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Wiesław SASIN

COUNTABLE CARTESIAN PRODUCT OF DIFFERENTIAL SPACES

Summary. In this paper we study some properties of a differential space (M,C) which is the countable Cartesian product of differential spaces (M_i,C_i), $i \in N$, in the sense of Sikorski. It is proved that the tanget space to the countable Cartesian product of differential spaces is the direct pruduct of the tangent spaces to each factor. The C-modul $\mathfrak{X}_i(M)$ of all smooth vector fields tangent to (M,C) parallel to (M_i,C_i) is defined and investigated. One proves that C-module $\mathfrak{X}(M)$ of all smooth vector fields tangent to (M,C) is isomorphic to the direct product of the C-modules $\mathfrak{X}_i(M)$, $i \in N$.
Some properties of the countable Cartesian product of differential spaces of constant differential dimension are presented. It is proved that then in the graded algebra A(M) of the pointwise forms there exists a unique operation which satisfies the well-known axioms of exterior derivative. Some sufficient conditions for the existence of a linear connection in a C-module $\mathfrak{X}(M)$ are presented. If (M,C) is a countable Cartesian product of compact differential spaces of constant differential dimension, there exists a linear connection in the C-Module $\mathfrak{X}(M)$. The notions of a smooth tensor of type (n,1), a vector field and a connection projectible onto (M_i,C_i) allow us to study some properties of a connection and tensors on (M,C) by investigation of the properties of its projection. In this way is proved that the curvature tensor of the connection ∇ in the Countable Cartesian product of parallelizable differential space is equal to 0.

In this paper we study some properties of the countable Cartesian product of differential space in the sense of Sikorski [4].
In particular the countable Cartesian product of differential manifolds of class C^∞ may be considered as a differential space. In section 1 we review some of the standart facts on Sikorski's differential spaces [3], [4]. It is possible in a natural way to introduce a diferential structure on the Cartesian product of differential spaces. InSection 2 we describe some basic notions and facts concerning the countable Cartesian product of differential spaces considered as a differential space.

1. PRELIMINARIES

Let M be a non-empty set and C a set of real functions defined on M. We denote by τ_C the weakest topology on M such that all functions belonging to C are continnnnous. For an arbitrary subset $A \subset M$ we denote by C_A the set of all functions $g\colon A \longrightarrow R$ such that for each point $p \in A$ there exist an open neighbourhood $U \in \tau_C$ of p and a function $f \in C$ such that $g|A \cap U = f|A \cap U$.

We denote by scC the set of all real functions of the form $\omega \circ (\alpha_1, \dots, \alpha_n)$, where $\omega \in \varepsilon_n$, $\alpha_1, \dots, \alpha_n \in C$, $n \in N$ and ε_n is the set of all real smooth functions of class C^∞ on R^n.

A set C is said to be a differential structure on M if

a) the set C is closed with respect to localization, i.e., $C = C_M$,

b) the set C is closed with respect to composition with smooth functions, i.e., $C = scC$.

By a differential space we shall mean any pair (M,C), where M is a set and C is a differential structure on M.

For a set C_o of real functions defined on M the set $(scC_o)_M$ is the smallest differential structure on M including the set C_o.
A differential structure C is said to be generated by C_o iff $C = (scC_o)_M$.

$\varphi \in (scC_o)_M$ if for any point $p \in M$ there exist an open in τ_{C_o} set $U \ni p$ and functions $\omega \in \varepsilon_n$, $\alpha_1, \dots, \alpha_n \in C_o$ such that $\varphi|U = \omega \circ (\alpha_1, \dots, \alpha_n)|U$.

If (M,C) is a differential space and $A \subset M$, then (A, C_A) is also a differential space called the differential subspace of (M,C) [5].

By a vector tangent to a differential space (M,C) at a point $p \in M$ we shall mean any linear mapping $v\colon C \longrightarrow R$ such that

$$v(\alpha \cdot \beta) = v(\alpha) \cdot \beta(p) + \alpha(p) \cdot v(\beta) \quad \text{for all } \alpha, \beta \in C. \tag{1}$$

The set T_pM of all tangent vectors at a given point $p \in M$ is a linear space. For any $v \in T_pM$ we have the formula

$$v(\omega \circ (\alpha_1, \dots, \alpha_n)) = \sum_{i=1}^{n} \omega'|_i(\alpha_1(p), \dots, \alpha_n(p)) \cdot v(\alpha_i) \tag{2}$$

for $\alpha_1, \dots, \alpha_n \in C$ and $\omega \in \varepsilon_n$.

Let (M,C) and (N,D) be differential spaces. A mapping $f\colon M \longrightarrow N$ is said to be smooth iff $f^*(\alpha) := \alpha \circ f \in C$ for every $\alpha \in D$.

If $f\colon M \longrightarrow N$ is smooth and $v \in T_pM$, then the formula

$$(f_{*p}v)(\alpha) = v(\alpha \circ f) \quad \text{for} \quad \alpha \in D$$

defines a vector $f_{*p}v$ tanget to N at $f(p)$.

Now we prove the following lemma which will be usefull in the sequal.

Lemma 1. Let C be a differential structure on M generated by a set C_o and $p \in M$ be an arbitrary point. Let $w_o\colon C_o \longrightarrow R$ be a mapping satysfying the following condition:

(*) for arbitrary $\alpha_i,\dots,\alpha_n \in C_o$ and $\omega \in \varepsilon_n$ if $\omega \circ(\alpha_1,\dots,\alpha_n) = 0$ then

$$\sum_{i=1}^{n} \omega'_{|i}(\alpha_1(p),\dots\alpha_n(p))w_o(\alpha_i) = 0 .$$

Then there is unique vector $w \in T_pM$ tangent to (M,C) at p such that

$$w|C_o = w_o .$$

Proof. For an arbitrary function $\varphi \in C$ we put

$$w(\varphi) = \sum_{i=1}^{n} \omega'_{|i}(\alpha_1(p),\dots,\alpha_n(p)\cdot w_o(\alpha_i) , \tag{3}$$

where $\alpha_1,\dots,\alpha_n \in C_o$ and $\omega \in \varepsilon_n$ are smooth functions such that there exists an open sset in λ_C $U \ni p$ and $\varphi|U = \omega \circ(\alpha_1,\dots,\alpha_n)|U$.

From the condition (*) it is easy to see correctness of (3) and one can verify that $w\colon C \longrightarrow R$ is a vector tangent to (M,C) at p such that $w|C_o = w_o$.

If $\tilde{w}$ is another vector such that $\tilde{w}|C_o = w_o$ then by (2) we have

$$\tilde{w}(\varphi) = \sum_{i=1}^{n} \omega'_{|i}(\alpha_1(p),\dots,\alpha_n(p))\cdot \tilde{w}(\alpha_i) = w(\varphi).$$

So the vector w is unique.

By a smooth vector field to (M, C) we mean every R-linear mapping $X\colon C \longrightarrow C$ such that

$$X(\alpha \cdot \beta) = \alpha \cdot X(\beta) + X(\alpha)\cdot \beta \quad \text{for all} \quad \alpha,\ \beta \in C.$$

The set $\mathfrak{X}(M)$ of all smooth vector fields tangent to (M, C) is a C-module.

Now let (M, C) be a differential space and Φ be a mapping which assigns a real space $\Phi(p)$ to ant point $p \in M$. By a Φ-field on M we shall mean any function W which assigns an element $W(p) \in \Phi\,(p)$ to any $p \in M$ [4].

A C-module $\mathfrak{W}$ of Φ-fields on (M, C) is said to be a differential module of dimension m if

(a) $\mathfrak{W}$ is closed with respect to localization, i.e., $\mathfrak{W}$ $\mathfrak{W}_M$

(b) $\mathfrak{W}$ has locallt a vector basis composed of m fields, i,e., if every point $p \in M$ has a neighbourhood $U \in \tau_C$ and there exist $\Phi|U$-fields $W_1, \dots, W_m$ on U such that for every point $p \in U$ the sequence $W_1(p), \dots, W_m(p)$ is a basis of the linear space $\Phi(p)$ and $W_1, \dots, W_m$ is a basis of C_U - module $\mathfrak{W}_U$.

Every vector field $X \in \mathfrak{X}(M)$ may be interpreted as Φ-field, Φ being the function $\Phi(p) = T_pM$ for $p \in M$ and $X(p)(\alpha) = X(\alpha)(p)$ for $p \in M$ and $\alpha \in C$.

We say that a differential space (M, C) has a differential dimension m if C-module $\mathfrak{X}(M)$ is a differential module of dimension m [4].

Now for $k \in N$ let $\Omega^k(M)$ be a C-module of all skew-symmetric C-k-linear mappings of the form $\omega : \mathfrak{X}(M)\text{x}\dots\text{x}\mathfrak{X}(M) \longrightarrow C$. The direct sum $\Omega(M) = \bigoplus_{k\geq 0} \Omega^k(M)$, where $\Omega^o(M) := C$, together with the canonical operations of addition and exterior multiplication is a graded algebra over R. In the algebra $\Omega(M)$ there is the operation d of exterior derivation given by the well-known global formula [5], [2].

Now let $TM := \bigsqcup_{p\in M} T_pM$ be a disjoint union of tangent spaces to (M, C).

By TC we denote the differential structure on TM generated by the set $\{\alpha \circ \Pi : \alpha \in C\} \cup \{d\alpha : \alpha \in C\}$, where $\Pi : TM \longrightarrow M$ is the canonical projection and $d\alpha : TM \longrightarrow R$ is the function defined by

$$(d\alpha)(v) = v(\alpha) \quad \text{for} \quad a \in C \quad \text{and} \quad v \in TM.$$

For any $k \in N$ put

$$T^{M}M = \{(v_1,\dots,v_k)\in TMx\dots xTM : \Pi(v_1) = \Pi(v_k)\}$$

and

$$T^kC = (TCx\dots xTC)_{T^kM}.$$

We denote $A^k(M)$, $k \in N$, the set of all smooth mappings $\omega : T^kM \longrightarrow R$ such that the mapping $\omega|T_pMx\dots xT_pM$ is skew-symmetric k-linear for each point $p \in M$. The direct sum $A(M) = \oplus_{k\le 0} A^k(M)$, where $A^0(M) = C$, together with the canonical operations of addition and multiplication is a graded algebra over R. The mapping $h_M: A^k(M) \longrightarrow \Omega^k(M)$, $k \in N$, given by the formula

$$(h_M\omega)(X_1,\dots,X_k)(p) = \omega(X_1(p),\dots,X_k(p)) \quad \text{for} \quad \omega \in A^k(M),$$

$X_1,\dots,X_k \in \mathfrak{X}(M)$, is a homomorphism of graded algebras. If a differential space (M,C) has a constant dimension then h_M is an isomorphism.

Definition 1. A differential space (M,C) is said to have the property (P) if for any $v \in TM$ there is a smooth vector field $X \in \mathfrak{X}(M)$ such that $X(\Pi(v) = v$.

One can prove

Lemma 2. If a differential space (M,C) has the property (P) then h_M is a monomorphism.

If we want to define an operator of exterior derivation in the algebra $A(M)$ we meet some difficulties [2]. Let $m^k(M)$ for $k \ge 1$ be the set of all elemments $\omega \in A^k(M)$ such that for each point $p \in M$ there exists an open neighbourhood U of p and a finite family of smooth functions

$$\alpha_{i_1\dots i_{k-1}},\ \alpha_{i_1},\dots,\alpha_{i_{k-1}} \in C_U \quad \text{for } (i_1,\dots,i_{k-1})\in I \subset N^{k-1} \quad \text{such that}$$

$$\omega|_U = \sum_I d\alpha_{i_1\dots i_k} \wedge d\alpha_{i_1} \wedge\dots\wedge d\alpha_{i_{k-1}}$$

and

$$\sum_I \alpha_{i_1 \dots i_{k-1}} \cdot d\alpha_{i_1} \wedge \dots \wedge d\alpha_{i_{k-1}} = 0.$$

$\mathfrak{M}(M) \bigoplus_{k \geq 0} \mathfrak{M}^k(M)$, where $\mathfrak{M}^{\rho}(M) := \{0\}$, is a homogeneous ideal in the graded algebra $A(M)$.

One can prove [2]

Proposition 1. Let (M,C) be a differential space.
If the ideal $\mathfrak{M}(M) = \{0\}$ then in the graded algebra $A(M)$ there exists exactly operator $\tilde{d}\colon A^k(M) \longrightarrow A^{k+1}(M)$, $k \in N$, satisfying the wellknown condition of exterior derivation.

If ω has a local form $\omega|_U = \sum_I \alpha_{i_1 \dots i_k} \cdot d\alpha_{i_1} \wedge \dots \wedge d\alpha_{i_k}$ the

$$d\omega|_U = \sum_I d\alpha_{i_1 \dots i_k} \wedge d\alpha_{i_1} \wedge \dots \wedge d\alpha_{i_k}.$$

The following diagram commutes

$$\begin{array}{ccc} A^k(M) & \overset{o\ell}{\longrightarrow} & A^{k+1}(M) \\ h_M \big\downarrow & & \big\downarrow h_M \\ \Omega^k(M) & \overset{d}{\longrightarrow} & \Omega^{k+1}(M) \end{array}$$

By a covariant derivative in a C-module of Φ-fields $\mathfrak{B}$ [5] we shall mean a function $\nabla : \mathfrak{X}(M) \times \mathfrak{B} \longrightarrow \mathfrak{B}$ which assigns to every $X \in \mathfrak{X}(M)$ and to every $W \in \mathfrak{B}$ a Φ-field $\Delta_X W$ in such a way that $\Delta_X W$ is a C-linear function of the variable W and the following condition is fulfield:

$$\Delta_X(\alpha W) = X(\alpha) \cdot W + \alpha \cdot \Delta_X W \quad \text{for} \quad \alpha \in C, \quad X \in \mathfrak{X}(M), \quad W \in \mathfrak{B} \tag{4}$$

2. MAIN RESULTS

Let (M_i, C_i) $i \in N$ be a sequence of differential spaces. Let $M = \underset{i\in N}{X} M_i$ be the Cartesian product of the sets M_i, $i \in N$. We denote by pr_j: $M \longrightarrow M_j$ for $j \in N$ the natural projection onto the j-th coordinate.

For any $\alpha \in C_j$ and $j \in N$ let $\bar{\alpha} : M \longrightarrow R$ be the function given by

$$\bar{\alpha} = pr_j^*(\alpha) = \alpha \circ pr_j \tag{5}$$

Put $C_o = \bigcup_{i\in N} pr_i^*(C_i)$. Let C be the differential structure on M generated by the set C_o. Then the differential space (M,C) is said to be the countable Cartesian product of differential spaces (M_i, C_i), $i \in N$.

It is easy bo observe that the topology τ_C is Tichonov's topology of the topologies τ_{C_i}, $i \in N$.

We put $M(\hat{k}) = M_1 \times \ldots \times M_{k-1} \times M_{k+1} \times \ldots$ for $k \in N$. For an arbitrary $p = (p_i)_{i \in N} \in M$ let $p(\hat{k}) = (p_1, \ldots, p_{k-1}, p_{k+1}, \ldots)$, $k \in N$. Of course $p(\hat{k}) \in M(\hat{k})$. For an arbitrary $q \in M(\hat{k})$ let Jq: $M \longrightarrow M$ be the imbeding defined by

$$J_q(s) = (q_1, \ldots, q_{k-1}, s, q_k, \ldots) \quad \text{for} \quad s \in M_k \tag{6}$$

It is easy to verify the identities:

$$pr_k \circ J_q = id_{M_k} \tag{7}$$

$$(pr_i \circ J_q)(s) = q_i \quad \text{for any} \quad s \in M_k \quad \text{if} \quad i \neq k. \tag{8}$$

It follows easilt from (7) and (8) that J_q is a smooth mapping.

Now let $w \in T_pM$ be a vector tangent to (M,C) at the point $p = (p_i)_{i\in N}$. Put

$$v_i = pr_{i*p}w \quad \text{for} \quad i \in N. \tag{9}$$

So every vector $w \in T_pM$ determines the vectors $v_i \in T_{p_i}M_i$, $i \in N$ defined by (9). Conversely

Proposition 2. If $(v_i)_{i\in N}$ is a sequence of vectors $v_i \in T_{p_i}M_i$, $i \in N$ then there exists unique vector $w \in T_pM$ such that

$$v_i = pr_{i*p}w \quad \text{for} \quad i \in N.$$

Proof. Let $w_o\colon C_o \longrightarrow R$ be the mapping given by

$$w_o(\bar{\alpha}) = v_j(\alpha) \quad \text{for} \quad \alpha \in C_j, \quad j \in N. \tag{10}$$

It is easily seen that $\overset{*}{pr_i}(C_i) \cap \overset{*}{pr_j}(C_j)$ for $i \neq j$ consists of constant function on M. Thus the formula (10) is correct. It remains to prove that w_o satisfies the condition (*) of Lemma 1. Without of loss of generality assume that $\omega \cdot(\bar{\alpha}_1,\dots,\bar{\alpha}_n) = 0$, where $\omega \in \varepsilon_n$, $\alpha_i \in C_i$ for $i = 1.2,\dots,n$. Consider the vector $v \in T_pM$ defined

$$v = J_{p(\hat{1})*p_1}v_1 +\dots+ J_{p(\hat{n})*p_n}nv .$$

Of course $v\ (\omega\cdot(\bar{\alpha}_1,\dots,\bar{\alpha}_n)) - 0$. Hence by (2) we have

$$\sum_{i=1}^{n} \omega'_{|i}(\bar{\alpha}_1(p),\dots,\bar{\alpha}_n(p))\cdot v)\bar{\alpha}_i) = 0.$$

Clearly $v\ (\bar{\alpha}_i) = (J_{p(\hat{i})*p_i}v_i(\bar{\alpha}_i) = v_i(\alpha_i) = w_o(\bar{\alpha}_i)$ for $i + 1,2,\dots,n$.

Thus $\sum_{i=1}^{n} \omega'_{|i}(\bar{\alpha}_1(p),\dots,\bar{\alpha}_n(p))\cdot w_o(\bar{\alpha}_i) = 0$. In view of Lemma 1 there is unique vector $w \in T_pM$ such that $w|C_o = w_o$. Observe that $w\ (\bar{\alpha}) = w_o(\bar{\alpha}) = v_i(\alpha)$ for for any $\alpha \in C_o$. Hence $v_i = pr_{i*p}w$ for $i \in N$.

Now we may prove

Proposition 3. The mapping $K\colon T_pM \longrightarrow \underset{i\in N}{X}\ T_{p_i}M_i$ defined by the formula

$$K(w) = (pr_{i*p}w) \quad \text{for} \quad w \in T_pM \tag{11}$$

is an isomorphism of T_pM and the direct product of $T_{p_i}M_i$, $i \in N$.

is an isomorphism of T_pM and the direct product of $T_{p_i}M_i$, $i \in N$.

The proof is immediate.

Now for a vector $w \in T_pM$ put

$$w_i = (J_{p(\hat{i})} \circ pr_i)_{*p}w \quad \text{for} \quad i \in N. \tag{12}$$

Of course $w_i \in T_pM$ for $i \in N$ and the vector w_i satisfies the folowing condition:

$$w_i(\bar{\alpha}_j) = 0 \quad \text{for} \quad \alpha_j \in C_j \quad \text{and} \quad j \neq i. \tag{13}$$

The vector w_i is said to be the i-th component of the vector w.

Definition 2. A vector $v \in T_pM$ is said to be parallel to (M_k, C_k) if $v(\bar{\alpha}) = 0$ for any $\alpha \in C_j$ and $j \neq k$.

Clearly the i-th component w_i of a vector $w \in T_pM$ is parallel to (M_i, C_i).

Lemma 3. The foloowing conditions are equivalent:

(i) $w \in T_pM$ is a vector parallet to (M_k, C_k)

(ii) the k-th component w_k is equal to w

(iii) $w \in J_{p(\hat{k})*p}(T_{p_k}M_k)$.

The proof is straightforward.

It is easy to see that the subspace $J_{p(\bar{k})*p_k}(T_{p_k}M_k)$ of all vectors from T_pM paralel to (M_k, C_k) is isomorphic to $T_{p_k}M_k$. The maping

$J_{p(\hat{k})*p_k} : T_{k_p}M_k \longrightarrow T_pM$ is an isomorphism onto image.

Proposition 4. If $v_i \in T_pM$, $i \in N$ is a sequence of vectors paralel to (M_i, C_i) respectively then there is unique vector $w \in T_pM$ which has the i-th component $w_i = v_i$ for $i \in N$.

Proof. It followsk from Proposition 2 that for the sequence $pr_{i*p}v_i \in T_p M_i$, $i \in N$ there is unique vector $w \in T_pM$ such that

$$pr_{i*p}w = pr_{i*p}v_i \quad \text{for} \quad i \in n.$$

Of course $w_i + (J_{p(\hat{i})} \circ pr_i)_{*p}w = J_{p(\hat{i})}p_i(pr_{i*p}v_i) = v_i$ for $i \in N$.

This finishes the proof.

Now let $w \in T_pM$ be an arbitrary vector. Let $\varphi \in C$ be a smooth function. There exists an open neighbourhood U of p and functions $\beta_{i_1},\dots,\beta_{i_n} \in C_o$, $\omega \in \varepsilon_n$ such that $\varphi|_U = \omega \circ (\beta_{i_1},\dots,\beta_{i_n})|U$.
From (2) and (13) it follows that $w_i(\varphi) = 0$ for $i \notin \{i_1,\dots,i_n\}$.
One can verify the identity

$$w(\beta_{i_k}) = w_i(\beta_{i_k}) \quad \text{if} \quad \beta_{i_k} \in pr_i^*(C_i). \tag{14}$$

Thus by (2) we have

$$w)\varphi) = \sum_{k=1}^{n} w_{i_k}(\varphi) \; .$$

So we may uniquely present the vector w as a formal sum of its components:

$$w = \sum_{i\in N} w_i \; .$$

In the sequal the vector w in Proposition 4 corresponding to a sequence $(v_i)_{i\in N}$ of vectors parallel to (M_i,C_i) respectively we will denote by

$$\sum_{i\in N} v_i .$$

Proposition 5. If (M_i,C_i), $i \in N$ is a sequence of differential manifolds of class C^∞ then for an arbitrary vector $w \in T_pM$ there exists a smooth curve $c\colon (-\varepsilon, \varepsilon) \longrightarrow M$ such that $c_{*o}\left.\frac{\partial}{\partial s}\right|_o = w$ and $c(0) = p$.

Proof. Choose $\varepsilon > 0$. Since M_i for $i \in N$ is a differential manifold of class C^∞, for the vector $pr_{i*p}w$ there exists a smooth curve $c_i\colon (-\varepsilon,\varepsilon) \longrightarrow M_i$ such that $pr_{i*p}w = c_{i*o}\left.\frac{\partial}{\partial s}\right|_o$ and $c_i(0) = p_i$.
Let $c\colon (-\varepsilon, \varepsilon) \longrightarrow M$ be the smooth curve defined by

$$c(t) = (c_i(t))_{i\in N} \quad \text{for} \quad t \in (-\varepsilon, \varepsilon).$$

Of course $c(0) = (p_i)_{i\in N} = p$. It is easy to see that $pr_{i*p}w = = pr_{i*p}(c_{*o}\frac{\partial}{\partial s}\Big|_o$ for $i \in N$. From Proposition 2 it follows that

$$w = c_{*o}\frac{\partial}{\partial s}\Big|_o .$$

Now, let $Z \in \mathfrak{X}(M)$ be an arbitrary vector field tangent to (M,C). We will denote by Z_i for $i \in N$ the vector field tangent to (M,C) given by

$$Z_i(p) = (J_{p(\hat{i})} \circ pr_i)_{*p}\, Z(p) \quad \text{for} \quad p \in M. \tag{15}$$

One can prove the identities:

$$Z_i(\bar{\alpha}) = Z(\bar{\alpha}) \quad \text{for} \quad \alpha \in C_i \tag{16}$$

$$Z_i(\bar{\alpha}) = 0 \quad \text{for} \quad \alpha \in C_j \quad \text{if} \quad j \neq i/ \tag{17}$$

So the vector field Z_i is smooth. The vector field Z_i is called the i-th component of Z.

Let $\varphi \in C$ be a smooth function. For an arbitrary point $p \in M$ there exist an open neighbourhood U and functions $\beta_i, \dots, \beta_i \in C_o$, $\omega \in \varepsilon_n$ such that $\varphi|U = \omega \circ (\beta_{i_1}, \dots, \beta_{i_n})|U$. Then

$$Z(\varphi)|U = Z_{i_1}(\varphi)|U + \dots + Z_{i_n}(\varphi)|U. \tag{18}$$

The sequence $(Z_i(\varphi))_{i\in N}$ is locally finite. We may write Z as a formal sum of its components: $Z = \sum_{i\in N} Z_i$. From (16) it follows that the components Z_i, $i \in N$ are C-linearly independent.

Definition 3. A vector field $Z \in \mathfrak{X}(M)$ is said to be parallel to (M_k, C_k) if for every $p \in M$ the vector $Z(p)$ is parallel to (M, C_k).

In a similar way as Lemma 3 one can prove

Lemma 4. Let $Z \in \mathfrak{X}(M)$. The foloowing condition are equivalent:

(i) Z is parallel to (M_k, C_k)

(ii) $Z(p) \in J_{p(\hat{k})*p_k}(T_{p_k} M_k)$ for each $p \in M$

(iii) $Z = Z_k$

(iv) $Z(\bar{\alpha}) = 0$ for any $\alpha \in C_j$ if $j \neq k$.

We denote by $\mathfrak{X}_k(M)$ the set of all smooth vector fields tangent to (M,C) which are parallel to (M_k, C_k). It is clear that $\mathfrak{X}_k(M)$ is a C-submodule of the C-module $\mathfrak{X}(M)$.

Lemma 5. If $X_i \in \mathfrak{X}_i(M), i \in N$ is a sequence of smooth vector fields parallel to (M_i, C_i) respectively then there exists unique vector field $Z \in \mathfrak{X}(M)$ such that $Z_i = X_i$ for $i \in N$.

Proof. Let $Z \in \mathfrak{X}(M)$ be the vector field given by

$$Z(p) = \sum_{i \in N} X_i(p) \quad \text{for} \quad p \in M. \tag{19}$$

For any $p \in M$ and $\alpha \in C_k$ we have the equality $Z(p)(\bar{\alpha}) = X_k(p)(\bar{\alpha})$. Thus $Z(\bar{\alpha}) = X_k(\bar{\alpha})$. So Z is smooth. It is clear that $Z_i(p) = X_i(p)$ for $i \in N$ and $p \in M$. Hence $Z_i = X_i$ for $i \in N$.

In the sequal the vector field Z defined by (19) we will denote by $\sum_{i \in N} X_i$.

Proposition 6. The mapping $L\colon \mathfrak{X}(M) \longrightarrow \underset{i \in N}{\mathrm{X}} \mathfrak{X}_i(M)$ defined by

$$L(Z) = (Z_1, Z_2, \dots) \quad \text{for} \quad Z \in \mathfrak{X}(M) \tag{20}$$

is an isomorphism of the C-module $\mathfrak{X}(M)$ and the direct product of the C-modules $\mathfrak{X}_i(M)$, $i \in N$.

Proof. It is clear that L is a homomorphism of C-modules. Let $Z \in \mathfrak{X}(M)$ be a vector field such that $L(Z) = 0$. Then $Z_i = 0$ for $i \in N$. In view of Lemma 5 $Z = 0$. Thus $\ker L = \{0\}$. By Lemma 5 the homomorphism L is "onto". Therefore L is an isomorphism.

Now we will give some characterisation of a smooth vector field tangent to M,C) parallel to (M_k, C_k).

Definition 4. Indexed set $(X^q)_{q\in M(\hat{k})}$ of smooth vector field $X^q \in \mathfrak{X}(M_k)$ is said to be smooth if the function $\psi : M \longrightarrow TM_k$ given by

$$\psi(p) = X^{p(\hat{k})}(p_k) \quad \text{for} \quad p \in M$$

is a smooth mapping of (M,C) into (TM_k, TC_k).

Let us observe that a smooth indexed set$(X^q)q\in M(\hat{k})$ of smooth vector fields $X^q \in \mathfrak{X}(M_k)$ determines the smooth vector field $X \in \mathfrak{X}_k(M)$ parallel to (M_k, C_k) by the formula

$$X(p) = J_{p(\hat{k})*p_k} X^{p(\hat{k})}(p_k) \quad \text{for} \quad p \in M. \tag{21}$$

Conversely if $X \in \mathfrak{X}_k(M)$ then there exist the smooth idenxed set $(X^q)_{q\in M(\hat{k})}$ of smooth vector fields tangent to (M_k, C_k) defined by

$$X^q(s) = pr_{k*p} X(p) \quad \text{for} \quad g \in M(\hat{k}), \quad s \in M_k, \tag{22}$$

where $p = (p_i)_{i\in N}$ is such a point of M that $p(\hat{k}) = q$ and $p_k = s$.

So we may write

Proposition 7. A vector field $X \in \mathfrak{X}(M)$ is parallel to (M_k, C_k) if and only if there exists a smooth indexed set $(X^q)_{q\in M(\hat{k})}$ of smooth vector fields tangent to (M_k, C_k) such that

$$X(p) = J_{p(\hat{k})*pk} X^{p(\hat{k})}(p_k) \quad \text{for} \quad p \in M.$$

Moreover there is one-to-one correspondence betwen smooth vector fields parallel to (M_k, C_k) and smooth indexed sets of smooth vector fields tangent to (M_k, C_k).

Noww, let $X \in \mathfrak{X}(M_k)$ be an arbitrary smooth vector field tangent to (M_k, C_k). Let $\bar{X}: M \longrightarrow \bigcup_{p\in M} T_pM$ be the mapping given by

$$\bar{X}(p) = J_{p(\hat{k})*p} X_{p_k} \quad \text{for} \quad p \in M. \tag{23}$$

It is easy to verify that $\bar{X}$ is a smooth vector field tangent to (M,C) parallel to (M_k, C_k). The corresponding indexed set $(\bar{X}^q)_{q\in M(\hat{k})}$ is constant, i.e., $\bar{X}^q =$ for ant $q \in M(\hat{k})$.

Lemma 6. If (M_k, C_k) is a differential space of dimension n then the C-module $\mathfrak{X}_k(M)$ is an n-dimensional differential module of Φ-fields, where $\Phi(p) = J_{p(\hat{k})*p_k}(T_{p_k}M_k)$ for $p \in M$.

Proof. The closeness of $\mathfrak{X}_k(M)$ with respect to localization is evident. Let $p = (p_i)_{i\in N}$ be an arbitrary point of M. Let $V \in \tau_0$ be an open neighbourhood of p_k such that there is on V a local vector basis $W_1, \ldots, W_n$ of the C_k-module $\mathfrak{X}(M_k)$. Consider the vector fields $\bar{W}_1, \ldots, \bar{W}_n$ defined by (23) on the open set $\bar{V} = M_1 x \ldots x M_{k-1} x V x M_{k+1} x \ldots$.

We will show that the sequence $\bar{W}_1, \ldots, \bar{W}_n$ os a vector basis of the C-module $\mathfrak{X}_k(M)$. Of course for any $q \in \bar{V}$ the mapping $J_{q(\hat{k})*q_k} : T_{q_k}M_k \longrightarrow \Phi(q)$ is an isomoorphism. Since $W_1(q_k), \ldots, W_n(q_k)$ is a basis of the vector space $T_{q_k}M_k$ the sequence $\bar{W}_1(q) = J_{q(\hat{k})*qk} W_1(q_k), \ldots, \bar{W}_n(q) = J_{q(\hat{k})*q_k} W_n(q_k)$ is a basis of $\Phi(q)$. It remains to show that for any $Z \in \mathfrak{X}_k(M)$ the restriction $Z|\bar{V}$ may be presented as a $C_{\bar{V}}$-linear combination of $\bar{W}_1, \ldots, \bar{W}_n$. Let $(Z^s)_{s\in M(\hat{k})}$ be the smooth indexed set of smooth vector fields from $\mathfrak{X}(M_k)$ corresponding to Z and $\psi : M \longrightarrow TM_k$ be the smooth function given by Definition 4. Since $\mathfrak{X}_k(M)$ is a differential module of dimension n we may write $Z^s|V = \sum_{i=1}^{n} \varphi_s^i W_i$ for $s \in M(\hat{k})$, where $\varphi_s^i \in C_{kV}$ for $i = 1,2,\ldots,n$.

Put

$$\bar{\varphi}^i(q) = \varphi^i_{q(\hat{k})}(q_k) \quad \text{for} \quad q \in \bar{V} \quad \text{and} \quad i = 1,\ldots,n. \tag{24}$$

Let $W_i^* : TV \longrightarrow R$ be a smooth function defined by

$$W_i^*(W_j(x)) = \delta_{ij} \quad \text{for} \quad x \in V, \quad i,j = 1,\ldots,n.$$

It is easy to see that $\bar{\varphi}_i = W_i^* \circ (\psi|V)$ for $i = 1,\ldots,n$. Thus $\bar{\varphi}_i \in C_V$ for $i = 1,\ldots,n$. An easy computation shows that

$$Z(q) = J_{q(\hat{k})*q_k} Z^{q(\hat{k})}(q_k = J_{q(\hat{k})*q_k}\left(\sum_{j=1}^{n} \varphi^j_{q(\hat{k})}(q_k)W_j(q_j)\right) =$$

$$= \sum_{j=1}^{n} \bar{\varphi}^{j}(q)\bar{W}_j(q) \quad \text{for} \quad q \in \bar{V}.$$

Hence $Z|\bar{V} = \sum_{j=1}^{n} \bar{\varphi}^{j}.\bar{W}_j$. This finiishes the proof.

From Lemma 6 and Proposition 6 it folows the following corollary.

Corollary 1. If (M_i,C_i) $i \in N$ is a sequence of differential spaces of cpmstamt dofferential dimension then the the C-module $\mathfrak{X}(M)$ is isomorphism to the direct product $\underset{i\in N}{X}\ \mathfrak{X}_i(M)$ of differential modules.

Now using Definition we prove the following lemma.

Lemma 7. If (M_i,C_i), $i \in N$ is a sequence of differential spaces having the property (P) then the Cartesian product (M,C) has the property (P).

Proof. Let $w \in T_pM$ be an arbitrary vector tangent to (M,C) at $p = (p_i)_{i\in N}$. Consider the sequence of vectors $v_i = pr_{i*p}w$, $i \in N$. Of course $v_i \in T_{p_i}M_i$ for $i \in N$. Since (M_i,C_i) has the property (P) there is a vector field $X_i \in \mathfrak{X}(M_i)$ such that $v_i = X_i(p)$ for $i \in N$. Thus $pr_{i*p}w = X_i(p)$ $i \in N$. Hence $J_{p(\hat{i})*p_i}(pr_{i*p}w) = J_{p(\hat{i})*p_i}X_i(p_i)$ or equivalently by (12) and (23) we have

$$w_i = \bar{X}_i(p) \quad \text{for} \quad i \in N.$$

Thus $w = \sum_{i\in N} w_i = \sum_{i\in N} \bar{X}_i(p)$. Therefore the vector field $\sum_{i\in N} \bar{X}_i$ is such a smooth vector field tangent to (M,C) that $w = \left(\sum_{i\in N} \bar{X}_i\right)(p)$.

We have proved that (M,C) has the property (P).

From Lemma 7 and Lemma 2 it follows

Corollary 2. If (M_i, C_i) $i \in N$ is a sequence of differential spaces of constant differential dimension then for the Cartesian product (M,C) the homomorphism $h_M\colon A^k(M) \longrightarrow \Omega^k(M)$ for $k \in N$ is a monomorphism.

Corollary 3. If (M_i, C_i) $i \in N$ is a sequence of differential spaces of constant differential dimension then in the graded algebra $A(M)$ there exists exactly one exterior derivation $\tilde{d}\colon A^k(M) \longrightarrow A^{k+1}(M)$, $k \in N$ satisfying following conditions:

(i) $\tilde{d}$ is R-linear

(ii) $\tilde{d}\,\alpha = d\alpha$ for $\alpha \in C$

(iii) $\tilde{d}\,(\omega \wedge \eta) = \tilde{d}\omega \wedge \eta + (-1)^{\deg\omega}\omega \wedge \bar{d}\eta$ for $\omega, \eta \in A(M)$

(iv) $\tilde{d} \circ \tilde{d} = 0$.

Proof. From Corollary 2 it follows that $\ker h_M = \{0\}$. In [2] one has proved that $\mathfrak{M}(M) \subset \ker h_M$. Hence the ideal $\mathfrak{M}(M) = \{0\}$. Proposition 1 now shows Corollary 3.

The following diagram commutes

$$\begin{array}{ccc} A^k(M) & \xrightarrow{\tilde{d}} & A^{k+1}(M) \\ h_M \downarrow & & \downarrow h_M \\ \Omega^k(M) & \xrightarrow{\tilde{d}} & \Omega^{k+1}(M) \end{array}$$

Proposition 8. Let (M_i, C_i) for every $i \in N$ be a connected differential manifold of class C^∞. Then for an arbitrary function $\alpha \in C$ if $d\alpha = 0$ then α is a constant function.

Proof. Let $\alpha \in C$ be such a function that $d\alpha = 0$. Consider the mapping J_q defined by (6). For any $q \in M(\hat{k})$ and $k \in N$ we have $J_q^*(d\alpha) = 0$. Hence $d(J_q^*\alpha) = 0$. Since M_k is a connected differential manifold of class C^∞ the mapping $J_q^*\alpha$ is constant for any $q \in M(\hat{k})$ and $k \in N$. Therefore $\alpha \in C$ is a constant function.

Lemma 8. If ∇^i is a covariant derivative in the C-module $\mathfrak{X}_i(M)$ for $i \in N$ then the mapping $\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ given by

$$\nabla_X Y = \sum_{i\in N} \nabla^i_X Y_i \quad \text{for} \quad X,\ Y \in \mathfrak{X}(M) \tag{25}$$

is a covariant derivative in the C-module $\mathfrak{X}(M)$.

The proof is straightforward.

Proposition 9. If (M_k, C_k) is a parallelizable differential space then in the C-module $\mathfrak{X}_k(M)$ there exists a covariant derivative.

Proof. Let $V_{k,1},\dots,V_{k,n_k}$ be a global basis of the C_k-module $\mathfrak{X}(M_k)$. According to the proof of Lemma 6 we conclude that the sequence $\bar{V}_{k,1},\dots,\bar{V}_{k,n}$ is a vector basis of the C-module $\mathfrak{X}_k(M)$.

It is easy to check that the mapping $\nabla^k\colon\ \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}_k(M)$ defined by the formula

$$\nabla^k_X Y = \sum_{i=1}^{n_k} X(\varphi^i)\bar{V}_{k,1} \quad \text{for} \quad X \in \mathfrak{X}(M) \quad \text{and} \quad Y \in \mathfrak{X}_k(M)\ , \tag{26}$$

where $Y = \sum_{i=1}^{n_k} \varphi^i \bar{V}_{k,i}$, is a covariant derivative.

From Lemma 8 and Proposition 9 it follows

Corollary 4. If (M_i, C_i) $i \in N$ is a sequence of parallelizable differential spaces then in the C-module $\mathfrak{X}(M)$ there exists a covariant derivative.

Proof. From Proposition 9 we conclude that for every $k \in N$ in the C-module $\mathfrak{X}_k(M)$ there exists the covariant derivative defined by (26). From Lemma 8 it follows that $\nabla\colon\ \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ given by (26)

$$\nabla_X Y = \sum_{i\in N} \nabla^i_X Y_i \quad \text{for} \quad X,\ Y \in \mathfrak{X}(M)$$

is a covariant derivative in the C-module $\mathfrak{X}(M)$.

Example. Let G_i, $i \in N$ be a sequence of Lie groups. Of course every G_i is parallelizable. From Corollary 4 we conclude that in the module $\mathfrak{X}(\underset{i\in N}{X} G_i)$ there exists a covariant derivative.

Proposition 10. If (M_i, C_i), $i \in N$ is a sequence of compact differential spaces of constant differential dimension then in the C-module $\mathfrak{X}(M)$ there exist a covariant derivative.

Proof. In a view of Lemma 6 for any $i \in N$ the C-module $\mathfrak{X}_i(M)$ is a differential module over the compact differential space (M, C). Using smooth partition of unity [1] one can show in a standart way that there exists a covariant derivative ∇^i in $\mathfrak{X}_i(M)$ for $i \in N$. By Lemma 8 there exists a covariant derivative ∇ in $\mathfrak{X}(M)$ defined by (25).

Now we make the following definitions

Definition 5. A vector field $Z \in \mathfrak{X}_k(M)$ parallel to (M_k, C_k) is said to be projectile onto (M_k, C_k) if there exists a smooth vector field $X \in \mathfrak{X}(M_k)$ such that $\bar{X} = Z$.

Denote by $\mathfrak{X}_k^{pr}(M)$ the subset of $\mathfrak{X}_k(M)$ of all projectile vector fields onto (M_k, C_k). It is easy to see that the map H: $\mathfrak{X}(M_k) \longrightarrow \mathfrak{X}_k^{pr}(M)$ given by

$$H(X) = \bar{X} \quad \text{for} \quad X \in \mathfrak{X}(M_k) \tag{27}$$

is an isomorphism of C_k-module $\mathfrak{X}(M_k)$ and $pr_k^*(C_k)$-module $\mathfrak{X}_k^{pr}(M)$.

Definition 6. A covariant derivative ∇: $\mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ is said to be projectile onto (M_k, C_k) if for any $X, Y \in \mathfrak{X}(M_k)$ the vector field $\nabla_{\bar{X}}\bar{Y} \in \mathfrak{X}_k^{pr}(M)$.

It is easy to prove

Proposition 11. If ∇ is a covariant derivative in $\mathfrak{X}(M)$ which is projectile onto (M_k, C_k) then ∇^k: $\mathfrak{X}(M_k)$ xx $\mathfrak{X}(M_k) \longrightarrow \mathfrak{X}(M_k)$ defined by

$$\nabla^k_X Y = H^{-1}(\nabla_{\bar{X}}\bar{Y}) \tag{28}$$

is a covariant derivative in C_k-module $\mathfrak{X}(M_k)$.

Definition 7. A C-n-linear map λ: $\mathfrak{X}_k(M)x\ldots x\mathfrak{X}_k(M) \longrightarrow \mathfrak{X}_k(M)$ is said to be projectile onto (M_k, C_k) if for any $X_1,\ldots,X_n \in \mathfrak{X}(M_k)$ the vector field $\lambda(\bar{X}_1,\ldots,\bar{X}_n) \in \mathfrak{X}_k^{pr}(M)$.

If λ: $\mathfrak{X}_k(M)x\ldots x\mathfrak{X}_k(M) \longrightarrow \mathfrak{X}_k(M)$ is a C-n-linear mapping projectile onto (M_k, C_k) then the mapping $pr^k(\lambda)$: $\mathfrak{X}(M_k)x\ldots x\mathfrak{X}(M_k) \longrightarrow \mathfrak{X}(M_k)$ given by

$$pr^k(\lambda)(X_1,\ldots,X_n) = H^{-1}(\lambda(\bar{X}_1,\ldots,\bar{X}_n)) \quad \text{for} \quad X_1,\ldots,X_n \in \mathfrak{X}(M) \tag{29}$$

is a tensor of type (n,1) on (M_k, C_k).

Lemma 9. If (M_k, C_k) is a differential space of constant dimension then for an arbitrary tensor μ : $\mathfrak{X}(M_k)x\ldots x\mathfrak{X}(M_k) \longrightarrow \mathfrak{X}(M_k)$ of type (n,1) there exists unique C-n-linear mapping

$$\tilde{\mu} : \mathfrak{X}_k(M)x\ldots x\mathfrak{X}_k(M) \longrightarrow \mathfrak{X}_k(M)$$

such that $\tilde{\mu}$ is projectile onto (M_k, C_k) and $\mu = pr^k(\tilde{\mu})$.

Proof. Let $\tilde{\mu}$: $\mathfrak{X}_k(M)x\ldots x\mathfrak{X}_k(M) \longrightarrow \mathfrak{X}_k(M)$ be the mapping given by

$$\tilde{\mu}(Z_1,\ldots,Z_n)(p) = J_{p(\hat{k})*p_k}\mu(Z_1^{p(\hat{k})},\ldots,Z_n^{p(\hat{k})})(p_k) \tag{30}$$

for any $Z_1,\ldots,Z_n \in \mathfrak{X}_k(M)$ and $p \in M$.
It is easy to see that $\tilde{\mu}(\bar{X}_1,\ldots,\bar{X}_n) = \overline{\mu(X_1,\ldots,X_n)}$ for any $X_1,\ldots X_n \in \mathfrak{X}(M_k)$.
If $V_{k,1},\ldots,V_{k,n_k}$ is a local vector basis of the C_k-module $\mathfrak{X}(M_k)$ then $\bar{V}_{k,1},\ldots,\bar{V}_{k,n_k}$ is a local vector basis of the C-module $\mathfrak{X}_k(M)$. Of course $\tilde{\mu}(\bar{V}_{k,j_1},\ldots,\bar{V}_{k,j_n}) = \overline{\mu(V_{k,j_1},\ldots,V_{k,j_n})}$ for any $j_1,\ldots,j_n \in \{1,\ldots,n_k\}$. Thus $\tilde{\mu}(Z_1,\ldots,Z_n)$ defined by (30) is a smooth vector field. A trivial vertification shows that $\tilde{\mu}$ is unique and $\mu = pr^k(\tilde{\mu})$.

Definition 8. A tensor λ: $\mathfrak{X}(M)x\ldots x\mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ of type (n,1) is said to be strongly projectile if for any $k \in N$ and for any $X_1,\ldots,X_n \in \mathfrak{X}(M_k)$ the vektor field $\lambda(\bar{X}_1,\ldots,\bar{X}_n)$ is projectile onto (M_k, C_k).

If λ is a strongly projectile tensor of type (n,1) then the formula (29) defines for an arbitrary $k \in N$ the tensor $pr^k(\lambda)$ of type (n,1).

Proposition 12. Let (M_i, C_i) be for every $i \in N$ a differential space of constant differential dimension. If $(\lambda_i)_{i\in N}$ is a sequence of tensors of type (n,1) on (M_i, C_i), $i \in N$ respectively then there is unique strongly projectile tensor $\lambda: \mathfrak{X}(M)x\ldots x\mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ of type (n,1) such that $pr^i(\lambda) = \lambda_i$ for $i \in N$. Moreover the correspondance $\lambda \longrightarrow pr^i(\lambda)$ between projectille tensors of type (n,1) on (M,C) and sequences of tensors of type (n,1) (M_i, C_i) for $i \in N$ is one-to-one.

Proof. Let $\Theta_i: \mathfrak{X}_i(M) \rightarrow \mathfrak{X}_i M$ for $i \in N$ be the projection of a smooth vector field tangent to (M,C) onto its i-th component. Denotes by $\bar{\lambda}_i$ for $i \in N$ the C-n-linearmapping defined by (30).

Let $\lambda: \mathfrak{X}(M)x\ldots x\mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ be the mapping given by

$$\lambda(Z_1, \ldots, Z_n) = \sum_{i\in N} \tilde{\lambda}_i\left(\Theta_i(Z_1), \ldots, \Theta_i(Z_n)\right) \tag{31}$$

for $Z_1, \ldots, Z_n \in \mathfrak{X}(M)$.
From Lemma 9 it follows that λ is unique strongly projectile tensor of type (n,1) such that $pr^i(\lambda) = \lambda_i$ for $i \in N$.

Lemma 10. Let ∇ be a covariant derivative in the C-module $\mathfrak{X}(M)$ projectile onto (M_k, C_k) for every $k \in N$ and satisfying the following condition

$$\nabla_X \bar{Y} = \quad \text{for} \quad X \in \mathfrak{X}(M_k) \quad \text{and} \quad Y \in \mathfrak{X}(M_l), \quad k \neq l \tag{32}$$

Then the torsion tensor T and the curvature tensor R of ∇ are strongly projectile. Moreover $pr^k(T) = T^k$ and $pr^k(R) = R^k$ for $k \in N$, where T^k and R^k are the torsion tensor and the curvature tensor of ∇^k defined by (28).

The proof is straighforward.

Corollary 5. If (M_i, C_i), $i \in N$ is a sequence of parallelizable differential spaces then the covariant derivative ∇ defined by (26) is projectile onto (M_k, C_k) for every $k \in N$ and the curvature tensor of ∇ $R = 0$.

Proof. Any easy computation shows that ∇ is projectile onto (M_k, C_k) for $k \in N$ and satisfies the condition (32). From Lemma 10 it follows that $pr^k(R) = R^k$ for $k \in N$. Of course $R^k = 0$ for every $k \in N$. Thus $pr^k(R) = 0$ for any $k \in N$. From Proposition 12 it follows that $R = 0$.

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S t r e s z c z e n i e

W pracy badamy własności przestrzeni różniczkowej (M,C), która jest przeliczalnym produktem kartezjańskim przestrzeni różniczkowych (M_i, C_i), $i \in N$, w sensie Sikorskiego. Przestrzeń styczna do przeliczalnego produktu kartezjańskiego przestrzeni różniczkowych jest produktem prostym prestrzeni stycznych do poszczególnych czynników. Definiujemy i badamy C-moduł $\mathfrak{X}_i(M)$ gładkich pól stycznych do (M,C) równoległych względem (M_i, C_i). Dowodzi się, że C-moduł $\mathfrak{X}(M)$ gładkich pól wektorowych stycznych do (M,C) jest izomorficzny z produktem prostym C-modułów $\mathfrak{X}_i(M)$, $i \in N$. Omówione są własności przeliczalnego produktu kartezjańskiego przestrzeni różniczkowych stałego wymiaru różniczkowego. Udowodniono, że w tym przypadku istnieje w algebrze z gradacją A(M) form punktowych dokładnie jedna operacja różniczkowania zewnętrznego spełniająca dobrze znane aksjomaty. Przedstawione są warunki dostateczne na istnienie koneksji w C-module $\mathfrak{X}(M)$. Jeżeli (M,C) jest przeliczalnym produktem kartezjańskim zwartych przestrzeni różniczkowych stałego wymiaru różniczkowego to w C-module $\mathfrak{X}(M)$ istnieje koneksja liniowa. Wprowadzone pojęcia gładkiego tensora typu (n,1), pola wektorowego i koneksji rzutowalnych na (M_i, C_i) pozwalają badać własności koneksji i tensorów na (M, C) poprzez badanie własności ich rzutów. W ten sposób pokazano, że tensor krzywizny koneksji ∇ w przeliczalnym produkcie kartezjańskim paraleryzowalnych przestrzeni różniczkowych jest równy 0.

Р е з ю м е

В настоящей работе исследуются свойства дифференциального пространства (M, C), являющеюся счётным декартовым произведением дифференциальных пространств (M_i, C_i), в смысле Сикорского. Показано, что касательное пространство счетного декартова произведения дифференциальных пространств есть прямое произведение касательных пространств каждого из сомножителей. Определяется и исследуется С-модуль $\mathfrak{X}_i(M)$ гладких векторных полей на (M, C) параллельных относительно (M_i, C_i). Доказано, что С-модуль $\mathfrak{X}(M)$ гладких векторных полей на (M, C) изоморфен прямоиу произведению С-модулей $\mathfrak{X}_i(M)$, $i \in N$. Устаиавливаются также свлйства декартова произведения дифференциальных пространств конкчной дифференциальной размерности. Доказано также, что в этом случае в градуированной алгебре $A(M)$ поточечных дифференциальных форм существует в точности одна операция внешного дифференцирования, удовлетворяющая хорошо известным аксиомам. Представлены достаточные условия существования связности в С-модуле $\mathfrak{X}(M)$ Если (M, C) есть счетное декартново произведение компактных дифференциальных пространств конкчной дифференциальной размерности, то в С-модуле $\mathfrak{X}(M)$ существует линейная связность. Введены понятия гладкого тензора типа $(n, 1)$, векторного поля и связности, проектируемых на (M_i, C_i), позволяющие исследовать свойства связности и тензоров на M, C посредсвом исследования свойств их проекций. Этим приемом показано, что тензор кривизны связности ∇ в счетном декартовом произведении параллелизуемых дифференциальных пространств равен 0.