarrow 0$, while $\varepsilon \longrightarrow 0$, uniformly on compact subsets of U.

Functions f_1 and f_2 called respectively the first and the second variation can be effectively derived if one has concrete functions

v_1 and v_2.
Notice that near ∂U

$$z^*(z) = f^{*-1}[f(z) + \varepsilon v_1(f(z)) + \varepsilon^2 v_2(f(z)) + o(\varepsilon^2)] \tag{4}$$

is well defined analytic function and maps ∂U into itself.
Thus z^* has the representation

$$z^*(z) = z\cdot\exp(i(\varepsilon\psi_1(z) + \varepsilon^2\psi_2(z) + o(\varepsilon^2))),$$

where ψ_1 and ψ_2 are analytic functions in a neighbourhood of the disk U and real on ∂U. Then the superposition $f^*\circ z^*$ has the form

$$f^*(z^*(z)) = f(z) + \varepsilon\,(f_1(z) + i\psi_1(z)\varphi(z)) + \tag{5}$$

$$+ \varepsilon^2(f_2(z) + i\psi_1(z)zf_1'(z) + \frac{1}{2}z\varphi'(z)[i\psi_1(z)]^2 + i\psi_2(z)\varphi(z)) + o(\varepsilon^2)$$

Comparing coefficients in formulas (4) and (5) we obtain following conditions:

$$v_1(f(z)) = f_1(z) + i\psi_1(z)\varphi(z), \tag{6}$$

$$v_2(f(z)) = f_2(z) + i\psi_1(z)zf_1'(z) + \frac{1}{2}[i\psi_1(z)]^2 z\varphi'(z) + i\psi_2(z)\varphi(z) .$$

Using above formulas, functions f_1, f_2 and ψ_1, ψ_2 can be led out. We know that f_1, f_2 are analytic in the disk U and ψ_1, ψ_2 are analytic in a neighbourhood of ∂U and real on ∂U. We can also see, by the first formula (6), that for $z\in\partial U$

$$\mathrm{Re}\left\{\frac{f_1(z)}{\varphi(z)}\right\} = \mathrm{Re}\left\{\frac{v_1(f(z))}{\varphi(z)}\right\} . \tag{7}$$

The right side of this equation is known; from the form of the function f^* it follows that $f_1(0)=0$. It is enough to determine f_1.
Let, like in [4]

$$v_1(w) = e^{i\alpha}\frac{w^2}{w-w_0},$$

where w_0 is an arbitrary but fixed point of the domain D, α is an arbitrary real number.
Thus

$$v_1(f(z)) = e^{i\alpha}\frac{f^2(z)}{f(z)-f(\zeta)},$$

where ζ is an arbitrary but fixed point of the disk U.
The function f_1, analytic in the disk U and satisfying conditions: $f_1(0)=0$ and (7) on the circle ∂U has the form

$$f_1(z) = e^{i\alpha}\left\{\frac{f^2(z)}{f(z)-f(\zeta)} - \frac{1}{2}\frac{f^2(\zeta)}{\varphi^2(\zeta)}\varphi(z)\frac{z+\zeta}{z-\zeta}\right\} + \tag{8}$$

$$+\frac{1}{2}e^{-i\alpha}\frac{\overline{f^2(\zeta)}}{\overline{\varphi^2(\zeta)}}\varphi(z)\frac{1+\bar{\zeta}z}{1-\bar{\zeta}z}.$$

The function ψ_1 led out from the first of the conditions (6) is the function of the form

$$i\psi_1(z) = \frac{1}{2}e^{i\alpha}\frac{f^2(\zeta)}{\varphi^2(\zeta)}\frac{z+\zeta}{z-\zeta} - \frac{1}{2}e^{-i\alpha}\frac{\overline{f^2(\zeta)}}{\overline{\varphi^2(\zeta)}}\frac{1+\bar{\zeta}z}{1-\bar{\zeta}z}.$$

The second of the mentioned conditions (6) allows us to determine functions f_2 i ψ_2. We shall show that through suitable choice of the function v_2, we can achieve $\psi_2=0$. Now v_2 needs to be chosen in this way that the function

$$v_2(f(z)) - i\psi_1(z)zf_1'(z) - \frac{1}{2}[i\psi_1(z)]^2z\varphi'(z)$$

is analytic in $\bar{U}$. Then the function ψ_2 as a function which is analytic in $\bar{U}$ and real on ∂U has an analytic continuation to $\mathbb{C}$. Thus the function ψ_2, according to Liouville's theorem is a constant which can be chosen as 0. The second of the conditions (6) can be written in the form

$$v_2(f(z)) = f_2(z) + e^{2i\alpha}A(z) + B(z) + e^{-2i\alpha}C(z) + i\psi_2(z)\varphi(z), \tag{9}$$

where

$$A(z) = \frac{1}{2}\frac{f^2(\zeta)}{\varphi^2(\zeta)}\frac{[f(z)-2f(\zeta)]f(z)\varphi(z)}{[f(z)-f(\zeta)]^2}\frac{z+\zeta}{z-\zeta} -$$

$$-\frac{1}{8}\frac{f^4(\zeta)}{\varphi^4(\zeta)}z\frac{d}{dz}\left[\varphi(z)\frac{(z+\zeta)^2}{(z-\zeta)^2}\right],$$

$$B(z) = -\frac{1}{2}\frac{\overline{f^2(\zeta)}}{\overline{\varphi^2(\zeta)}}\frac{[f(z)-2f(\zeta)]f(z)\varphi(z)}{[f(z)-f(\zeta)]^2}\frac{1+\bar{\zeta}z}{1-\bar{\zeta}z} +$$

$$+\frac{1}{4}\frac{|f(\zeta)|^4}{|\varphi(\zeta)|^4}\, z\frac{d}{dz}\left[\varphi(z)\frac{(z+\zeta)(1+\bar{\zeta}z)}{(z-\zeta)(1-\bar{\zeta}z)}\right],$$

$$C(z) = -\frac{1}{8}\frac{\overline{f^4(\zeta)}}{\overline{\varphi^4(\zeta)}}\, z\frac{d}{dz}\left[\varphi(z)\frac{(1+\bar{\zeta}z)^2}{(1-\bar{\zeta}z)^2}\right].$$

The function C is analytic in the unit disk U, functions A and B are analytic in $U-\{\zeta\}$ and they have poles in the point ζ. We find their principal parts

$$A(z) = \frac{1}{2}\frac{f^4(\zeta)}{\varphi^4(\zeta)}\left\{\zeta^3\varphi'(\zeta)\frac{1}{(z-\zeta)^2} + \left[\zeta^2\varphi'(\zeta) - \right.\right.$$

$$\left.\left. -\frac{1}{3}\zeta^3\varphi(\zeta)\{f(\zeta),\zeta\} + 2\zeta\frac{\varphi^3(\zeta)}{f^2(\zeta)}\right]\frac{1}{z-\zeta}\right\} + \dots,$$

$$B(z) = \frac{|f(\zeta)|^4}{|\varphi(\zeta)|^4}\frac{\zeta|\zeta|^2}{(1-|\zeta|^2)^2}\varphi(\zeta)\frac{1}{z-\zeta} + \dots .$$

So we can see that v_2 needs to be chosen as

$$v_2(f(z)) =$$

$$= e^{2i\alpha}\frac{f^4(\zeta)}{\varphi^4(\zeta)}\left\{\frac{1}{2}\frac{\zeta\varphi^2(\zeta)\varphi'(\zeta)}{f(\zeta)[d-f(\zeta)]^2}\frac{[f(z)-d]^2 f(z)}{[f(z)-f(\zeta)]^2} + \right.$$

$$+\left[\frac{1}{2}\frac{\zeta^2\varphi'^2(\zeta)}{f^2(\zeta)} - \frac{1}{6}\frac{\zeta^2\varphi^2(\zeta)}{f^2(\zeta)}\{f(\zeta),\zeta\} + \frac{\varphi^4(\zeta)}{f^4(\zeta)} - \right.$$

$$\left.\left. -\frac{1}{2}\frac{\zeta\varphi^2(\zeta)\varphi'(\zeta)}{f^3(\zeta)} + \frac{\zeta\varphi^2(\zeta)\varphi'(\zeta)}{f^2(\zeta)[d-f(\zeta)]}\right]\frac{f^2(z)}{f(z)-f(\zeta)}\right\} +$$

$$+\ \frac{|f(\zeta)|^4}{|\varphi(\zeta)|^4}\ \frac{\varphi^2(\zeta)}{f^2(\zeta)}\ \frac{|\zeta|^2}{(1-|\zeta|^2)^2}\ \frac{f^2(z)}{f(z)-f(\zeta)}\ ,$$

where $d=f(c)$.

Thus considering (9) and $\psi_2=0$ we come to a conclusion that

$$f_2(z) = \tag{10}$$

$$= e^{2i\alpha}\ \frac{f^4(\zeta)}{\varphi^4(\zeta)} \left\{ \frac{1}{2}\ \frac{\zeta\varphi^2(\zeta)\varphi'(\zeta)}{f(\zeta)[d-f(\zeta)]^2}\ \frac{[f(z)-d]^2 f(z)}{[f(z)-f(\zeta)]^2} + \right.$$

$$+ \left[\frac{1}{2}\ \frac{\zeta^2\varphi'^2(\zeta)}{f^2(\zeta)} - \frac{1}{6}\ \frac{\zeta^2\varphi^2(\zeta)}{f^2(\zeta)}\ \{f(\zeta),\zeta\} + \frac{\varphi^4(\zeta)}{f^4(\zeta)} - \right.$$

$$\left. - \frac{1}{2}\ \frac{\zeta\varphi^2(\zeta)\varphi'(\zeta)}{f^3(\zeta)} + \frac{\zeta\varphi^2(\zeta)\varphi'(\zeta)}{f^2(\zeta)[d-f(\zeta)]} \right] \frac{f^2(z)}{f(z)-f(\zeta)} -$$

$$- \frac{1}{2}\ \frac{\varphi^2(\zeta)}{f^2(\zeta)}\ \frac{[f(z)-2f(\zeta)]f(z)\varphi(z)}{[f(z)-f(\zeta)]^2}\ \frac{z+\zeta}{z-\zeta} +$$

$$\left. + \frac{1}{8}\ z\ \frac{d}{dz} \left[\varphi(z)\ \frac{(z+\zeta)^2}{(z-\zeta)^2} \right] \right\} +$$

$$+ \frac{|f(\zeta)|^4}{|\varphi(\zeta)|^4} \left\{ \frac{\varphi^2(\zeta)}{f^2(\zeta)}\ \frac{|\zeta|^2}{(1-|\zeta|^2)^2}\ \frac{f^2(z)}{f(z)-f(\zeta)} + \right.$$

$$+ \frac{1}{2}\ \frac{\varphi^2(\zeta)}{f^2(\zeta)}\ \frac{[f(z)-2f(\zeta)]f(z)\varphi(z)}{[f(z)-f(\zeta)]^2}\ \frac{1+\bar\zeta z}{1-\bar\zeta z} -$$

$$\left. - \frac{1}{4}\ z\ \frac{d}{dz} \left[\varphi(z)\ \frac{(z+\zeta)(1+\bar\zeta z)}{(z-\zeta)(1-\bar\zeta z)} \right] \right\} +$$

$$+ e^{-2i\alpha}\ \frac{\overline{f^4(\zeta)}}{\overline{\varphi^4(\zeta)}}\ \frac{1}{8}\ z\ \frac{d}{dz} \left[\varphi(z)\ \frac{(1+\bar\zeta z)^2}{(1-\bar\zeta z)^2} \right].$$

The function f^*, given by formulas (3), (8) and (10) was received by the assumption that the function f is analytic also on the circle ∂U.

Because f_1 as well as f_2 depend only upon the functional behavior at the interior point ζ, we can extend this result by the uniform convergence to all functions from the class S_c.

The received variation f^* needn't satisfy the condition $f^{*\prime}(c)=1$, so we have to define a function $\tilde{f}$ which belongs to the class S_c:

$$\tilde{f} = \frac{f^*}{f^{*\prime}(c)} .$$

We can express the variation $\tilde{f}$ in the form (1).

Let L be a functional from the space H'(U) such that $L|_{S_c}$ is not constant. Applying the first and the second variation to the problem of finding the maximum of the functional Re{L} we shall achieve several conditions satisfied by a maximal function, that is a function realizing maximum of Re{L}.

We shall prove the following theorem:

THEOREM 2. Let L be a functional from the space H'(U) such that $L|_{S_c}$ is not constant. Then there is a function f where Re{L} attains its maximal value on S_c and it satisfies the differential equation

$$\frac{\varphi^2(\zeta)}{f(\zeta)} L\left\{ \frac{\{[d-2f(\zeta)]d+f(\zeta)f\}f}{[d-f(\zeta)]^2[f-f(\zeta)]} \right\} = q(\zeta), \tag{11}$$

where

$$q(\zeta) = \frac{1}{2}\left[L\left\{ \varphi\frac{z+\zeta}{z-\zeta} - f\frac{d}{dz}\left[\varphi\frac{z+\zeta}{z-\zeta}\right]_{z=c} \right\} - \overline{L\left\{ \varphi\frac{1+\bar{\zeta}z}{1-\bar{\zeta}z} - f\frac{d}{dz}\left[\varphi\frac{1+\bar{\zeta}z}{1-\bar{\zeta}z}\right]_{z=c} \right\}} \right],$$

$d=f(c)$, ζ is an arbitrary point from the ring $R=\{\ \zeta:\ r<|\zeta|<1\ \}$. r=

$=\max(r_3, |c|)$, $r_1<r_3<1$, ' r_1 is defined in the formula (14)
The function q is an analytic function of a variable ζ in R and has an analytic continuation in the ring $\{\ \zeta\colon r<|\zeta|<1/r\ \}$ and $q(\zeta)\le 0$ for $\zeta\in\partial U$.

PROOF. Since the class S_c is, as was proved by W.Janowski in [4], a compact normal family, there exists at least one function which maximizes Re{L}. Let f be such a function i.e. $\mathrm{Re}\{L(f)\}=\max_{g\in S_c}\mathrm{Re}\{L(g)\}$
Using the formula (3) for the maximal function f we get

$$\mathrm{Re}\{L(f_1)\} = 0. \tag{12}$$

The above equation, since α is an arbitrary real number, leads to

$$L\left\{ \frac{\varphi^2(\zeta)}{f(\zeta)}\ \frac{\{[d-2f(\zeta)]d+f(\zeta)f\}f}{[d-f(\zeta)]^2[f-f(\zeta)]} - \frac{1}{2}\varphi\frac{z+\zeta}{z-\zeta} + \frac{1}{2} f \frac{d}{dz}\left[\varphi(z)\frac{z+\zeta}{z-\zeta}\right]_{z=c}\right\} +$$

$$+\ \overline{L\left\{\frac{1}{2}\bar\varphi\frac{1+\bar\zeta z}{1-\bar\zeta z} - \frac{1}{2}\bar f\frac{d}{dz}\left[\varphi(z)\frac{1+\bar\zeta z}{1-\bar\zeta z}\right]_{z=c}\right\}} = 0. \tag{13}$$

According to the Caccioppoli-Köthe theorem about the general form of a functional from the space H'(U) ([6]), L has the form

$$L(h) = \frac{1}{2\pi i}\int_{|z|=r_1} h(z)\ g(z)\ dz\ , \tag{14}$$

where g is analytic in the domain $\{\ z : |z|>r_2\ \}$, $0<r_2<1$, $g(\infty)=0$, r_1 is an arbitrary but fixed number from $(r_2,1)$. So we can see that the left side of the (13) is an analytic function of a variable ζ in $\{\ \zeta\colon r_1<|\zeta|<1\ \}$.
The mentioned Caccioppoli-Köthe theorem shows us also that the functional L can be extended to all meromorphic functions having their poles in the ring $\{z\colon r_3<|z|<1\}$, where r_3 is an arbitrary number from the interval $(r_1,1)$. So we can write the equation (13) in the form (11).
It is easy to see that q is real on the circle ∂U. Furthermore, using again the Caccioppoli-Köthe theorem we draw the conclusion

that q is analytic at least in the ring $\{\zeta : r<|\zeta|<1/r\}$, where $r=\max\{r_3, |c|\}$.
In order to prove that the function q is nonpositive on ∂U, we shall use the second variation. We can write the function $\tilde{f}_2$ given by the third of the formulas (1) in the form

$$\tilde{f}_2(z) = e^{2i\alpha} A + B + e^{-2i\alpha} C$$

The maximality of f, (1), (12) and the above formula lead to

$$\mathrm{Re}\left\{ e^{2i\alpha} L(A) + L(B) + e^{-2i\alpha} L(C) \right\} \le 0 .$$

This inequality is valid for all real α, therefore for α such that

$$e^{2i\alpha} = \left|L(A)+\overline{L(C)}\right| \Big/ \left[L(A)+\overline{L(C)}\right].$$

we get:

$$\left|L(A)+\overline{L(C)}\right| \le - \mathrm{Re}\{L(B)\} ,$$

and finally

$$\left| \frac{1}{2} \frac{\zeta\varphi^2(\zeta)\varphi'(\zeta)}{f(\zeta)[d-f(\zeta)]^2} L\left(\frac{[f-d]^2 f}{[f-f(\zeta)]^2} \right) + \left[\frac{1}{2} \frac{\zeta^2\varphi'^2(\zeta)}{\varphi^2(\zeta)} - \right.\right. \tag{15}$$

$$- \frac{1}{6} \zeta^2\{f(\zeta),\zeta\} - \frac{1}{2} \frac{\zeta\varphi'(\zeta)}{f(\zeta)} + \frac{\zeta\varphi'(\zeta)}{d-f(\zeta)} + \left. \frac{\varphi^2(\zeta)}{[d-f(\zeta)]^2} \right] q(\zeta) -$$

$$- \frac{1}{2} \frac{\varphi^2(\zeta)}{[d-f(\zeta)]^2} L\left\{ \left[\frac{f\varphi}{f-f(\zeta)} - \frac{\{[d-2f(\zeta)]d+f(\zeta)f\}\varphi}{[f-f(\zeta)]^2} \right] \cdot \frac{z+\zeta}{z-\zeta} \right\} +$$

$$+ \frac{1}{8} L\left\{ z \frac{d}{dz} \left[\varphi \frac{(z+\zeta)^2}{(z-\zeta)^2} \right] \right\} + \frac{1}{2} \frac{d}{dz} \left[\varphi(z) \frac{z+\zeta}{z-\zeta} \right]_{z=c} q(\zeta) -$$

$$- \frac{1}{4} \frac{d}{dz} \left[\varphi(z) \frac{z+\zeta}{z-\zeta} \right]_{z=c} L\left(\varphi \frac{z+\zeta}{z-\zeta} \right) + \left[\frac{1}{4} \left[\frac{d}{dz} \left[\varphi(z) \frac{z+\zeta}{z-\zeta} \right]_{z=c} \right]^2 + \right.$$

$$+ \frac{\varphi^2(\zeta)c}{[d-f(\zeta)]^3} \frac{c+\zeta}{c-\zeta} - \frac{1}{8} \frac{d}{dz} \left[\varphi(z) \frac{(z+\zeta)^2}{(z-\zeta)^2} \right]_{z=c} -$$

$$- \frac{1}{8}\, c\, \frac{d^2}{dz^2}\left[\varphi(z)\frac{(z+\zeta)^2}{(z-\zeta)^2}\right]_{z=c}\Bigg] L(f) + \frac{1}{8}\, \overline{L\left\{z\frac{d}{dz}\left[\varphi\frac{(1+\bar{\zeta} z)^2}{(1-\bar{\zeta} z)^2}\right]\right\}} -$$

$$- \frac{1}{4}\, \overline{\frac{d}{dz}\left[\varphi(z)\frac{1+\bar{\zeta} z}{1-\bar{\zeta} z}\right]_{z=c}}\ \overline{L\left\{\varphi\frac{1+\bar{\zeta} z}{1-\bar{\zeta} z}\right\}} +$$

$$+ \Bigg[\frac{1}{4}\left[\overline{\frac{d}{dz}\left[\varphi(z)\frac{1+\bar{\zeta} z}{1-\bar{\zeta} z}\right]_{z=c}}\right]^2 - \frac{1}{8}\,\overline{\frac{d}{dz}\left[\varphi(z)\frac{(1+\bar{\zeta} z)^2}{(1-\bar{\zeta} z)^2}\right]_{z=c}} -$$

$$- \frac{1}{8}\,\bar{c}\,\overline{\frac{d^2}{dz^2}\left[\varphi(z)\frac{(1+\bar{\zeta} z)^2}{(1-\bar{\zeta} z)^2}\right]_{z=c}}\Bigg]\,\overline{L(f)}\ \Bigg| \le$$

$$\le -\,\mathrm{Re}\Bigg\{\frac{|\zeta|^2}{(1-|\zeta|^2)^2}\, q(\zeta) +$$

$$+ \frac{1}{2}\,\frac{\varphi^2(\zeta)}{[d-f(\zeta)]^2}\, L\left\{\left[\frac{f\varphi}{f-f(\zeta)} - \frac{\{[d-2f(\zeta)]d+f(\zeta)f\}\varphi}{[f-f(\zeta)]^2}\right]\frac{1+\bar{\zeta} z}{1-\bar{\zeta} z}\right\} -$$

$$- \frac{1}{4}\, L\left\{z\frac{d}{dz}\left[\varphi\frac{(z+\zeta)(1+\bar{\zeta} z)}{(z-\zeta)(1-\bar{\zeta} z)}\right]\right\} - \frac{1}{2}\,\frac{d}{dz}\left[\varphi(z)\frac{1+\bar{\zeta} z}{1-\bar{\zeta} z}\right]_{z=c} q(\zeta) +$$

$$+ \frac{1}{4}\,\frac{d}{dz}\left[\varphi(z)\frac{z+\zeta}{z-\zeta}\right]_{z=c} L\left\{\varphi\frac{1+\bar{\zeta} z}{1-\bar{\zeta} z}\right\} +$$

$$+ \frac{1}{4}\,\frac{d}{dz}\left[\varphi(z)\frac{1+\bar{\zeta} z}{1-\bar{\zeta} z}\right]_{z=c} L\left\{\varphi\frac{z+\zeta}{z-\zeta}\right\} + \Bigg[-\frac{\varphi^2(\zeta)c}{[d-f(\zeta)]^3}\,\frac{1+\bar{\zeta} c}{1-\bar{\zeta} c} -$$

$$- \frac{1}{2}\,\frac{d}{dz}\left[\varphi(z)\frac{z+\zeta}{z-\zeta}\right]_{z=c}\frac{d}{dz}\left[\varphi(z)\frac{1+\bar{\zeta} z}{1-\bar{\zeta} z}\right]_{z=c} +$$

$$+\frac{1}{4}\frac{d}{dz}\left[\varphi(z)\frac{(z+\zeta)(1+\bar{\zeta}z)}{(z-\zeta)(1-\bar{\zeta}z)}\right]_{z=c} +$$

$$+\frac{1}{4}c\frac{d^2}{dz^2}\left[\varphi(z)\frac{(z+\zeta)(1+\bar{\zeta}z)}{(z-\zeta)(1-\bar{\zeta}z)}\right]_{z=c}\Bigg] L(f)\Bigg\}.$$

Multiplying the above inequality by $(1-|\zeta|^2)^2$ and converging with $|\zeta|$ to 1 we obtain the result: $q(\zeta) \le 0$ for $\zeta \in \partial U$.

Converging with ζ to 0 in (13) and (15) and then with ζ to c in (13) and applying the additional variation:

$$f^{*}(z) = \frac{f(rz)}{rf'(rc)} \quad (\ 0<r<1\ )$$

we get the following corollary:

COROLLARY. Let L be a functional from the space $H'(U)$ such that $L|_{S_c}$ is not constant, $f(z) = b_1z + b_2z^2 + \ldots$ – the function where $\mathrm{Re}\{L\}$ attains its maximal value on S_c. Then

(i) $$L\Big[\ \varphi - \varphi'(c)f\ \Big] \ge 0\ ,$$

(ii) $$L\left[\ b_1 - \frac{\varphi}{z} + \frac{\varphi'(c)-1}{c}f\ \right] + \overline{L\Big[\ z\varphi - [1+\varphi'(c)]cf\ \Big]} = 0\ ,$$

(iii) $$L\left\{\left[-\frac{1}{4}\varphi'^2(c) + \frac{2}{3}\varphi'(c) + \frac{1}{12} + \frac{2}{3}c\varphi''(c)\right]f + \right.$$

$$\left. + \frac{c^2}{d}\frac{f}{f-d} - \frac{1}{2}\varphi\frac{z+c}{z-c}\right\} +$$

$$+ \overline{L\left\{\left[-\frac{1}{2}\frac{1+|c|^2}{1-|c|^2}\varphi'(c) - \frac{|c|^2}{(1-|c|^2)^2}\right]f + \frac{1}{2}\varphi\frac{1+\bar{c}z}{1-\bar{c}z}\right\}} = 0,$$

(iv) $$\mathrm{Re}\Big\{L\Big[\ z\varphi' - 2\,\varphi'(c)\varphi + [\ 2\varphi'^2(c)-\varphi'(c)-c\varphi''(c)\ ]\ f\ \Big]\Big\} \ge 0,$$

where $d=f(c)$.

The distortion theorem and connection between functions from the class S_c and S give

$$\frac{(1-|c|)^2}{1+|c|} \le |b_1| \le \frac{(1+|c|)^2}{1-|c|},$$

thus the above and de'Branges' theorem ([1]) leads us to

$$|b_n| \le \frac{(1+|c|)^2}{1-|c|} n .$$

The equivalent of the Koebe function in the class S_c is

$$k_c(z) = e^{i(\gamma-\pi)} \frac{(1+|c|)^2}{1-|c|} k(e^{i(\pi-\gamma)} z) =$$

$$= \frac{(1+|c|)^2}{1-|c|} \left[z + 2 e^{i(\pi-\gamma)} z^2 + \dots + n e^{i(n-1)(\pi-\gamma)} z^n + \dots \right],$$

where

$$k(z) = \frac{z}{(1-z)^2}, \ \gamma = \arg c \ i \ \gamma \in (0,2\Pi] .$$

The function k_c is a function realizing maximum of the functionals $Re\{b_n\}$ for n=1 and for $n \in \mathbb{N}$ such that $(n-1)(\Pi-\gamma) = 2m\Pi$, $m \in \mathbb{N}$.

Adapting theorems given by P.Duren for the class S ([3], pp.284-288, 304-306), we shall show that the function where Re{L} attains its maximal value on S_c likewise as in the class S, maps the unit disk onto the complement of a single analytic arc. At first we shall prove that the range of f is dense in $\mathbb{C}$. Furthermore Schiffer's theorem will convince us that $\mathbb{C}-f(U)$ is the union of analytic arcs. Finally utilizing the specific property of the class S_c we shall prove that this set is a single analytic arc.

LEMMA 1. Let L be a functional from the space H'(U) such that $L|_{S_c}$ is not constant, f - the function where Re{L} attains its maximal value on S_c. Then the range of f is dense in $\mathbb{C}$.

PROOF. The proof of the theorem is analogous with that of the theorem 9.4 in [3], we shall change only the function G and define another functional h. Namely :

$$G(w) = \frac{F(w)-F(0)}{F'(d)} ,$$

$$M(h) = L(h) - L(1)h(0) - L(f)h'(c) \ , \ h \in H(U) \ .$$

LEMMA 2. Let L be a functional from the space H'(U) such that $L|_{S_c}$ is not constant, f – the function where Re{L} attains its maximal value on S_c. Then f maps the unit disk onto the complement of the union of finitely many analytic arcs w=w(t) satisfying the differential equation

$$\frac{1}{w(w-d)^2} L\left[\frac{[(d-2w)d+wf]f}{f-w} \right] dw^2 > 0 \ , \text{ where } d=f(c).$$

PROOF. We proceed as in the proof of the theorem 10.2 in [3] changing only the normalization of the function F_r, now

$$G_r(w) = \frac{F_r(w)-F_r(0)}{F_r'(d)} = w + \lambda_r \frac{[(d-2w_0)d+w_0w]w}{w_0(d-w_0)^2(w-w_0)} + o(r^2) \ .$$

It appears that if f maximizes the functional Re{L} then Γ may have at most one common point with each circle $|w-d|=r$, $r\in\mathbb{R}$.

LEMMA 3. If a function f from the class S_c omits two different values α and β such that $|d-\alpha|=|d-\beta|$, d=f(c), then f doesn't maximize Re{L} for $L\in H'(U)$.

PROOF. The proof of the lemma is analogous with that of the theorem 9.5 and corollary 2 in [3], with the exception that we define functions ψ_1 and ψ_2 in another way , so we have

$$\psi_1(w) = \frac{w+\psi(w)-\psi(0)}{1+\psi'(d)} \quad \text{and} \quad \psi_2(w) = \frac{w-\psi(w)+\psi(0)}{1-\psi'(d)} \ , \text{ where } d=f(c).$$

The above lemmas allow us to prove that Γ is a single analytic arc.

Theorem 3. Let L be a functional from the space $H'(U)$ such that $L|_{S_c}$ is not constant, f - the function where Re{L} attains its maximal value on S_c. Then f maps the unit disk onto the complement of a single analytic arc $w = w(t)$ satisfying the differential equation

$$\frac{1}{w(w-d)^2} L\left(\frac{[(d-2w)d+wf]f}{f-w}\right)dw^2 > 0, \quad \text{where} \quad d = f(c).$$

Each point $w \in \Gamma$ different from the ends of the arc has the property that the tangent line makes with the line joining points w and d the angle of less than $\Pi/4$.

Proof. Using lemmas 1-3 and considering the function

$$g = \frac{(w-d)^2}{w}\,\frac{f}{w-f},$$

we conclude, as in the proof of the theorem 10.3 in [3], that Γ satisfies the thesis of the theorem.

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METODA WARIACYJNA W KLASIE S_c

S t r e s z c z e n i e

W pracy rozważa się rodzinę S_c funkcji holomorficznych i jednolistnych w kole jednostkowym spełniających warunki: $f(0) = 0$ oraz $f'(c) = 1$ dla ustalonego c z koła jednostkowego.

Stosując metodę wariacyjną A. Changa, MM. Schiffera i G. Schobera przedstawioną w [2], wyprowadzono wzory na pierwszą i drugą wariację dla klasy S_c. Wzory te pozwalają rozważać problem znajdowania w klasie S_c maksimum funkcjonału Re{L}, gdzie L jest liniowym i ciągłym funkcjonałem określonym w przestrzeni funkcji holomorficznych w kole jednostkowym. Pierwsza wariacja prowadzi do równania Schiffera druga wariacja została zastosowana do wykazania, że prawa strona równania Schiffera jest niedodatnia na okręgu koła jednostkowego.

Otrzymano również pewne warunki spełniane przez funkcję maksymalną. Stosując wariację brzegową, podobnie jak u Durena w [3], udowodniono, że funkcja maksymalna koło jednostkowe na płaszczyznę rozciętą wzdłuż jednego łuku analitycznego.

МЕТОД ВАРИЯЦИЙ В КЛАССЕ S_c

Р е з ю м е

В работе рассматривается класс S_c голоморфных и однолистных функции в еденичном круге, которые удовлетбаряют условиям: $f(0)=0$ и $f'(c)=1$ для фиксированного с из еденичного круга.

При помощи вариационного метода А.Чанга, М.М.Шифера и Г.Шобера введенного в [2], выделяются формулы первой и второй вариации для класса S_c. Эти формулы позволяют рассматривать проблему нахождения максимума функционала Re{L}, где L – линейный и непрерывный функционал определенный в пространстве функции голоморфных в еденичном круге. Первая вариация ведет к уравнению Шифера, вторая вариация использована к выказанию что правая сторона уравнения Шифера неотрицательна на оденичном круге. Получены также некоторые условия, которым удовлетваряет максимальная функция.

Применяя береговую вариацию, аналогично Дурену ([3]), доказано, что максимальная функция отображает еденичный круг на плоскость разрезанную вдоль некоторой аналитической дуги.