ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1992
Seria: MATEMATYKA-FIZYKA z. 67 Nr kol. 1115

Michał LORENS

# SKALAR DIFFERENTIAL CONCOMITANTS OF THE THIRD ORDER OF A METRIC TENSOR IN THE TWO-DIMENSIONAL SPACE AND THEIR APPLICATIONS

**Summary.** All scalar differential committants of the third order for a metric tensor in a two-dimensional space have been determined in the paper (Theorem 1). A full classification of two-dimensional Riemann's space of almost constant curvature has been created by means of these committants. The two-dimensional Riemann's space of almost constant curvature has been defined and its specification has been determined.

## INTRODUCTION

Let $V^n$ be the n-dimensional Riemannian space with a metric $g_{ij}$ (which need not be positive definite). As is known (cf. [3], p. 127 and also [4], p. 138), if a purely differential geometric object of the first class is a differential concomitant oforder s of $g_{ij}$, then this objects is an algebraic concomitant of the tensors

$$g_{ij}, \quad R_{ijkl}, \quad \nabla_{\upsilon_1} R_{ijkl}, \dots, \nabla_{\upsilon_{s-2}} \dots {}_{\upsilon_1} R_{ijkl},$$

where $R_{ijkl}$ and $\nabla$ denote the covariant curvature tensor and the covariant derivative determined by $g_{ij}$.

The determination of scalar differential concomitants is important from the geometric point of view, because one can characterize the spaces $V^n$ by means of those scalars.

It is well known that in $V^2$ every scalar differential concomitant of the second order of $g_{ij}$ is an arbitrary function of the Gauss curvature and the signature of $g_{ij}$.

In the pressent note we determine in $V^2$ all scalar differential concomitants of the third order of $g_{ij}$ and we give a certain classification of spaces $V^2$ (§1 and §2). Finaly, we investigate one of the classes found in §2.

**1. LET US CONSIDER A RIEMANNIAN SPACE $V^2$ WITH A METRIC $g_{ij}$**

The problem of finding scalar concomitants of the third order of $g_{ij}$ leads to that of solving the equation

$$F(g_{ij}, R_{ijkl}, \nabla_p R_{ijkl}) = F(g_{i'j'}, R_{i'j'k'l'}, \nabla_{p'} R_{i'j'k'l'}), \quad (1.1)$$

$$i,j,k,l,p,i',j',k',l',p' = 1,2.$$

In space $V^2$ the curvature tensor has only one essential component $R_{1212}$. It is known that

$$R_{1212} = K \det(g_{ij}),$$

$$\nabla_p R_{1212} = (\partial_p K)\det(g_{ij}), \quad p = 1,2, \quad (1.2)$$

where $K$ is the Gauss curvature in $V^2$. From (1.2) it follows that

$$F(g_{ij}, R_{ijkl}, \nabla_p R_{ijkl}) = H(g_{ij}, K, \partial_p K).$$

Thus we have the following

**Corollary 1.** Every scalar differential concomitant of the third order of the tensor $g_{ij}$ in $V^2$ is a function of the tensor $g_{ij}$, the Gauss curvature $K$ and the gradient of $K$.

Hence equation (1.1) has the form

$$H(g_{ij}, K, \partial_p K) = H(g_{i'j'}, K, \partial_{p'} K), \quad i,j,p,i',j',p' = 1,2.$$

Now it is easily seen that our problem is equivalent to the determination of scalar concomitants of the pair

$$(g_{ij}, \partial_i K),$$

or of the pair

$$(g_{ij}, \partial_i K \cdot \partial_j K)$$

(cf. [2], p. 57).

Let us put

$$G = (g_{ij}), \quad W = (\partial_i K \cdot \partial_j K)$$

and consider the matrix bundle

$$G\lambda + W . \tag{1.3}$$

(Our considerations are based on the results of [5], p. 10-18, 21-23). We notice that the bundle (1.3) is regular and stricly equivalent to the bundle $E\lambda + G^{-1}W$, E being the unit 2 x 2 matrix. Hence the bundles $G\lambda + W$ and $E\lambda + G^{-1}W$ have identical elementary divisors. The matrix $G^{-1}W$ has the following eigenvalues: $\mu = 0, \rho = trG^{-1}W = \Delta_1 K$ ($\Delta_1 K$ is the Beltrami's differential parameter of the first order of the function K). It is known that the scalars $\mu$, $\rho$ and the Weierstrass characteristic of the matrix $G^{-1}W$ uniquely determine elementary divisors of the bundle $E\lambda + G^{-1}W$.

Thus we arrive at the following.

**Theorem 1.** Ewery scalar differential concomitant of the third order of the metric tensor $g_{ij}$, for n = 2, is an arbitrary function of the partial signature of the canonical form of the bundle (1.3), of the Weierstrass characteristic of the matrix $G^{-1}W$, of the Gauss curvature K and of the Beltrami's parameter $\Delta_1 K$.

## 2. WE USE RESULTS OF § 1 TO GIVE A CERTAIN CLASSIFICATION OF SPACES $V^2$

The Weierstrass characteristic of the matrix $G^{-1}W$ can have the following forms:

$1^o$ [1,1],

$2^o$ [2].

In cases $1^o$, $2^o$ we have the following canonical forms of the bundle (1.3) and of the matrix $G^{-1}W$:

$1^{\circ}$ a) $\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \rho \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & \rho \end{bmatrix}$, $\rho \ge 0$,

b) $\lambda\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\rho \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & \rho \end{bmatrix}$, $\rho \le 0$,

c) $\lambda\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \rho \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & \rho \end{bmatrix}$, $\rho \ge 0$,

d) $\lambda\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\rho \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & \rho \end{bmatrix}$, $\rho \le 0$,

$2^{\circ}$ $\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

**Remark 1.** The matrix W is semi-positive definite and rank $W \le 1$. On account of the above, we obtain the following.

**Corollary 2.** If the Weierstrass characteristic of the matrix $G^{-1}W$ is equal to [1,1] at every point $p \in V^2$, then:
$V^2$ is a space of constant curvature $\Leftrightarrow$ $\Delta_1 K = 0$, for every point $p \in V^2$.

**Corollary 3.** If the Weierstrass characteristic of $G^{-1}W$ is equal to[2] at every point $p \in V^2$, then the signature of the metric tensor is {+,-}, $\Delta_1 K=0$ and $(\partial_1 K)^2 + (\partial_2 K)^2 + (\partial_2 K)^2 \neq 0$ at every point $p \in V^2$.

**Remark 2.** If the Weierstrass characteristic of the matrix $G^{-1}W$ is equal to [2] at some point $p \in V^2$, then there exists a neighbourhood $U_p$ of p such that for every point $q \in U_p$ the Weierstrass charakteristic of $G^{-1}W$ is equal to [2].

Using corollaries 1 and 2 we formulate the following.

**Definition.** A Riemannian space $V^2$ for which the Weierstrass characteristic of the matrix $G^{-1}W$ is equal to [2] at every point $p \in V^2$ is called a space of almost constant curvature.

## 3. WE CONSIDER A SPACE $V^2$ OF ALMOST CONSTANT CURVATURE

We assume in this section that all functions are $C^\infty$. It follows from Corollary 2 that then:

(i) the gradient of the Gauss curvature is an isotropic vector field,

(ii) the gradient of the Gauss curvature is a non-zero vector at every point $p \in V^2$.

Let us put

$$v^i = g^{is} v_s \ .$$

The field X is isotropic and $X_p \neq 0$, $p \in V^2$. A coordinate system (x,y) can be chosen so that

$$v^1 = 1, \ v^2 = 0$$

(cf. [1], p. 82). Thus we obtain

$$g_{11} = g_{ij} v^i v^j = 0 \ , \tag{3.1}$$

$$\partial_1 K = g_{1s} v^s = g_{11} = 0 \ ,$$
$$\partial_2 K = g_{2s} v^s = g_{12} \ , \tag{3.2}$$

where $\partial_1 K = \frac{\partial K}{\partial x}$, $\partial_2 K = \frac{\partial K}{\partial y}$.

From (3.1) and (3.2) it follows that

$$K(x,y) = f(y), \quad y \in \Delta \ (\Delta \text{ is an open interval of } R),$$
$$g_{12}(x,y) = f'(y) \neq 0 \quad \text{for} \quad y \in \Delta \ . \tag{3.3}$$

Therefore in the chart (x,y) we have

$$(g_{ij}(x,y)) = \begin{pmatrix} 0 & f'(y) \\ f'(y) & h(x,y) \end{pmatrix} .$$

By an elementary calculation we obtain

$$\left\{ {1 \atop 11} \right\} = 0, \quad \left\{ {1 \atop 12} \right\} = \frac{1}{2f'} \frac{\partial h}{\partial x}, \quad \left\{ {1 \atop 22} \right\} = \frac{1}{2f'} \left( \frac{h}{f'} \frac{\partial h}{\partial x} - 2 \frac{h}{f'} f'' + \frac{\partial h}{\partial y} \right),$$

$$\left\{ {2 \atop 11} \right\} = 0, \quad \left\{ {2 \atop 12} \right\} = 0, \qquad \left\{ {2 \atop 22} \right\} = \frac{f''}{f'} - \frac{1}{2f'} \frac{\partial h}{\partial x},$$

$$R_{1212} = \frac{1}{2} \frac{\partial^2 h}{\partial x^2},$$

$$K(x,y) = - \frac{\dfrac{\partial^2 h(x,y)}{\partial x^2}}{2[f'(y)]^2} \tag{3.4}$$

Comparing (3.3) with (3.4) we get

$$\frac{\partial^2 h(x,y)}{\partial x^2} = - 2f(y)[f'(y)]^2.$$

Hence we have

$$h(x,y) = - f(y)[f'(y)]^2 x^2 + k(y)x + l(y),$$

where k, l are arbitrary real functions with the domain $\Delta$. Therefore, in a space $V^2$ of almost constant curvature the metric $ds^2$ has (locally) the form

$$ds^2 = 2f'(y)dxdy + \{-f(y)[f'(y)]^2 x^2 + k(y)x + l(y)\}dy^2. \tag{3.5}$$

On the other hand, it is easily seen that a space $V^2$ with metric (3.5) is of almost constant curvature.

In this way we have proved.

**Theorem 2.** The space $V^2$ is a space of almost constant curvature if and only if the metric $ds^2$ in $V^2$ has (locally) the form

$$ds^2 = 2f'(y)dxdy + \{- f(y)[f'(y)]^2 x^2 + k(y)x + l(y)\}dy^2,$$

where $f, k, l$ are arbitrary real $C^{\infty}$ functions in an open interwal $\Delta \subset \mathbb{R}$ and $f'(y) \neq 0$, $y \in \Delta$.

Remark 3. A space $V^2$ of almost constant curvature is not compact.

Institute of Mathematics
College of Pedagogics
35-310 Rzeszów, Poland

## REFERENCES

[1] Eisenhart L.P.: Riemannian Geometry, Princeton, 1966.

[2] Lorens M.: Kilka uwag o wyznaczaniu komitant skalarnych obiektów geometrycznych, Rocznik Naukowo-Dydaktyczny WSP w Rzeszowie, Matematyka, Z. 5/41 (1979), p. 51-58.

[3] Łubczonok G.: On the reduction theorems, Ann. Pol. Math. XXVI (1972), p. 125-133.

[4] Schouten J.A. und Struik D.J.: Einfuhrung in die neuren Methoden der Differentialgeometrie, Groningen, 1935.

[5] Zajtz A. Komitanten der Tensoren zweiter Ordnung, Zeszyty Naukowe UJ w Krakowie, Prace Mat. 8 (1964).

## SKALARNE KOMITANTY RÓŻNICZKOWE TRZECIEGO RZĘDU TENSORA METRYCZNEGO W DWUWYMIAROWEJ PRZESTRZENI I ICH ZASTOSOWANIA

S t r e s z c z e n i e

W pracy wyznaczono wszystkie skalarne komitanty różniczkowe trzeciego rzędu tensora metrycznego w dwuwymiarowej przestrzeni (Twierdzenie 1). Podano pewną klasyfikację dwuwymiarowych przestrzeni Riemanna przy pomocy tych komitant. Zdefiniowano dwuwymiarową przestrzeń Riemanna o prawie stałej krzywiźnie i wyznaczono jej metrykę.

## СКАЛЯРНЫЕ КОМИТАНТЫ ТРЕТЬЕГО ПОРЯДКА МЕТРИЧЕСКОГО ТЕНЗОРА ДВУМЕРНОГО ПРОСТРАНСТВА И ИХ ПРИЛОЖЕНИЯ

Р е з ю м е

В работе автор нашёл в двумерном пространстве все скалярные комитанты третьего порядка метрического тензора (Теорема 1). Далее с помощью этих скаляров он провел некоторую классификацию двумерных пространств. В заключение работы он определил пространство почти постоянной кривизны и нашёл его метрику.