ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1992
Seria: MATEMATYKA-FIZYKA z. 67 Nr kol. 1115

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SUBGROUPS OF CLIFFORD'S GROUP

Summary. Let K be a commutative field. The set $H = (K\setminus\{0\}) \times K$ equipped in the operation:

$$(a,b)(c,d) = (ac, ad + cb)$$

is called Clifford's group of the field K. It is isomorphous with the multiplicative group of matrices in the form of $\begin{bmatrix} a & 0 \\ b & a \end{bmatrix}$, where $a,b \in K$, $a \neq 0$.

Let $F_1 \subset K\setminus\{0\}$, $F_2 \subset K$ be a non-empty sets and $f_1: F_1 \longrightarrow K$, $f_2: F_2 \longrightarrow K\setminus\{0\}$.

Theorem. (i) The set $\{(a, f_1(a)) : a \in F_1\}$ is a sb-group of the group H only provided that F_1 is a multiplicative subgroup of the field K and the function $F_1 \ni a \longrightarrow a^{-1}f_1(a) \in K$ satisfies the functional equation (6).

(ii) The set $\{(f_2(a), a) : a \in F_2\}$ is a subgroup of the group $(H,\cdot)$ only provided that additive subgroup of the field K as well as an function $k: A \longrightarrow K\setminus\{0\}$ which satisfies conditions (7), (8) exist and they are such that:

$$F_2 = \{k(a)a : a \in A), \quad f_2k(a)a) = k(a) \quad \text{for} \quad a \in A.$$

In particular, the general solution of functional equation (3) is given in the paper.

Let K be a commutative field. We define a binary operation $: H \times H \longrightarrow H$, where $H = (K\setminus\{0\}) \times K$, as follows:

$$(a,b)(c,d) = (ac, ad + cb). \tag{1}$$

The set H endowed with operation (1) is called Clifford's group of the field K (cf. [2] p. 281). It is isomorphic to the multiplicative group of matrices of the form $\begin{bmatrix} a & 0 \\ b & a \end{bmatrix}$, where $a,b \in K$, $a \neq 0$.

Using the well known method of finding subgroups (e.g. see [1]-[3]) we are going to give necessary and sufficient conditions for non-empty sets $F_1 \subset K\setminus\{0\}$, $F_2 \subset K$ and functions $f_1: F_1 \to K$, $f_2: F_2 \to K\setminus\{0\}$ in order that the sets

$$D_1 = \{(a, f_1(a)): a \in F_1\},$$

$$D_2 = \{(f_2(a), a): a \in F_2\}$$

were subgroups of the group $(H, \cdot)$. The case where $K = R$, $F_1 = R\setminus\{0\}$, $F_2 = R$ (where R denotes the set of all real numbers), and f_1, f_2 are continuous is solved in [2].

Let us start with the following

Lemma 1. The following conditions hold:

(i) D_1 is a subgroup of the group $(H, \cdot)$ if and only if F_1 is a multiplicative subgroup of K and f_1 satisfies the functional equation

$$g(ab^{-1}) = b^{-2}(bg(a)-ag(b)), \tag{2}$$

(ii) D_2 is a subgroup of the group $(H, \cdot)$ if and only if there exists a function $f_o: K \to K$ satisfying the following conditional functional equation

$$\text{if } f(a)f(b) \neq 0, \text{ then } f(f(b)^{-2}(f(b)a-f(a)b)) = f(a)f(b)^{-1} \tag{3}$$

such that $F_2 = \{a \in K: f_o(a) \neq 0\}$ and $f_2(a) = f_o(a)$ for $a \in F_2$.

Proof. (i) Observe that

$$(a,b)^{-1} = (a^{-1}, -ba^{-2}) \quad \text{for} \quad (a,b) \in H$$

and consequently

$$(c,d)\,(a,b)^{-1} = (ca^{-1}, a^{-2}(ad-cb)) \quad \text{for} \quad (a,b), (c,d) \in H.$$

Thus D_1 is a subgroup of the group $(H,\cdot)$ if and only if

$$(a,f_1(a))\ (b,f_1(b))^{-1} = (ab^{-1},\ b^{-2}(bf_1(a)-af_1(b))) \in D_1 \quad \text{for} \quad a,b \in F_1,$$

which implies the assertion.

(ii) In the same way, as above, we obtain that D_2 is a subgroup of the group $(H,\cdot)$ if and only if

$$(f_2(a),a)\ (f_2(b),b)^{-1} =$$

$$= (f_2(a)f_2(b)^{-1},\ f_2(b)^{-2}(f_2(b)a-f_2(a)b)) \in D_2 \quad \text{for} \quad a,b \in F_2.$$

It is equivalent to the following condition

$$f_2(b)^{-2}(f_2(b)a-f_2(a)b) \in F_2 \quad \text{and}$$

$$f_2(f_2(b)^{-2}(f_2(b)a-f_2(a)b)) = f_2(a)f_2(b)^{-1} \quad \text{for} \quad a,b \in F_2. \tag{4}$$

Assume that D_2 is a subgroup of $(H,\cdot)$. Then (4) holds. Consequently the function $f_o\colon K \longrightarrow K$,

$$f_o(a) = \begin{cases} f_2(a) & \text{for} \quad a \in F_2 ; \\ 0 & \text{otherwise} , \end{cases}$$

satisfies (3). Moreover, $F_2 = \{a \in K\colon\ f_o(a) \neq 0\}$ and $f_2(a) = f_o(a)$ for $a \in F_2$.

On the other hand, if there exists a function $f : K \longrightarrow K$ such that (3) holds, $F_2 = \{a \in K\colon f(a) \neq 0\}$, and $f_2(a) = f(0)$ for $a \in F_2$, then F_2 and f_2 fulfil (4). As a result D_2 is a subgroup of the group $(H,\cdot)$, which completes the proof.

We are going to solve functional equations (2) and (3) in a more general case, i.e. in the classes of functions $f\colon L \longrightarrow E$ and $g\colon E \longrightarrow K$, where E is a linear space over the commutative field K and L is a multiplicative subgroup of K. We have the following

Lemma 2. Let $f\colon E \longrightarrow K$ be a function satisfying (3) and $F = \{x \in E\colon f(x) \neq 0\}$. Then the following conditions hold:

(i) if $F \neq \phi$, then $f(0) = 1$,

(ii) if $x, y \in F$ and $f(x)^{-1}x = f(y)^{-1}y$, then $x = y$,

(iii) $f(f(x)y+f(y)x) = f(x)f(y)$ for $x, y \in F$,

(iv) $A = \{f(x)^{-1}x\colon x \in F\}$ is an additive subgroup of E.

Proof. (i) Fix a $z \in F$. Then, by (3),

$$f(0) = f(f(z)^{-2}(f(z)z-f(z)z)) = 1.$$

(ii) Fix any $x, y \in F$ and assume that $f(x)^{-1}x = f(y)^{-1}y$. Then

$$f(y)x-f(x)y = 0. \tag{5}$$

Thus, in virtue of (3),

$$f(0) = f(f(y)^{-2}(f(y)x-f(x)y)) = f(x)f(y)^{-1}.$$

Since, by (i), $f(0) = 1$, we obtain $f(x) = f(y)$. Hence (5) implies that $x = y$.

(iii) The proof is strightforward.

(iv) It is enough to prove that $a-b \in A$ for every $a, b \in A$. Fix any $x, y \in F$ and denote $z = f(y)^{-2}(f(y)x-f(x)y)$. Then, by (3), $f(z) = f(x)f(y)^{-1} \neq 0$. Thus $f(z)^{-1}z \in A$. It is easy to verify that $f(x)^{-1}x-f(y)^{-1}y = f(z)^{-1}z$. Hence $f(x)^{-1}x-f(y)^{-1}y \in A$, which completes the proof.

Proposition 1. The following conditions are valid:

(i) a function $g\colon L \to E$ satisfies functional equation (2) if and only if the function $h\colon L \to K$, $h(a) = a^{-1}g(a)$ satisfies the functional equation

$$h(ab) = h(a) + h(b), \tag{6}$$

(ii) a function $f\colon E \to K$ satisfies functional equation (3) if and only if there exist an additive subgroup A of E and a function $k\colon A \to K\setminus\{0\}$ such that

$$k(x+y) = k(x)k(y) \qquad \text{for } x, y \in A, \tag{7}$$

$$\text{if } x, y \in A \text{ and } k(x)x = k(y)y, \text{ then } x = y, \tag{8}$$

$$f(x) = \begin{cases} k(y) & \text{if} \quad x = k(y)y, \quad y \in A \,; \\ 0 & \text{otherwise} \qquad\qquad \text{for} \quad x \in E \,. \end{cases} \tag{9}$$

Proof. (i) It is enough to notice that a function $g\colon L \longrightarrow E$ satisfies (2) if and only if

$$g(ab) = ag(b) + bg(a) \qquad \text{for} \quad a,b \in L.$$

(ii) Let $f\colon E \longrightarrow K$ be a function given by (9). We are going to prove that f fulfils (3). Fix $x,y \in E$. The case where $f(x)f(y) = 0$ is trivial. Assume that $f(x)f(y) \neq 0$. Then there exist $z,w \in A$ such that $x = k(z)z$ and $y = k(w)w$. Since $f(x) = k(z)$ and $f(y) = k(w)$, we have

$$f(f(y)^{-2}(f(y)x-f(x)y)) = f(k(z)k(w)^{-1}(z-w)) =$$

$$= f(k(z-w)(z-w)) = k(z-w) = f(x)f(y)^{-1}.$$

Thus f satisfies (3).

Now, assume that a function $f\colon E \longrightarrow K$ fulfils (3). Put $A = \{f(x)^{-1}x\colon x \in E\backslash f^{-1}(\{0\})\}$. By Lemma 2(iv) A is an additive subgroup of E. Define a function $k\colon A \longrightarrow K$ as follows:

$$k(f(x)^{-1}x) = f(x) \qquad \text{for} \quad x \in E\backslash f^{-1}(\{0\}).$$

It results from Lemma 2(ii) that k is well defined. We want to prove that k satsfies (7) and (8). Fix $z,w \in A$. Then there exist $x,y \in E$ such that $f(x)f(y) \neq 0$ and $z = f(x)^{-1}x$, $w = f(y)^{-1}y$. According to the definition of k we have $k(w) = f(y)$ and $k(z) = f(x)$.
Thus in view of Lemma 2(iii)

$$k(z+w) = k(f(x)^{-1}f(y)^{-1}(f(y)x+f(x)y)) = f(x)f(y) = k(z)k(w).$$

In this way we have proved that k fulfils (7). Further, suppose that $k(z)z = k(w)w$. Then $x = k(z)z = k(w)w = y$ and consequently $z = f(x)^{-1}x = f(y)^{-1}y = w$. Hence (8) holds, too.

Let

$$f_o(x) = \begin{cases} k(y) & \text{if } x = k(y)y, \quad y \in A ; \\ 0 & \text{otherwise} \end{cases}$$

for $x \in E$. In order to complete the proof it is enough to show that $f_o = f$. Fix an $x \in E$. If $f(x) \neq 0$, then $f(x)^{-1}x \in A$ and $k(f(x)^{-1}x) = f(x)$. Thus

$$f_o(x) = f_o(k(f(x)^{-1}x)f(x)^{-1}x) = k(f(x)^{-1}x) = f(x).$$

On the contrary, if $f(x) = 0$, then $x \notin \{k(z)z\colon z \in A\}$. Thus $f_o(x) = 0$ and consequently $f_o(x) = f(x)$. Hence $f_o = f$, which completes the proof.

Lemma 1 and Proposition 1 yield the following

Theorem. The following conditions hold:

(i) D_1 is a subgroup of the group $(H,\cdot)$ if and only if F_1 is a multiplicative subgroup of K and the function

$$F_1 \ni a \longrightarrow a^{-1}f_1(a)$$

satisfies functional equation (6),

(ii) D_2 is subgroup of the group $(H,\cdot)$ if and only if there exist an additive subgroup A of K and a function $k\colon A \longrightarrow K\setminus\{0\}$ satisfying (7) and (8) and such that $F_2 = \{k(a)a\colon a \in A\}$, $f_2(k(a)a) = k(a)$ for $a \in A$.

The subsequent proposition and corollary explain condition (8) a little.

Proposition 2. Let A be an additive subgroup of E. Let $k\colon A \longrightarrow K\setminus\{0\}$ be a function satisfying (7). Then k fulfils (8) if and only if

$$z \notin (1-k(z))A \qquad \text{for} \quad z \in A\setminus\{0\} . \tag{10}$$

Proof. Assume that k satisfies (8). For an indirect proof suppose that there exist $z,w \in A\setminus\{0\}$ such that $(1-k(z))w = z$ and denote $y = w-z$. Then $y \neq w$ and $(1-k(z))w = w-y$. Since $z = w-y$ and, by (7), $k(w-y) = k(w)k(y)^{-1}$, we have $k(w)w = k(y)y$, which contradicts (8).

Now, assume that (10) holds. For an indirect proof suppose that k does not satisfy (8). Then there exist $z, w \in A\setminus\{0\}$, $z \neq w$ such that $k(z)z = k(w)w$. Denote $y = w-z$. Then $k(w)w = k(w-y)(w-y)$ and consequently, by (7), $w = k(-y)(w-y)$. Since, on account of (7), $k(-y) = k(y)^{-1}$, we obtain that $y = (1-k(y))w$, a contradiction.

Corollary. The following conditions hold.

(i) Let A be an additive subgroup of E. Let B a multiplicative subgroup of K. If

$$A \cap ((1-B)A) = \{0\} , \tag{11}$$

then every function $k\colon A \longrightarrow B$ satisfying (7) fulfils also (8).

(ii) If a function $k\colon E \longrightarrow K\setminus\{0\}$ satisfies (7) and (8), then $k \equiv 1$.

(iii) If A is an additive subgroup of K and

$$B := K\setminus\{1-ab^{-1}\colon a,b \in A\setminus\{0\}\},$$

then every function $k\colon A \longrightarrow B$ satisfying (7) fulfils (8), too.

Condition (ii) is due to Z. Wilczyński; the author wish to thank him for that at this place.

Proof. (i) Since (11) holds and $k(A) \subset B$, we obtain that (10) is valid, which, by Proposition 2, completes the proof.

(ii) For an indirect proof suppose that there exists a $y \in E$ such that $k(y) \neq 1$. Then $y \in (1-k(y))E$. Thus (10) does not hold, a contradiction.

(iii) Let us observe that, for every $b \in B$ and $a_1, a_2 \in A\setminus\{0\}$, we have $b \neq 1-a_1a_2^{-1}$ and consequently $a_1 \neq a_2(1-b)$. Thus, on account of Proposition 2, we get the assertion.

Remark. By Corollary (i), (iii) one can find numerous examples of functions satisfying (7) and (8). For instance, if $K = R$, then the function

$$Z \ni n \longrightarrow \pi^n$$

fulfils (7) and (8), where R and Z denote the sets of all real numbers and integers, respectively.

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PODGRUPY GRUPY CLIFFORDA

S t r e s z c z e n i e

Niech K będzie ciałem przemiennym. Zbiór $H = (K\setminus\{0\})\times K$ wyposażony w działanie

$$(a,b)(c,d) = (ac, ad + cb)$$

nazywamy grupą Clifforda ciała K. Jest ona izomorficzna z multiplikatywną grupą macierzy postaci $\begin{bmatrix} a & 0 \\ b & a \end{bmatrix}$, gdzie $a,b \in K$, $a \neq 0$.

Niech $F_1 \subset K\setminus\{0\}$, $F_2 \subset K$ będą niepustymi zbiorami i $f_1: F_1 \to K$, $f_2: F_2 \to K\setminus\{0\}$.

Twierdzenie. (i) Zbiór $\{(a, f_1(a)): a \in F_1\}$ jest podgrupą grupy $(H,\cdot)$ wtedy i tylko wtedy, gdy F_1 jest podgrupą multiplikatywną ciała K i funkcja $F_1 \ni a \to a^{-1}f_1(a) \in K$ spełnia równanie funkcyjne (6).

(ii) Zbiór $\{(f_2(a), a): a \in F_2\}$ jest podgrupą grupy $(H,\cdot)$ wtedy i tylko wtedy, gdy istnieje podgrupa addytywna A ciała K i funkcja $k: A \to K\setminus\{0\}$ spełniająca warunki (7) i (8) takie, że $F_2 = \{k(a)a: a \in A\}$, $f_2(k(a)a) = k(a)$ dla $a \in A$.

W szczególności, w pracy podane jest rozwiązanie ogólne równania funkcyjnego (3).

ПОДГРУППЫ ГРУППЫ КЛИФФОРДА

Р е з ю м е

Пусть К будет полем. Множество $H = (K\setminus\{0\}\times K$ с бинарной операцей

$$(a,b)(c,d) = (ac, ad + cb)$$

называется группой Клиффорда поля К. Она изоморфна мультипликативной группе матриц вида $\begin{bmatrix} a & 0 \\ b & a \end{bmatrix}$, где $a,b \in K$, $a \neq 0$.

Пусть $F_1 \subset K\setminus\{0\}$, $F_2 \subset K$ будут непустыми множествами и $f_1 : F_1 \longrightarrow K$, $f_2 : F_2 \longrightarrow K\setminus\{0\}$.

Теорема. (I) Множество $\{(a, f_1(a)) : a \in F_1\}$ является подгруппой группы $(H,\cdot)$ тогда и только тогда, если F_1 является мультипликативной подгруппой К и функция $F_1 \ni a \longrightarrow a^{-1}f_1(a) \in K$ удовлетворяет функциональному уравнению (6).

(II) Множество $\{(f_2(a), a) : a \in F_2\}$ является подгруппой группы $(H,\cdot)$ тогда и только тогда, если существует аддитивная подгруппе А поля К и функция $k: A \to K\setminus\{0\}$ для которых выполняются условия (7), (8) и такие что $F_2 = \{k(a)\, a : a \in A\}$, $f_2(k(a)a) = k(a)$ для $a \in A$.

В особенности представлено общее решение функционального уравнения (3).