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ON A CERTAIN BOUNDARY PROBLEM FOR THE EQUATION $(\Delta + a^2)(\Delta + b^2)u(X) = 0$

Summary. In this paper is given the construction and synthesis of the equation $(\Delta + a^2)(\Delta + b^2)u(X) = 0$, where $X = (x_1,\dots,x_n)$ is a point of the Euclidean n - space $(n \geq 2)$, in the set $\Omega = \{X\colon x_i > 0,\ i = 1,\dots,n\}$, satysfying on the subsets $S_i^+ = \{X\colon x_i = 0,\ x_k > 0, k \in \{1,\dots,n\}\setminus\{i\}\}$ $(i = 1,\dots,n)$ of the boundary $\partial\Omega$ the boundary conditions of the first type.

To this construction the convenient Green function is applied.

1. INTRODUCTION

The purpose of this note is to construct the solution u of a certain boundary problem for the equation

$$(\Delta + a^2)(\Delta + b^2)u(X) = 0, \tag{1}$$

where $X = (x_1,\dots,x_n)$ denotes a point of the n-dimensional Euclidean space E_n $(n \geq 2)$, $\Delta = \sum_{i=1}^{n} D^2_{x_i}$ is the Laplace operator, a, b are positive constants, $a \neq b$, in the set

$$\Omega = \{X\colon x_i > 0,\ i = 1,\dots,n\}.$$

Let

$$S_i^+ = \{X : x_i = 0,\ x_k > 0,\ k \in \{1,\dots,n\}\setminus\{i\}\} \quad (i = 1,\dots,n).$$

We shall look for the solution of the equation (1) satisfying on the subset S_i^+ ($i = 1,\dots,n$) of the boundary of Ω the boundary conditions of the first type.

Analogous problem for the equation (1) in the set $\{X : x_n > 0\}$ was solved in [1].

Applying the results of [1] we shall solve our boundary problem. Following [2] we introduce the operation o and we derive with its help the formulas representing the Green function and the boundary problem for (1) in Ω.

2. THE FUNDAMENTAL SOLUTION AND THE FUNDAMENTAL FORMULA FOR THE EQUATION (1)

In this chapter applying the results of [1] we shall give some lemmas connected with the fundamental solution of the equation (1).

Let $X \in E_n$, $Y = (y_1,\dots,y_n) \in E_n$. Further let us write

$r = |Y - X| = \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2}$. Let us consider the function

$$U(r) = -a^{-2\nu}(br)^{-\nu} Y_\nu(br) + b^{2\nu-\nu}(ar)^{-\nu} Y_\nu(ar), \tag{2}$$

where $\nu = \frac{n-2}{2}$ and $Y_\nu(z)$ is the Bessel function of the second kind [3]. We call the function (2) a fundamental solution of the equation (1).

Lemma 1. [1] If $r > 0$, then the function $U(r)$ given by formula (2) satisfies the equation (1).

Theorem 1. [1] Let D be a bounded domain whose boundary we denote by S. Let S consist of finite number of piecewisesmooth hypersurfaces. Let u be a function of class C^4 in D and of class C^3 in $D \cup S$ satisfying the equation (1) in D. Then

$$\frac{1}{\gamma_n} \int_S \{uD_n[\Delta U + (a^2 + b^2)U] - D_n u[\Delta U + (a^2 + b^2)U] + $$
$$+ \Delta u D_n U - D_n \Delta u U\} dS_Y = \begin{cases} u(X) & \text{for } X \in D \\ 0 & \text{for } X \in E_n \setminus (D \cup S) \end{cases}$$

where

$$\gamma_n = \Theta_n\, 2^{\nu+1}\Gamma(\nu + 1)\Pi^{-1}(ab)^{-2\nu}(a^2 - b^2)$$

and Θ_n is the surface of the n-dimensional unit sphere; D_n denotes the inward normal derivative to S at $Y \in S$.

3. THE DEFINITION OF THE OPERATION o AND ITS PROPERTIES

Let us consider the sets $N = \{1,\dots,n\}$ $W = \{0,1\}$ $C = W\times\dots\times W$ (n-times). Let us denote by $(e_1,\dots,e_n)$ the basis of the space E_n of the form $e_i = (e_{i1},\dots,e_{in})$ with $e_{ii} = 1$ and $e_{ik} = 0$ for $i \neq k$ $(i,k \in N)$.
The elements $c = (c_1,\dots,c_n) \in C$ will be identified with the vectors $\sum_{i=1}^{n} c_i e_i$.

Let $X \in E_n$, $X_c = (x_1^{c_1},\dots,x_n^{c_n})$, where

$$x_i^{c_i} = \begin{cases} x_i & \text{for } c_i = 0 \\ -x_i & \text{for } c_i = 1 \end{cases}$$

and $X_{(0,\dots,0)} = X$. Let $Y \in E_n$ and $r_c = |Y - X_c|$, $r_{(0,\dots,0)} = r = |Y-X|$. Let $V(r)$ be a function defined for $r > 0$. We shall write V_c for $V(r_c)$ $(c \in C)$.

Definition 1. In the set of all functions V_c, $c \in C$, we define the operation o as follows: $V_c \circ V_{c'} = V_{c+c'}$ for c, c', $c+c' \in C$.

In virtue of the definition 1 we have that the operation o is commutative and associative and has a neutral element V. We shall assume that this operation is also distributive with respect to addition and that the fixed factors may be taken outside the operation and multiplied.

4. CONSTRUCTION OF GREEN'S FUNCTIONS FOR THE EQUATION (1) AND FOR THE SET Ω UNDER BOUNDARY CONDITIONS OF THE FIRST TYPE

We shall now define some auxiliary functions which will be helpful in the formulation of the boundary problem for the equation (1) and the set Ω.
Let us put $\Omega_i = \{X : x_i > 0\}$, $S_i = \{X : x_i = 0\}$ $(i \in N)$, $\hat{\Omega} = \Omega \cup S_1^+ \cup \ldots$
$\ldots \cup S_n^+$.
Let us consider the following functions

$$G_{e_i} = G_{e_i}(X,Y) = U - U_{e_i} \quad (i \in N) \tag{3}$$

and the function

$$G = G(X,Y) = G_{e_1} \circ \ldots \circ G_{e_n} \tag{4}$$

Theorem 2. The functions G_{e_i} $(i \in N)$ given by (3) have the following properties:

1° G_{e_i} are of the class C^∞ for $X \neq Y$; $(X,Y) \in \Omega_i \times (\Omega_i \cup S_i)$ $[(\Omega_i \cup S_i) \times \Omega_i]$;

2° G_{e_i} satisfy the equation (1) as the functions of the point $X \in \Omega_i$, $X \neq Y$, $Y \in \Omega_i \cup S_i$;

3° (a) $\Delta^s G_{e_i} \longrightarrow 0$ when $X \longrightarrow X_i \in S_i$, $X \in \Omega_i$, Y is fixed in Ω_i $X \neq Y$ $(i \in N;\ s = 0,1)$;

b) $\Delta^s G_{e_i} \longrightarrow 0$ when $Y \longrightarrow Y_i \in S_i$, $Y \in \Omega_i$, X is fixed in Ω_i, $Y \neq X$ $(i \in N;\ s = 0,1)$.

Theorem 2 follows from Lemma 1 and the formula (3).

Theorem 3. The function G given by (4) has the following properties:

1° G is of the class C^∞ for $(X,Y) \in \Omega \times \hat{\Omega}$ $[\hat{\Omega} \times \Omega$, $(\Omega \cup S_i^+)\times(\hat{\Omega}\setminus S_i^+)$, $(\hat{\Omega}\setminus S_i^+)\times(\Omega \cup S_i^+)$, $i \in N]$, $X \neq Y$;

2° G satisfies the equation (1) as the function of the point $X \in \Omega$ with fixed $Y \in \hat{\Omega}$, $Y \neq X$;

3° (a) $\Delta^{s} G \longrightarrow 0$ when $X \longrightarrow X_i \in S_i^{+}$, $X \in \Omega$, $Y \in \hat{\Omega} \setminus S_i^{+}$, $X \neq Y$, $i \in N$, $s = 0, 1$;

b) $\Delta^{s} G \longrightarrow 0$ when $Y \longrightarrow Y_i \in S_i^{+}$, $Y \in \Omega$, $X \in \hat{\Omega} \setminus S_i^{+}$, $Y \neq X$, $i \in N$, $s = 0, 1$.

Proof. By (3), (4) the function G is the linear combination of the functions U_c, where $c \in C$. Since U_c, $c \in C$, satisfy the equation (1) as the functions of the point $X \in \Omega$, $X \neq Y$ with $Y \in \Omega$ fixed, theses 1°, 2° of our Theorem follow.

In order to prove 3° we will show only that the function G satisfies boundary conditions for $i = 1$. The proof that G satisfies boundary conditions 3° where $i \in N \setminus \{1\}$ is analogous. The function G is a linear combination of the functions $G_{e_1} \circ U_c$, where $c = (0, c_2, \ldots, c_n) \in C$.

By the definition of the operation $\circ$ we obtain

$$G_{e_1} \circ U_c = G_{e_1} \Big|_{x_i = x_i^{c_i}, \quad i \in N \setminus \{1\}} \tag{5}$$

for $X \in \Omega \cup S_1^{+}$, $Y \in \hat{\Omega} \setminus S_1^{+}$.

By 3° of Theorem 2 we obtain 3° for $i = 1$.

5. FORMULATION OF THE BOUNDARY PROBLEM FOR THE EQUATION (1) IN THE DOMAIN Ω

Applying formally the results of Theorem 1 and Theorem 3 we shall present now formulae for the solution of the boundary problem for the equation (1) in Ω.

Let $y^{i} = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$ denote the projection of the point Y on the plane $y_i = 0$ $(i \in N)$, if we identify this plane with the space E_{n-1}.

Let

$$D_i = \{y^{i} : y_k > 0, \quad k \in N \setminus \{i\}\}, \quad i \in N.$$

Let us consider the function

$$u(X) = \sum_{j=1}^{n} u^j(X), \tag{6}$$

where

$$u^j(X) = \frac{1}{\gamma_n} \int_{D_j} \{f_1^j(y^j) D_{y_j} [\Delta G(X,Y) + (a^2 + b^2)G(X,Y)] +$$

$$+ f_2^j(y^j) D_{y_j} G(X,Y)\}\Big|_{y_j = 0} dy^j \quad (j \in N) \tag{7}$$

f_k^j $(k = 1,2)$ being given functions defined on D_j $(j \in N)$.
We shall prove under suitable assumptions on the functions f_k^j $(k = 1,2$ $j \in N)$ that the function u given by (6) is the solution of the equation (1) in Ω satysfying the boundary conditions

$$u(X) = f_1(x^j), \quad \Delta u(X) = f_2(x^j) \tag{8}$$

for $X \in S_j^+$ $(j \in N)$.

6. THE SYNTHESIS OF THE SOLUTION (6) OF THE PROBLEM (1), (8) IN Ω

Let us consider the following functions

$$\begin{cases} \Delta^s D_{y_j} G(X,Y) \\ \Delta^s D_{y_j} [\Delta G(X,Y) + (a^2 + b^2)G(X,Y)] \end{cases} \tag{9}$$

$s = 0,1;$ $j \in N\setminus\{i\}$, $i \in N$, $(X,Y) \in \Omega \times S_j^+$.

Lemma 2. The functions defined by formula (9) tend to zero when $X \longrightarrow X_i \in S_i^+$.

Proof. Let us consider $i = 1$. For $i \neq 1$ the proof is analogous. The function G is a linear combination of the functions of the form (5). To get our thesis for $i = 1$, $j \in N\setminus\{1\}$ it is sufficient to show that the functions

$$D_{y_j} \Delta^s (G_{e_1} \circ U_c) \tag{10}$$

where $(X \times Y) \in \Omega \times S_j^+$, $c = (0, c_2, \ldots, c_n) \in C$ tend to zero when $X \longrightarrow X_1 \in S_1^+$, $X \in \Omega$.

Let $w^s(r) = r^{-1} D_r \Delta^s U$, $\tilde{G}^s_{e_1} = w^s - w^s_{e_1}$.

By definition of $\circ$ we have

$$D_{y_j} \Delta^s (G_{e_1} \circ U_c) = (-1)^{c_j+1} x_j [\tilde{G}^s_{e_1} \circ w^s_c] = (-1)^{c_j+1} x_j \tilde{G}^s_{e_1} \Big|_{\substack{x_k = x_k^{c_k} \\ k \in N\setminus\{1\}}}$$

for $(X, Y) \in \Omega \times S_j^+$.

Since $\tilde{G}^s_{e_1} \longrightarrow 0$ as $X \longrightarrow X_1 \in S_1$, $X \in \Omega_1$, $Y \in \Omega_1$, $Y \neq X$ the functions (9) tend to zero as $X \longrightarrow X_1 \in S_1^+$, $X \in \Omega$.

Lemma 3. If the function $f^j(y^j)$ is measurable and bounded in D_j and

$$\int_{D_j} |f^j(y^j)| dy^j < \infty \quad (j \in N)$$

then

1° the integrals $\int_{D_j} f^j(y^j) D_Y^{\beta} D_X^{\alpha} G(X, Y) \Big|_{y_j = 0} dy^j$

where $|\alpha|, |\beta| = 0, 1, 2, \ldots$ are locally uniformly convergent at every $X \in \hat{\Omega}\setminus S_j^+$ $(J \in N)$

2° $D_X^{\alpha} \int_{D_j} f^j(y^j) D_Y^{\beta} G(X, Y) \Big|_{y_j=0} dy^j = \int_{D_j} f^j(y^j) D_Y^{\beta} D_X^{\alpha} G(X, Y) \Big|_{y_j=0} dy^j$

for $X \in \hat{\Omega}\setminus S_j^+$, $j \in N$.

Proof. We consider the case $j = 1$. For $j \neq 1$ the reasoning is similar. By definition of G and o it is enough to show the local uniform convergence of the following integrals

$$I(X) = \int_{D_1} f^1(y^1) D_Y^{\beta} D_X^{\alpha} U_c \Big|_{y_1 = 0} dy^1 \quad \text{for} \quad c \in C$$

at every point $X \in \hat{\Omega}\setminus S_1^+$.

Let $K(\bar{X},\eta)$ be a sphere with the center $\bar{X} = (\bar{x}_1,\dots,\bar{x}_n) \in \hat{\Omega}\setminus S_1^+$ and the radius $\eta > 0$, $K(\bar{X},\eta) \subset \Omega_1$. The functions $D_X^{\alpha} D_Y^{\beta} U_c$, $c \in C$, are linear combinations of the functions of the form

$$(hr_c)^{-\nu} Y_{\nu+s}(hr_c)(r_c)^{-s} \prod_{j=1}^{s} (y_j - x_j^{c_j})^{\gamma_j}, \tag{11}$$

where $s \geq \sum_{j=1}^{n} \gamma_j$; $s, \gamma_j = 0, 1, \dots, |\alpha| + |\beta|$, $j \in N$.

Since

$$r_c \geq \bar{x}_1 - \eta > 0 \quad \text{for} \quad c \in C, \quad X \in K(\bar{X},\eta), \quad Y \in S_1^+ \tag{12}$$

thus by the asymptotical properties of the Bessel functions $Y_s(cr)$ as $r \longrightarrow \infty$ ([3], p. 132) and by (12) we obtain

$$|D_X^{\alpha} D_Y^{\beta} U_c| \leq M \quad \text{for} \quad X \in K(\bar{X},\eta), \quad Y \in S_1^+ \tag{13}$$

where M is the convenient positive constant. It follows from the assumptions of Lemma 3 and the formula (13) that the integrals $I(X)$ are locally uniformly convergent at the point $\bar{X} \in \hat{\Omega}\setminus S_1^+$.

2^o is a consequence of 1^o.

Lemma 4. If the functions $f_k^j(y^j)(k = 1,2)$ are measurable and bounded in D_j and

$$\int_{D_j} |f_k^j(y^j)| dy^j < \infty \quad (k = 0,1), \quad j \in N$$

then

1^o the function u^j given by formula (7) satisfies the equatioon (1) in the set Ω $(j \in N)$;

2^o the function u^j satisfies the following boundary conditions

$$\Delta^s u_j(X) = 0 \quad \text{for} \quad X \in S_i^+ \qquad i \in N\setminus\{j\}$$

for $s = 0,1;\ \ j \in N$.

Lemma 4 is an immediate consequence of Lemmas 2, 3 and thesis 2^o of Theorem 3.

Let us write:

$$L_i(X) = \frac{2}{\gamma_n} \int_{D_i} \{f_1^i(y^i) D_{y_i} [\Delta\, U(r) + (a^2 + b^2)U(r)] +$$

$$+ f_2^i(y^i) D_{y_i} U(r)\}\Big|_{y_i = 0} dy^i \tag{14}$$

Lemma 5. If the functions $f_k^i(y^i)$ $(k = 1,2)$ are measurable and bounded in D_i and continuous at the point

$$\tilde{x}^i = (\tilde{x}_1,\dots,\tilde{x}_{i-1},\ \tilde{x}_{i+1},\dots,\tilde{x}_n) \in D_i\ ,$$

$$\int_{D_i} |f_k^i(y^i)|dy^i < \infty \qquad (k = 1,2)\ ,$$

then

$$L_i(X) \longrightarrow f_1^i(\tilde{x}^i), \qquad \Delta L_i(X) \longrightarrow f_2^i(\tilde{x}^i)$$

as $X \longrightarrow \tilde{X}_i = (\tilde{x}_1,\dots,\tilde{x}_{i-1},0,\ \tilde{x}_{i+1},\dots,\tilde{x}_n) \in S_i^+$,

$X \in \Omega$ and $L_i(X)$ is given by (14) $(i \in N)$.

The proof of Lemma 5 is analogical to that of Lemma 3 in [1].

Let

$$L_i^c(X) = \int_{D_i} \{f_1^i(y^i) D_{y_i} [\Delta U(r_c) + (a^2 + b^2)U(r_c)] +$$

$$+ f_2^i(y^i) D_{y_i} U(r_c)\}\Big|_{y_i = 0} dy^i \tag{15}$$

for $c \in C\setminus\{(0,\dots,0)\ e_i\}$, $i \in N$.

Lemma 6. If the functions $f_k^i(y^i)$ $(k = 1,2)$ are measurable and bounded in D_i,

$$\int_{D_i} |f_k^i(y^i)|dy^i < \infty \quad (i \in N), \text{ then}$$

$$\Delta^s L_i^c(X) \longrightarrow 0 \quad (s = 0,1) \quad \text{when} \quad X \longrightarrow \tilde{X}_i \in S_i^+, \ X \in \Omega$$

and $L_i^c(X)$ is given by (15), $i \in N$.

Proof. Let $K(\tilde{X}_i,\delta)$ be a sphere with the center at $\tilde{X}_i \in S_i^+$ and such radius $\delta > 0$ that its projection on S_i is in S_i^+ $(i \in N)$. Then there exists a number $\delta_1 > 0$ such that

$$r_c \geq \delta_1 \quad \text{for} \quad X \in K(\tilde{X}_i,\delta) \cap (\Omega \cup S_i^+), \ Y \in S_i^+ \tag{16}$$

and $c \in C\setminus\{(0,\dots,0), e_i\}$, $(i \in N)$.
By (2), Lemma 1 and (16) we obtain

$$|D_{y_i} \Delta^k U(r_c)| \leq x_i M \quad \text{for} \quad X \in K(\tilde{X}_i,\delta) \cap (\Omega \cup S_i^+) \ , \tag{17}$$

$$Y \in S_i, \quad c \in C\setminus\{(0,\dots,0),e_i\} \quad (i \in N, \quad k = 0,1,2),$$

where M is a convenient positive constant.
By (17) and by assumptions of Lemma 6 we obtain the thesis of Lemma 6.
From Lemmas 3, 4, 5 an (6) follows

Theorem 4. Let the functions $f_k^j(y^j)$ $(k = 1,2)$ be continuous and bounded on D_j and

$$\int_{D_j} |f_k^j(y^j)|dy^j < \infty \qquad \text{for} \quad j \in N .$$

Then the function u given by (6), (7) satisfies the equation (1) in Ω and the boundary conditions (8).

REFERENCES

[1] Filar M.: On a certain boundary problem for the equation $(\Delta+a^2)(\Delta+b^2)u=0$, Comment. Math. XIX (1976) p. 55-66.

[2] Filar M.: Boundary problems for the homogeneous iterated Helmholtz equation in a certain unbounded domain of the Euclidean n-space. Comment. Math. XXV (1985), p. 31-48.

[3] Lebiediew N.: Funkcje specjalne i ich zastosowania, Warszawa 1957.

O PEWNYM PROBLEMIE BRZEGOWYM DLA RÓWNANIA $(\Delta + a^2)(\Delta + b^2)u(X) = 0$

S t r e s z c z e n i e

W pracy podana jest konstrukcja i synteza rozwiązania równania $(\Delta + a^2)(\Delta + b^2)u(X) = 0$, gdzie $X = (x_1,\dots,x_n)$ jest punktem przestrzeni euklidesowej n-wymiarowej $E_n (n \geq 2)$, w zbiorze $\Omega = \{X: x_i > 0, i = 1,\dots,n\}$ spełniającego na podzbiorach $S_i^+ = \{X: x_i = 0, x_k > 0, k \in \{1,\dots,n\}\setminus\{i\}\}$ $(i = 1,\dots,n)$ brzegu Ω warunki brzegowe pierwszego rodzaju.

Do konstrukcji rozwiązań użyto odpowiedniej funkcji Greena.

НЕКОТОРАЯ КРАЕВАЯ ЗАДАЧА ДЛЯ УРАВНЕНИЯ $(\Delta + a^2)(\Delta + b^2)u(X) = 0$

Р е з ю м е

В настоящей работе решено уравнение $(\Delta + a^2)(\Delta + b^2)u(X) = 0$, где $X = (x_1,\dots,x_n)$ точка n-мерного эвклидого пространства E_n $(n \geq 2)$, в области $\Omega = \{X: x_i > 0, i = 1,\dots,n\}$, с краевыми условиями первого рода на подмножествах

$$S_i^+ = \{X: x_i = 0, x_k > 0, k \in \{1,\dots,n\}\setminus\{i\}\} \ (i = 1,\dots,n)$$

края области Ω

Конструкция этого решения сделана с помощью соответствующей функции Грина.