ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1992

Seria: MATEMATYKA-FIZYKA z. 67 Nr kol. 1115

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OF SOME DIFFERENTIAL FORMULAE FOR LEGENDRE FUNCTIONS OF THE FIRST KIND

Summary. Legendre functions are the solutions of Legendre differential equation:

$$(1-z)^2 \frac{d^2w}{dz^2} - 2z \frac{dw}{dz} + \left[(v+1) - \mu^2(1-z^2)^{-1}\right]w = 0$$

where z, ν, μ are arbitrary (see [1], page 126).

The paper comprises a series of new formulae for derivatives of the n-th order of $P_\nu^\mu(z)$ Legendre function of the first kind, e.g.

$$(1^o) \quad D^n\left[(z^2-1)^{\mu/2} P_\nu^\mu(z)\right] =$$

$$= (-1)^n(\nu-\mu+1)_n(-\mu-\nu)_n(z^2-1)^{(\mu-n)/2} P_\nu^{\mu-n}(z),$$

$$(4^o) \quad D^n\left[z^{n+(\mu+\nu-1)/2} (z-1)^{-\mu/2} P_\nu^\mu(\sqrt{z})\right] =$$

$$= \frac{1}{2^n} z^{(\mu+\nu-1)/2} (z-1)^{-(\mu+n)/2} P_{\nu+n}^{\mu+n}(\sqrt{z}).$$

Some applications of the formulae in summing the expressions that include Legendre $P_\nu^\mu(z)$ functions are given. For $\mu = 0$, $\nu = 2$ some of them reduce to the formulae valid for Legendre polynomials $P_n(z)$.

The Legendre functions are solutions of Legendre's differential equation

$$(1-z^2) \frac{d^2w}{dz^2} - 2z \frac{dw}{dz} + \left[\nu(\nu+1) - \mu^2(1-z^2)^{-1}\right]w = 0 \tag{I}$$

z, ν, μ, being unrestricted. Under substitution $w = (z^2-1)^{\mu/2}v$, (I) becomes

$$(1-z^2)\frac{d^2v}{dz^2} - 2(\mu+1)z\frac{dv}{dz} + (\nu-\mu)(\nu+\mu+1)v = 0 .$$

And, with $\xi = 1/2-z/2$ as new independent variable, the differential equation becomes

$$\xi(1-\xi)\frac{d^2v}{d\xi^2} + (\mu+1)(1-2\xi)\frac{dv}{d\xi} + (\nu-\mu)(\nu+\mu+1)v = 0.$$

This is Gauss' equation. Thus a solution of (I) is of the form

$$w = P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\mu/2} F(-\nu,\ \nu+1;\ 1-\mu;\ 1/2-z/2), \qquad |1-z|<2,$$

where

$$F(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n\,z^n}{(c)_n\,n!}, \quad \text{with } (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)},$$

is a Gauss hypergeometric series (see [1], pp. 126, 109 and 69). The function $P_\nu^\mu(z)$ is known as the Legendre function of the first kind. It is one-valued and regular in the z-plane supposed cut along the real axis from 1 to ∞. We assume $|\arg(z\pm1)|<\pi$ (see [1], p. 127).

Formulae for the first derivative of the Legendre functions $P_\nu^\mu(z)$,

$$\frac{d}{dz}P_\nu^\mu(z) = (z^2-1)^{-1}\Big[(\nu-\mu+1)P_{\nu+1}^\mu(z)-(\nu+1)z\,P_\nu^\mu(z)\Big], \tag{1}$$

$$\frac{d}{dz}P_\nu^\mu(z) = (z^2-1)^{-1}\Big[\nu z\,P_\nu^\mu(z) - (\mu+\nu)P_{\nu-1}^\mu(z)\Big], \tag{2}$$

$$\frac{d}{dz}P_\nu^\mu(z) = \frac{(\nu+\mu)(\nu-\mu+1)}{\sqrt{z^2-1}}P_\nu^{\mu-1}(z) - \frac{\mu z}{z^2-1}P_\nu^\mu(z), \tag{3}$$

are well known (see, for example, [1], 3.8.10. and 3.8.9., p.162). In this note we give a number of simple and compact formulae for the derivatives of n-th order for some expressions involving these functions, and application to calculation of some expressions with Legendre functions. In what follows the symbol D means d/dz.

We have the following new formulae:

$$D^n\left[(z^2-1)^{\mu/2}\ P_\nu^\mu(z)\right] = \tag{1°}$$

$$= (-1)^n(\nu-\mu+1)_n(-\mu-\nu)_n(z^2-1)^{(\mu-n)/2}\ P_\nu^{\mu-n}(z)\ ,$$

$$D^n\left[(z^2-1)^{-\mu/2}\ P_\nu^\mu(z)\right] = (z^2-1)^{-(\mu+n)/2}\ P_\nu^{\mu+n}(z)\ , \tag{2°}$$

$$D^n\left[z^{n-(\mu+\nu)/2-1}\ (z-1)^{\mu/2}\ P_\nu^\mu(\sqrt{z})\right] = \tag{3°}$$

$$= \frac{(-\mu-\nu)_{2n}}{2^n}\ z^{-(\mu+\nu)/2-1}\ (z-1)^{(\mu-n)/2}\ P_{\nu-n}^{\mu-n}(\sqrt{z}),$$

$$D^n\left[z^{n+(\mu+\nu-1)/2}\ (z-1)^{-\mu/2}\ P_\nu^\mu(\sqrt{z})\right] = \tag{4°}$$

$$= \frac{1}{2^n}\ z^{(\mu+\nu-1)/2}\ (z-1)^{-(\mu+n)/2}\ P_{\nu+n}^{\mu+n}(\sqrt{z})\ ,$$

$$D^n\left[z^{n+(\nu-\mu-1)/2}\ (z-1)^{\mu/2}\ P_\nu^\mu(\sqrt{z})\right] = \tag{5°}$$

$$= \frac{(\nu-\mu+1)_{2n}}{2^n}\ z^{(\nu-\mu-1)/2}\ (z-1)^{(\mu-n)/2}\ P_{\nu-n}^{\mu-n}(\sqrt{z}),$$

$$D^n\left[z^{n+(\mu-\nu)/2-1}(z-1)^{-\mu/2}\ P_\nu^\mu(\sqrt{z})\right] = \tag{6°}$$

$$= \frac{1}{2^n}\ z^{(\mu-\nu)/2-1}\ (z-1)^{-(\mu-n)/2}\ P_{\nu-n}^{\mu-n}(\sqrt{z}),$$

$$D^n\left[z^{n-\mu-1}\ (1-z^2)^{\mu/2}\ P_\nu^\mu(\tfrac{1}{z})\right] = \tag{7°}$$

$$= (\nu-\mu+1)_n(-\mu-\nu)_n\ z^{-\mu-1}(1-z^2)^{(\mu-n)/2}\ P_\nu^{\mu-n}(\tfrac{1}{z})\ ,$$

$$D^n\left[z^{n+\mu-1}(1-z^2)^{-\mu/2}P_\nu^\mu\left(\frac{1}{z}\right)\right] = z^{\mu-1}(1-z^2)^{-(\mu+n)/2}P_\nu^{\mu+n}\left(\frac{1}{z}\right), \tag{8°}$$

$$D^n\left[z^{-(\nu+1)/2}(1-z)^{\mu/2}P_\nu^\mu\left(\frac{1}{\sqrt{z}}\right)\right] = \tag{9°}$$

$$= \frac{(-1)^n}{2^n}(\nu-\mu+1)_{2n}\, z^{-(\nu+n+1)/2}(1-z)^{(\mu-n)/2}P_{\nu+n}^{\mu+n}\left(\frac{1}{\sqrt{z}}\right),$$

$$D^n\left[z^{\nu/2}(1-z)^{-\mu/2}P_\nu^\mu\left(\frac{1}{\sqrt{z}}\right)\right] = \tag{10°}$$

$$= \frac{(-1)^n}{2^n}z^{(\nu-n)/2}(1-z)^{-(\mu+n)/2}P_{\nu-n}^{\mu+n}\left(\frac{1}{\sqrt{z}}\right),$$

$$D^n\left[z^{\nu/2}(1-z)^{\mu/2}P_\nu^\mu\left(\frac{1}{\sqrt{z}}\right)\right] = \tag{11°}$$

$$= \frac{(-1)^n}{2^n}(-\mu-\nu)_{2n}\, z^{(\nu-n)/2}(1-z)^{(\mu-n)/2}P_{\nu-n}^{\mu-n}\left(\frac{1}{\sqrt{z}}\right),$$

$$D^n\left[z^{-(\nu+1)/2}(1-z)^{-\mu/2}P_\nu^\mu\left(\frac{1}{\sqrt{z}}\right)\right] = \tag{12°}$$

$$= \frac{(-1)^n}{2^n}z^{-(\nu+n+1)/2}(1-z)^{-(\mu+n)/2}P_{\nu+n}^{\mu+n}\left(\frac{1}{\sqrt{z}}\right),$$

$$D^n\left[z^{-(\nu+1)/2}(1-z)^{\nu+n}P_\nu^\mu\left(\frac{z+1}{2\sqrt{z}}\right)\right] = \tag{13°}$$

$$= (-1)^n(\nu-\mu+1)_n\, z^{-(\nu+n+1)/2}(1-z)^{\nu}P_{\nu+n}^{\mu}\left(\frac{z+1}{2\sqrt{z}}\right),$$

$$D^n\left[z^{\nu/2}(1-z)^{n-\nu-1}P_\nu^\mu\left(\frac{z+1}{2\sqrt{z}}\right)\right] = \tag{14°}$$

$$= (-1)^n(-\mu-\nu)_n\, z^{(\nu-n)/2}(1-z)^{-\nu-1}P_{\nu-n}^{\mu}\left(\frac{z+1}{2\sqrt{z}}\right),$$

$$D^n\left[(z^2-1)^{\nu/2}\, P^{\mu}_{\nu}\left(\frac{z}{\sqrt{z^2-1}}\right)\right] = \tag{15°}$$

$$= (-1)^n(\nu-\mu+1)_n(z^2-1)^{(\nu-n)/2}\, P^{\mu}_{\nu-n}\left(\frac{z}{\sqrt{z^2-1}}\right),$$

$$D^n\left[(z^2-1)^{-(\nu+1)/2}\, P^{\mu}_{\nu}\left(\frac{z}{\sqrt{z^2-1}}\right)\right] = \tag{16°}$$

$$= (-1)^n(\nu-\mu+1)_n(z^2-1)^{-(\nu+n+1)/2}\, P^{\mu}_{\nu+n}\left(\frac{z}{\sqrt{z^2-1}}\right).$$

These formulae are valid under the following conditions:

$\lvert\arg(-z \pm 1)\rvert < \pi$	for (1°) - (2°),
$\lvert\arg z\rvert,\quad \lvert\arg(z-1)\rvert < \pi$	for (3°) - (6°),
$\lvert\arg(-z^{-1} \pm 1)\rvert < \pi$	for (7°) - (8°),
$\lvert\arg z\rvert,\quad \lvert\arg(z^{-1}-1)\rvert < \pi$	for (9°) - (12°),
$\operatorname{Re}\dfrac{z+1}{2\sqrt{z}} > 0$	for (13°) - (14°),
$\operatorname{Re}\dfrac{z}{\sqrt{z^2-1}} > 0$	for (15°) - (16°).

Moreover, (3°) and (5°), (4°) and (6°), (9°) and (11°), (10°) and (12°), (13°) and (14°), (15°) and (16°) are connected by virtue of the relation $P^{\mu}_{\nu}(z) = P^{\mu}_{-\nu-1}(z)$ (see [1], p. 140). All these formulae can be obtained by differentation of the corresponding power series or by induction.

We prove, for example, the formula (2°). The proof is by induction. It is obviously valid for $n = 0$. Let us suppose that it is true also if $k = n > 0$. We shall show that it is valid if $k = n+1$. So we have to prove that

$$D^{n+1}\left[(z^2-1)^{-\mu/2}\, P^{\mu}_{\nu}(z)\right] = (z^2-1)^{-(\mu+n+1)/2}\, P^{\mu+n+1}_{\nu}(z).$$

The proof follows from the relations (1) and

$$(\nu-\mu+1)P^{\mu}_{\nu+1}(z)-(\nu+\mu+1)z\, P^{\mu}_{\nu}(z) = \sqrt{z^2-1}\;\; P^{\mu+1}_{\nu}(z) \tag{4}$$

(see [1], 3.8.8). Indeed, by the induction assumption and next according to the relations (1) and (4), we have in turn

$$D^{n+1}\left[(z^2-1)^{-\mu/2}\, P^{\mu}_{\nu}(z)\right] = D\left\{D^{n}\left[(z^2-1)^{-\mu/2}\, P^{\mu}_{\nu}(z)\right]\right\} =$$

$$= D\left[(z^2-1)^{-(\mu+n)/2}\, P^{\mu+n}_{\nu}(z)\right] =$$

$$= -(\mu+n)z(z^2-1)^{-(\mu+n)/2-1}\, P^{\mu+n}_{\nu}(z)+(z^2-1)^{-(\mu+n)/2}\, DP^{\mu+n}_{\nu}(z) =$$

$$= -(\mu+n)z(z^2-1)^{-(\mu+n)/2-1}\, P^{\mu+n}_{\nu}(z) +$$

$$+ (z^2-1)^{-(\mu+n)/2}\left\{\left[\nu-(\mu+n)+1\right] P^{\mu+n}_{\nu+1}(z)-(\nu+1)z\, P^{\mu+n}_{\nu}(z)\right\}(z^2-1)^{-1} =$$

$$= (z^2-1)^{-(\mu+n)/2-1}\left\{\left[\nu-(\mu+n)+1\right] P^{\mu+n}_{\nu+1}(z)-\left[\nu+(\mu+n)+1\right]z\, P^{\mu+n}_{\nu}(z)\right\} =$$

$$= (z^2-1)^{-(\mu+n+1)/2}\, P^{\mu+n+1}_{\nu}(z).$$

Thus the proof of the formula (2°) is finished.

Now we prove the formula (12°) as well. It is obviously valid for $n = 0$. Let us suppose that it is true also if $k = n > 0$. We shall show that it is valid if $k = n+1$. So we have to prove that

$$D^{n+1}\left[z^{-(\nu+1)/2}\,(1-z)^{-\mu/2}\, P^{\mu}_{\nu}\left(\frac{1}{\sqrt{z}}\right)\right] =$$

$$= \frac{(-1)^{n+1}}{2^{n+1}}\, z^{-(\nu+n+2)/2}\,(1-z)^{-(\mu+n+1)/2}\, P^{\mu+n+1}_{\nu+n+1}\left(\frac{1}{\sqrt{z}}\right).$$

The proof follows from the relations

$$P^{\mu+1}_{\nu-1}(z) - P^{\mu+1}_{\nu+1}(z) = -(2\nu+1)\sqrt{z^2-1}\;\; P^{\mu}_{\nu}(z)$$

(see [1], 3.8.3., p. 162). Indeed, we have in turn

$$D^{n+1}\left[z^{-(\nu+1)/2}\,(1-z)^{-\mu/2}\,P^{\mu}_{\nu}\left(\frac{1}{\sqrt{z}}\right)\right] = D\left\{D^{n}\left[z^{-(\nu+1)/2}\,(1-z)^{-\mu/2}\,P^{\mu}_{\nu}\left(\frac{1}{\sqrt{z}}\right)\right]\right\} =$$

$$= D\left[\frac{(-1)^n}{2^n}\,z^{-(\nu+n+1)/2}\,(1-z)^{-(\mu+n)/2}\,P^{\mu+n}_{\nu+n}\left(\frac{1}{\sqrt{z}}\right)\right] =$$

$$= \frac{(-1)^n}{2^n}\,D\left[z^{-(\nu+n)-1/2}\,z^{(\nu+n)/2}\,(1-z)^{-(\mu+n)/2}\,P^{\mu+n}_{\nu+n}\left(\frac{1}{\sqrt{z}}\right)\right] =$$

$$= \frac{(-1)^{n+1}}{2^{n+1}}\left[2(\nu+n)+1\right]\,z^{-(\nu+n)-3/2}\,z^{(\nu+n)/2}\,(1-z)^{-(\mu+n)/2}\,P^{\mu+n}_{\nu+n}\left(\frac{1}{\sqrt{z}}\right) +$$

$$+ \frac{(-1)^n}{2^n}\,z^{-(\nu+n)-1/2}\,D\left[z^{(\nu+n)/2}\,(1-z)^{-(\mu+n)/2}\,P^{\mu+n}_{\mu+n}\left(\frac{1}{\sqrt{z}}\right)\right] =$$

$$= \frac{(-1)^{n+1}}{2^{n+1}}\left\{\left[2(\nu+n)+1\right]z^{-(\nu+n+3)/2}\,(1-z)^{-(\mu+n)/2}\,P^{\mu+n}_{\nu+n}\left(\frac{1}{\sqrt{z}}\right) +\right.$$

$$\left.+ z^{-(\nu+n)-1/2}\,z^{(\nu+n-1)/2}\,(1-z)^{-(\mu+n+1)/2}\,P^{\mu+n+1}_{\nu+n-1}\left(\frac{1}{\sqrt{z}}\right)\right\} =$$

$$= \frac{(-1)^{n+1}}{2^{n+1}}\,(1-z)^{-(\mu+n+1)/2}\,z^{-(\nu+n+2)/2}\left\{\left[2(\nu+n)+1\right]\left(\frac{1-z}{z}\right)^{1/2}\,P^{\mu+n}_{\nu+n}\left(\frac{1}{\sqrt{z}}\right) +\right.$$

$$\left.+ P^{\mu+n+1}_{\nu+n-1}\left(\frac{1}{\sqrt{z}}\right)\right\} = \frac{(-1)^{n+1}}{2^{n+1}}\,(1-z)^{-(\mu+n+1)/2}\,z^{-(\nu+n+2)/2}\,P^{\mu+n+1}_{\nu+n+1}\left(\frac{1}{\sqrt{z}}\right).$$

This proves the formula (12°).

The formulae (1°) - (16°) can be applied to calculation of sums of Legendre functions.

Example. Summation of some expressions with Legendre functions.

1. The following formula is true:

$$\sum_{k=0}^{n} \binom{n}{k}(-\mu)_{n-k}\frac{(-1)^k}{2^k}(z^2-1)^{k/2}P^{\mu+k}_{\nu-k}(z) = \frac{(-1)^n}{2^n}(-\mu-\nu)_{2n}(z^2-1)^{n/2}P^{\mu-n}_{\nu-n}(z) . \tag{5}$$

Indeed, differentiating the product $z^{\nu/2}(1-z)^{\mu/2}P^{\mu}_{\nu}\left(\frac{1}{\sqrt{z}}\right)$ and next using formula (10°), we get in turn

$$D^n\left[z^{\nu/2}(1-z)^{\mu/2}P^{\mu}_{\nu}\left(\frac{1}{\sqrt{z}}\right)\right] = D^n\left[(1-z)^{\mu}z^{\nu/2}(1-z)^{-\mu/2}P^{\mu}_{\nu}\left(\frac{1}{\sqrt{z}}\right)\right] =$$

$$= \sum_{k=0}^{n}\binom{n}{k}D^{n-k}\left[(1-z)^{\mu}\right]\cdot D^k\left[z^{\nu/2}(1-z)^{-\mu/2}P^{\mu}_{\nu}\left(\frac{1}{\sqrt{z}}\right)\right] =$$

$$= \sum_{k=0}^{n}\binom{n}{k}(-\mu)_{n-k}(1-z)^{\mu-n+k}\frac{(-1)^k}{2^k}z^{(\nu-k)/2}(1-z)^{-(\mu+k)/2}P^{\mu+k}_{\nu-k}\left(\frac{1}{\sqrt{z}}\right) =$$

$$= \sum_{k=0}^{n}\binom{n}{k}(-\mu)_{n-k}\frac{(-1)^k}{2^k}(1-z)^{-n+(\mu+k)/2}z^{(\nu-k)/2}P^{\mu+k}_{\nu-k}\left(\frac{1}{\sqrt{z}}\right) .$$

Now taking into account the formula (11°) we have

$$\frac{(-1)^n}{2^n}(-\mu-\nu)_{2n}z^{(\nu-n)/2}(1-z)^{(\mu-n)/2}P^{\mu-n}_{\nu-n}\left(\frac{1}{\sqrt{z}}\right) =$$

$$= \sum_{k=0}^{n}\binom{n}{k}(-\mu)_{n-k}\frac{(-1)^k}{2^k}(1-z)^{-n+(\mu+k)/2}z^{(\nu-k)/2}P^{\mu+k}_{\nu-k}\left(\frac{1}{\sqrt{z}}\right) .$$

By the above, we get the relation:

$$\frac{(-1)^n}{2^n}(-\mu-\nu)_{2n}\left(\frac{1-z}{z}\right)^{n/2} P^{\mu-n}_{\nu-n}\left(\frac{1}{\sqrt{z}}\right) =$$

$$= \sum_{k=0}^{n} \binom{n}{k}(-\mu)_{n-k} \frac{(-1)^k}{2^k}\left(\frac{1-z}{z}\right)^{k/2} P^{\mu+k}_{\nu-k}\left(\frac{1}{\sqrt{z}}\right) .$$

Next making the substitution $z \longrightarrow \frac{1}{z^2}$ we have the formula (5).

2. The following relation is true:

$$\sum_{k=0}^{n} \frac{(-\mu)_{n-k}}{2^k\, k!\,(n-k-2\mu)_{n-k}} (z^2-1)^k\, C^{\mu-n+k+1/2}_{n-k}(z)\, P^{\mu+k}_{\nu}(z) =$$

$$= (-1)^n \frac{(\nu-\mu+1)_n\, (-\mu-\nu)_n}{2^n\, n!} (z^2-1)^n\, P^{\mu-n}_{\nu}(z) \tag{6}$$

where

$$C^{\lambda}_{n}(z) = \frac{(2\lambda)_n}{n!} F(-n,\ n+2\lambda;\quad \lambda+1/2;\quad 1/2-z/2)$$

is a Gegenbauer polynomial (see [1], p. 177).

We shall prove the relation (6). Differentiating the product $(z^2-1)^{\mu/2}P^{\mu}_{\nu}(z)$, we have

$$D^n\left[(z^2-1)^{\mu/2}\, P^{\mu}_{\nu}(z)\right] = D^n\left[(z^2-1)^{\mu}\, (z^2-1)^{-\mu/2}\, P^{\mu}_{\nu}(z)\right] =$$

$$= \sum_{k=0}^{n} \binom{n}{k} D^{n-k}\left[(z^2-1)^{\mu}\right]\, D^k\left[(z^2-1)^{-\mu/2}\, P^{\mu}_{\nu}(z)\right] =$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{2^{n-k}\, (n-k)!}{(n-k-2\mu)_{n-k}} (-\mu)_{n-k} (z^2-1)^{\mu-n+k}\, C^{\mu-n+k+1/2}_{n-k}(z)\, \cdot$$

$$\cdot\ (z^2-1)^{-(\mu+k)/2}\, P^{\mu+k}_{\nu}(z) =$$

$$= 2^n\, n!\, (z^2-1)^{\mu/2-3n/2} \sum_{k=0}^{n} \frac{(-\mu)_{n-k}}{(n-k-2\mu)_{n-k}} \frac{1}{2^k\, k!} (z^2-1)^k \cdot$$

$$\cdot\, C_{n-k}^{\mu-n+k+1/2}(z)\, P_\nu^{\mu+k}(z)\,,$$

where the formula (2^o) and the Rodrigues formula for Gegenbauer polynomials

$$C_n^\lambda(z) = \frac{(-1)^n\, (\lambda)_n}{n!\,(n+2\lambda)_n} (1-z^2)^{1/2-\lambda}\, D^n\left[(1-z^2)^{n+\lambda-1/2}\right]$$

were used. From this, by (1^o), we get the relation (6).

REFERENCES

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PEWNE WZORY RÓŻNICZKOWANIA FUNKCJI LEGENDRE'A $P_\nu^\mu(z)$ PIERWSZEGO RODZAJU

S t r e s z c z e n i e

Funkcje Legendre'a są rozwiązaniami równania różniczkowego Legendre'a

$$(1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + \left[\nu(\nu+1) - \mu^2(1-z^2)^{-1}\right]w = 0\,,$$

gdzie z, ν, μ są dowolne (zobacz [1], str. 126).

Praca zawiera szereg nowych wzorów na pochodne n-tego rzędu funkcji Legendre'a $P_\nu^\mu(z)$ pierwszego rodzaju, na przykład

$$(1^\circ)\quad D^n\left[(z^2-1)^{\mu/2}\, P_\nu^\mu(z)\right] =$$

$$= (-1)^n(\nu-\mu+1)_n(-\mu-\nu)_n(z^2-1)^{(\mu-n)/2}\, P_\nu^{\mu-n}(z),$$

$$(4^\circ)\quad D^n\left[z^{n+(\mu+\nu-1)/2}\,(z-1)^{-\mu/2}\, P_\nu^\mu(\sqrt{z})\right] =$$

$$= \frac{1}{2^n}\, z^{(\mu+\nu-1)/2}\,(z-1)^{-(\mu+n)/2}\, P_{\nu+n}^{\mu+n}(\sqrt{z}).$$

Podano także zastosowanie wzorów do sumowania wyrażeń z funkcjami Legendre'a $P_\nu^\mu(z)$. Dla $\mu = 0$, $\nu = n$ niektóre z nich sprowadzają się do wzorów dla wielomianów Legendre'a $P_n(z)$.