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ON THE CONVERGENCE OF SERIES OF THE FORM $\sum \min\{a_n,b_n\}$

Summary. Biler P. and Witkowski A. asked in [1] when the series of the form $\sum \min\{a_n,b_n\}$ is convergent, where $\{a_n\}$ and $\{b_n\}$ belong to the class d of all decreasing to zero sequences of positive reals such that $\sum a_n = \sum b_n = \infty$. In the paper, necessary and sufficient conditions for a fixed $\{a_n\} \in d$ are given in order that $\sum \min\{a_n,b_n\} < \infty$ for some $\{b_n\} \in d$. These conditions lead to large classes of concrete examples of sequences $\{a_n\} \in d$ such that the above property holds/does not hold.

Let d denote the class of all sequences $\{a_n\}$, $a_n > 0$ such that $a_n \geq a_{n+1}$ for $n \in N$, $a_n \longrightarrow 0$ and the series $\sum a_n$ is divergent. In [1] (Problem 3.11), the following question is formulated: what can be said about convergence of the series $\sum \min\{a_n,b_n\}$, where $\{a_n\} \in d$? The answer given in [1] is that the series can be convergent, i.e. there exist $\{a_n\},\{b_n\} \in d$ such that $\sum \min\{a_n,b_n\} < \infty$.

The aim of this note is to give more insight into this problem. Namely, we give necessary as well as sufficient conditions for a sequence $\{a_n\} \in d$ for which there exist a sequence $\{b_n\} \in d$ such that $\sum \min\{a_n,b_n\} < \infty$.

Theorem. Suppose that $\{a_n\} \in d$. If

$$\sum_{n\in N} \min\{a_n,b_n\} < \infty \tag{1}$$

for some sequence $\{b_n\} \in d$, then the sequence $\{a_n\}$ contains a subsequence $\{a_{n_i}\}$ satisfying the conditions

(i) $$\sum_{i\in N} (n_{2i} - n_{2i-1})a_{n_{2i}} < \infty ;$$

(ii) $$\sum_{i\in N} \sum_{n_{2i}\le n<n_{2i+1}} a_n < \infty ;$$

(iii) $$\sum_{i\in N} (n_{2i+1} - n_{2i})a_{n_{2i}-1} = \infty .$$

Conversely, if $\{a_n\}$ contains a subsequence $\{a_n\}$ satisfying conditions (ii), (iii) and the following, stronger than (i), condition

(iv) $$\sum_{n\in N} (n_{2i} - n_{2i-1})a_{n_{2i}-1} < \infty ,$$

then there exists a sequence $\{b_n\} \in d$ with the property (1).

In particular, if $\limsup_n a_{n-1}a_n^{-1} < \infty$, then the set of conditions (i), (ii) and (iii) (or, equivalently, (ii), (iii) and (iv)) is sufficient and necessary for the existence of a sequence $\{b_n\} \in d$ such that (1) holds.

Proof. Assume that (i) holds for a certain sequence $\{b_n\} \in d$. Since $\{a_n\}, \{b_n\} \in d$, there exists an increasing sequence of positive integers $\{n_i\}$ such that

$$\min\{a_n, b_n\} = \begin{cases} a_n & \text{if} \quad n \in N_{2i} , \quad i \in N , \\ b_n & \text{if} \quad n \in N_{2i-1} , \quad i \in N , \end{cases}$$

where $N_k = \{n \in N : n_k \le n < n_{k+1}\}$ for $k \in N$.
Evidently,

$$\sum_{i\in N} \sum_{n\in N_{2i}} a_n < \infty \quad \text{and} \quad \sum_{i\in N} \sum_{i\in N_{2i-1}} b_n < \infty$$

Since $a_{n_{2i}} \le b_{n_{2i}} \le b_n$ for $n < n_{2i}$ and $b_n \le b_{n_{2i}-1} \le a_{n_{2i}-1}$ for $n \ge n_{2i}$ we have

$$\sum_{i\in N}(n_{2i} - n_{2i-1})a_{n_{2i}} < \sum_{i\in N}\sum_{i\in N_{2i-1}} b_n < \infty$$

and

$$\sum_{i\in N}(n_{2i+1} - n_{2i})a_{n_{2i}-1} \geq \sum_{i\in N}\sum_{i\in N_{2i}} b_n = \infty .$$

Consequently, conditions (i)-(iii) are satisfied.

Suppose now that a subsequence $\{a_{n_i}\}$ of the sequence $\{a_n\}$ fulfills conditions (ii)-(iv). Put $b_n = a_{n_{2i}-1}$ for $n \in N_{2i-1} \cup N_{2i}$, $i \in N$ and $b_n = a_1$ for $n \in N$ such that $n < n_1$. Clearly, $b_n \geq b_{n+1}$ for $n \in N$, $b_n \longrightarrow 0$ and

$$\sum_{i\in N}\sum_{n\in N_{2i}} b_n = \sum_{i\in N} (n_{2i+1} - n_{2i})a_{n_{2i}-1} = \infty ,$$

in view of (iii). Hence $\{b_n\} \in d$. It is easy to see that

$$\min \{a_n,b_n\} = \begin{cases} a_{n_{2i}-1} & \text{for} \quad n \in N_{2i-1} , \quad i \in N \\ a_n & \text{for} \quad n \in N_{2i} , \quad i \in N \end{cases}$$

and thus, by (ii) and (iv),

$$\sum_{n\geq n_1} \min\{a_n,b_n\} = \sum_{i\in N} (n_{2i} - n_{2i-1})a_{n_{2i-1}} + \sum_{i\in N}\sum_{n\in N_{2i}} a_n < \infty ,$$

which completes the proof.

Corollary 1. For every $\{b_n\} \in d$ we have

$$\sum \min\{n^{-1}, b_n\} = \infty . \tag{2}$$

Proof. Suppose that there is a sequence $\{b_n\} \in d$, for which (2) does not hold. Then, by (ii) and (iii), there exists an increasing sequence $\{n_i\} \subset N$ such that

$$\sum_{i\in N} (n_{2i+1} - n_{2i})n_{2i+1}^{-1} < \infty \tag{3}$$

and

$$\sum_{i\in N} (n_{2i+1} - n_{2i})n_{2i}^{-1} = \infty \tag{4}$$

or, equivalently,

$$\sum_{i\in N} \varepsilon_i < \infty \tag{3'}$$

and

$$\sum_{i\in N} \varepsilon_i(1 - \varepsilon_i)^{-1} < \infty \tag{4'}$$

where $\varepsilon_i = (n_{2i+1} - n_{2i})n_{2i+1}^{-1}$ for $i \in N$.

Hence $\varepsilon_i \longrightarrow 0$, in view of (3'). But this means that conditions (3'), (4') cannot hold at the same time. Contradiction. The proof is thus completed.

Given $a > 0$, let us now denote

$$a(\alpha,0) = \alpha; \quad a(\alpha,1) = a^{\alpha}; \quad a(\alpha,k) = a^{a(\alpha,k-1)};$$

$$L_1x = \ln x \ (x > 0); \quad L_k(x) = \ln(L_{k-1}x) \quad (x > e(0,k-1))$$

for $\alpha \in R$ and $k = 2,3,\dots$

Corollary 2. For arbitrary $k \in N$ there exists a sequence $\{b_n\} \in d$ such that

$$\sum_{n>e(1,k)} \min \{f(n), b_n\} < \infty ,$$

where $f(x) = x^{-1}(L_1x)^{-1} \dots (L_kx)^{-1}$ for $x > e(1,k)$.

Proof. Fix $k \in N$. By theorem, it suffices to show that there exists an increasing sequence $\{n_i\}$ of positive integers satisfying the conditions:

$$\text{(I)} \quad \sum_{i \in N} (n_{2i} - n_{2i-1}) f(n_{2i}) < \infty ;$$

$$\text{(II)} \quad \sum_{i \in N} \sum_{n_{2i} \le n < n_{2i+1}} f(n) < \infty ;$$

$$\text{(III)} \quad \sum_{i \in N} (n_{2i+1} - n_{2i}) f(n_{2i}) = \infty .$$

Let $\{m_i\}$ be an arbitrary increasing sequence of positive integers such that

$$2(m_1, k) > e(1,k); \quad m_{i+1} - m_i \ge 2 \quad \text{for} \quad i \in N;$$

and

$$\sum_{i \in N} m_i^{-1} < \infty. \tag{5}$$

Define

$$n_{2i-1} = \begin{cases} 2(m_j^2, k) & \text{if} \quad i = 2j-1, \\ 2((m_j+1)^2, k) & \text{if} \quad i = 2j, \end{cases}$$

and

$$n_{2i} = \begin{cases} 1+n_{2i-1} & \text{if} \quad i = 2j-1, \\ n_{2i+1}-1 & \text{if} \quad i = 2j. \end{cases}$$

For the above sequence $\{n_i\}$ we have

$$\sum_{i\in N} (n_{2i} - n_{2i-1})f(n_{2i}) \le \sum_{i\in N} \ln^{-1} n_{2i} =$$

$$= \sum_{j\in N} \{\ln^{-1}[2(m_j^2,k) + 1] + \ln^{-1}[2(m_{j+1}^2,k) - 1]\}$$

$$\le \sum_{j\in N} [m_j^{-2}\ln^{-1}2 + (m_{j+1}^2\ln 2-1)^{-1}] < \infty ,$$

i.e. condition (I) is fulfilled.

Before checking condition (III) notice that the following inequalities can be obtained by induction with respect to k:

$$2(\alpha+1,k) \ge [2(\alpha,k)]^2 \ge 4 \cdot 2(\alpha,k)$$

for $\alpha \ge 1$ and $k \in N\setminus\{1\}$. Therefore

$$\sum_{\substack{i\in N\\ i\text{-odd}}} (n_{2i+1} - n_{2i})f(n_{2i}) =$$

$$= \sum_{j\in N} [2((m_j+1)^2,k)-2(m_j^2,k)-1]f(2(m_j^2,k)+1) \ge$$

$$\ge \sum_{j\in N} 2(m_j^2+2m_j,k)[2(m_j^2,k)+1]^{-1}\ln^{-k}[2(m_j^2,k)+1]$$

$$\geq \sum_{j\in N} 2(m_j^2+2m_j-1,k)\cdot 2(m_j^2+2m_j-1,k)\cdot[2(m_j^2,k)+1]^{-1}[2(m_j^2,k-1)\ln 3]^{-k} = \infty.$$

This means that condition (III) is satisfied.

Now notice that

$$\sum_{i\in N}\ \sum_{n_{2i}\leq n<n_{2i+1}} f(n) \leq \sum_{i\in N} \int_{p_i}^{r_i} f(x)dx$$

$$\leq \sum_{i\in N} \int_{p_i}^{r_i} x^{-1}(L_1x)^{-1}\dots(L_kx)^{-1}dx = \sum_{i\in N} \ln[(L_kr_i)\cdot(L_k^{-1}p_i)] ,$$

where $p_i = n_{2i}-1$ and $r_i = n_{2i+1}-1$. On the other hand,

$$L_k(r_i+1)\cdot(L_kp_i)^{-1} = L_k2((m_j+1)^2,k)\cdot(L_k2(m_j^2,k))^{-1} = 1 + \gamma_j/m_j$$

for $i = 2j-1$, and

$$L_k(r_i+1)\cdot(L_kp_i)^{-1} = L_k2(m_j^2,k)\cdot(L_k(2(m_j^2,k)-\ 2)^{-1} = 1 + \delta_j/m_j^2$$

for $i = 2j-2$, where $\{\gamma_j)$ and $\{\delta_j\}$ are certain bounded sequences of positive reals. By (5) the above relations imply that

$$\sum_{i\in N} \ln[(L_kr_i)\cdot(L_kp_i)^{-1}] < \infty ,$$

i.e. condition (II) holds.

The proof of Corollary 2 is thus completed.

Let us notice that Corollaries 1 and 2 deliver many concrete examples of sequences related to the question of p. Biler and A. Witkowski. Corollary 1 shows in particular, that the sequences of the form $\{n^{-1}\}$ does not satisfy

(1) for any $\{b_n\} \in d$. On the other hand, Corollary 2 gives a large class sequences $\{a_n\} \in d$ such that (1) holds for some $\{b_n\} \in d$. In a very special cases, we see that the sequences $\{a_n\}$ of the form

$$\left\{\frac{1}{n\cdot\ln(n)}\right\}, \quad \left\{\frac{1}{n\cdot\ln(n)\cdot\ln(\ln(n))}\right\}, \ldots$$

belong to this class.

REFERENCE

[1] Biler Piotr, Witkowski Alfred: Zadania z gwiazdką z analizy, t. 2, Wydawnictwo Uniwersytetu Wrocławskiego, Wrocław 1986.

O ZBIEŻNOŚCI SZEREGÓW POSTACI $\sum \min\{a_n, b_n\}$.

S t r e s z c z e n i e

Biler P. i Witkowski A. stawiają w [1] problem zbieżności szeregów postaci $\sum \min\{a_n, b_n\}$, gdzie $\{a_n\}$ i $\{b_n\}$ należą do klasy d ciągów malejących do zera liczb dodatnich, dla których $\sum a_n = \sum b_n = \infty$. W niniejszej pracy podajemy dla ustalonego ciągu $\{a_n\} \in d$ warunki konieczne i wystarczające na to, by $\sum\min\{a_n, b_n\} < \infty$ dla pewnego $\{b_n\} \in d$. Warunki te prowadzą do szerokich klas konkretnych przykładów ciągów $\{a_n\} \in d$, odpowiednio mających powyższą własność i nie mających jej.

О СХОДИМОСТИ РЯДОВ ВИДА

Р е з ю м е

Биллер П. и Витковски А. спрашивают в 1 о сходимости рядов вида $\sum \min\{a_n, b_n\}$ где $\{a_n\}$ and $\{b_n\}$ принадлежат классу d убывающихся последовательностей положительных чисел, таких что $\sum a_n = \sum b_n = \infty$. В следующей работе представлены необходимые и достаточные условия для фиксированной последовательности $\{a_n\}$ на то, что $\sum \min\{a_n, b_n\}$ для некоторой последовательности $\{b_n\} \in d$. Эти условия определяют широкие классы примеров последовательностей $|\{a_n\} \in d$, таких что свойство имеет место не имеет места.