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Piotr ANTOSIK
Józef BURZYK

A THEOREM ON CONTINUOUS CONVERGENCE

In this note we prove the following

Theorem. Assume that X is a topological group, $\{x_n\}$ is a sequence in X, f_n for $n \in N$ and f are functionals on X. If the following conditions hold:

1° $|f_n(x+y)| \leq |f_n(x)| + |f_n(y)|$ for $n \in N$ and $x, y \in X$;

2° f_n for $n \in N$ are continuous functionals;

3° f is continuous;

4° $f(o) = 0$;

5° $f_n(x) \to f(x)$ for every $x \in X$

and

6° for each subsequence $\{u_n\}$ of $\{x_n\}$ there is a subsequence $\{v_n\}$ of $\{u_n\}$ and $v \in X$ such that

$$\sum_{n=1}^{\infty} v_n = v,$$

then

$$\lim_{n\to\infty} f_n(x_n) = 0$$

In particular, if X is a complete metric group, f_n and f are continuous quasi-norms on X, $f_n(x) \to f(x)$ for $x \in X$ and $x_n \to 0$ in X, then $f_n(x_n) \to 0$. Assume that $\mathcal{F}$ is a pointwise bounded family of semi--norms on a Banach space X, $p_n \in \mathcal{F}$ for $n \in N$ and $u_n \to 0$ in X. Let $\{\alpha_n\}$ be a scalar sequence such that $\alpha_n \to \infty$ and $\alpha_n u_n \to 0$. We note that $p_n(u_n) = \alpha_n^{-1} p_n(\alpha_n u_n)$ for $n \in N$. Assuming in the theorem $f_n = \alpha_n^{-1} p_n$, $f=0$ and $x_n = \alpha_n u_n$ for $n \in N$ we see that $f_n(x_n) \to 0$ or, equivalently, $p_n(u_n) \to 0$. In other words, pointwise bounded families of semi-norms on Banach spaces are equicontinuous.

The proof of the theorem presented in this note is based on some properties of convergence of sequences in a topological group. This remark allow us to formulate a generalization of the theorem and the result in [1] which is given for convergence groups satisfying special conditions FLUSH. Other proof of the theorem based on topological properties in given in [2].

<u>Lemma 1</u>. If a_{mn} for m, n are nonnegative numbers such that $a_{mn} \to a_n$ as $m \to \infty$ for $n \in N$ and $a_n \to 0$ as $n \to \infty$, then for every positive number ε there exists a subsequence $\{p_n\}$ of $\{n\}$ such that for every subsequence $\{q_n\}$ of $\{p_n\}$ we have

$$\sum_{n=1}^{m-1} a_{q_m q_n} < \varepsilon$$

for $m \in N$.

<u>Proof</u>. Since $a_n \to 0$, there exists a subsequence $\{a_m\}$ of $\{a_n\}$ such that

$$\sum_{n=1}^{\infty} a_{m_n} < \varepsilon .$$

To avoid subscripts we assume that $m_n = n$ for $n \in N$. Then we have

$$\sum_{n=1}^{\infty} a_n < \varepsilon .$$

Assume $p_1 = 1$. Since $a_{mp_1} \to a_{p_1}$ as $m \to \infty$ and $a_{p_1} < \varepsilon$, there exists an index p_2 such that $a_{p_2 p_1} < \varepsilon$. Since $a_{mp_1} + a_{mp_2} \to a_{p_1} + a_{p_2}$ and $a_{p_1} + a_{p_2} < \varepsilon$, there exists an index p_3 such that $p_2 < p_3$ and

$$a_{p_3 p_1} + a_{p_3 p_2} < \varepsilon .$$

By induction we select a subsequence $\{p_n\}$ of $\{n\}$ such that

$$a_{p_n p_1} + \dots + a_{p_n p_{n-1}} < \varepsilon$$

for $n \in N$. Hence the lemma follows.

<u>Lemma 2</u>. If X is a topological group, f is a continuous functional on X and $x_n \to 0$, then for every positive ε there exists a subsequence

$\{p_n\}$ of $\{n\}$ such that for every subsequence $\{q_n\}$ of $\{p_n\}$ and for every finite subset M of N we have

$$|f(\sum_{n\in M} x_{q_n}) - f(o)| < \varepsilon.$$

Proof. Assume that

$$\mathcal{U} = \{x : |f(x) - f(o)| < \varepsilon\}.$$

Since $f(x_n)-f(o)\to 0$, there exists an index p_1 such that $x_{p_1}\in\mathcal{U}$. Since $f(x_{p_1}+x_n)\to f(x_{p_1})$, $f(x_n)\to 0$ and $|f(x_{p_1})-f(o)|<\varepsilon$, there is an index p_2 such that $p_i<p_2$, $|f(x_{p_2})-f(o)|<\varepsilon$ and $|f(x_{p_1}+x_{p_2})-f(o)|<\varepsilon$. Consequently, $x_{p_1}, x_{p_2}, x_{p_1}+x_{p_2}\in\mathcal{U}$. By induction, we select a subsequence $\{p_n\}$ of $\{n\}$, such that

$$\sum_{n\in M} x_{p_n} \in \mathcal{U}$$

for every finite subset M of N. This implies that for every subsequence $\{q_n\}$ of $\{p_n\}$ and for every finite subset M of N we have

$$\sum_{n\in M} x_{q_n} \in \mathcal{U}$$

or, equivalently,

$$|f(\sum_{n\in M} x_{q_n}) - f(O)| < \varepsilon$$

which was to be proved.

Lemma 3. Assume that X is a topological group f_n for $n\in N$ are continuous functionals on X and $x_n\to 0$ in X. Then there exists a subsequence $\{p_n\}$ of $\{n\}$ such that for every subsequence $\{q_n\}$ of $\{p_n\}$, for every $n\in N$ and for every finite subset M of N we have

$$|f_{q_n}(\sum_{k\in M} x_{q_{n+k}}) - f_{q_n}(O)| < 1/n.$$

Proof. By Lemma 2 there exists a subsequence $\{m_{1n}\}$ of $\{n\}$ such that $m_{11} = 1$ and

$$|f_{m_{11}}(\sum_{k\in M} x_{m_{1k}}) - f_{m_1}(O)| < 1$$

whenever M is a finite set of positive integers and $\min M > 1$. Again, by Lemma 2 there exists a subsequence $\{m_{2n}\}$ of $\{m_{1n}\}$ such that $m_{21} = m_{12}$ and

$$|f_{m_{21}}(\sum_{k\in M} x_{m_{2k}}) - f_{m_{21}}(0)| < 1/2$$

whenever M is a finite subset of N and $\min M > 1$. By induction we select a sequence of sequences $\{m_{in}\}$ such that for every $i \in N$ $\{m_{i+1,n}\}$ is a subsequence of $\{m_{in}\}$, $m_{i+1,1} = m_{12}$ and

$$|f_{m_{i1}}(\sum_{k\in M} x_{m_{ik}}) - f_{m_{i1}}(0)| < 1/i \tag{0}$$

whenever M is a finite subset of N and $\min M > 1$. We put $p_i = m_{i1}$ for $i \in N$. Let $\{q_i\}$ be a subsequence of $\{p_i\}$. Assume that $q_i = p_{r_i}$ for $i \in N$ where $\{r_i\}$ is a subsequence of $\{i\}$. We note that for every $i \in N$, $\{q_{i+k}\}$ is a subsequence of $\{m_{r_i,k}\}$ with $q_{i+k} > m_{r_i 1}$ for $k \in N$. Hence, by (0), we get

$$|f_{q_i}(\sum_{k\in M} x_{q_{i+k}}) - f_{q_i}(0)| < 1/i$$

for $i \in N$ and every finite subset M of N. Replacing in the last inequality i by n we get the lemma.

Proof of the Theorem. At first we note that if 6^0 holds, then $x_n \to 0$ and $-x_n \to 0$. Let $\{m_i\}$ be a subsequence of $\{i\}$ and let ε be a positive number. In view of conditions 6^0, 3^0 and 4^0 and Lemma 2 there is a subsequence $\{p_i\}$ of $\{m_i\}$ such that for every subsequence $\{n_i\}$ of $\{p_i\}$ we have

$$|f(-\sum_{k=1}^{j} x_{n_k})| < \varepsilon/3 \quad \text{and} \quad |f(\sum_{k=1}^{j} x_{n_k})| < \varepsilon/3 \quad \text{for } j \in N. \tag{1}$$

From 6^0, 1^0, 5^0 and (1) it follows that there is a subsequence $\{q_i\}$ of $\{p_i\}$ such that for every subsequence $\{n_i\}$ of $\{q_i\}$ we have

$$|f_{n_i}(-\sum_{k=1}^{i-1} x_{n_k})| < \varepsilon/3 \quad \text{for } i \in N. \tag{2}$$

By 2^0, 6^0 and Lemma 3, there exists a subsequence $\{r_i\}$ of $\{q_i\}$ such that for every subsequence $\{n_i\}$ of $\{r_i\}$ we have

$$|f_{n_i}(-\sum_{k=i+1}^{j} x_{n_k}) - f_{n_i}(0)| < \varepsilon/3 \quad \text{for } i,j \in N. \tag{3}$$

And finally, by 6^0, there is a subsequence $\{n_i\}$ of $\{r_i\}$ and $x \in X$ such that

$$\sum_{j=1}^{\infty} x_{nj} = x \tag{4}$$

Thus we may assume that $\{n_i\}$ is a subsequence of $\{m_i\}$ such that (1), (2), (3) and (4) hold. From (4) we get

$$x_{n_i} = x - \sum_{j=1}^{i-1} x_{n_j} - \sum_{j=i+1}^{\infty} x_{n_j}$$

for $i \in N$. Hence, by 1^0, we can write

$$|f_{n_i}(x_{n_i})| \leqslant |f_{n_i}(x)| + |f_{n_i}(-\sum_{j=1}^{i-1} x_{n_j})| + |f_{n_i}((-\sum_{j=i+1}^{\infty} x_{n_j}) - f_{n_i}(0)| + |f_{n_i}(0)|$$

for $i \in N$. Hence, by 5^0, (2) and (3) we get

$$\limsup_{i\to\infty} |f_{n_i}(x_{n_i})| \leqslant |f(x)| + 2\varepsilon/3.$$

From (4) and the second part of (1) we get $|f(x)| < \varepsilon/3$. Consequently, we can write

$$\limsup_{i\to\infty} |f_{n_i}(x_{n_i})| < \varepsilon. \tag{6}$$

In this way we have shown that every subsequence $\{f_{m_i}(x_{m_i})\}$ and for every $\varepsilon > 0$, there is a subsequence $\{f_{n_i}(x_{n_i})\}$ of $\{f_{m_i}(x_{m_i})\}$ such that (6) holds or, equivalently

$$\lim_{i\to\infty} f_i(x_i) = 0$$

which was to be proved.

REFERENCES

[1] Antosik, P., Swartz, Ch.: Matrix Methods in Analysis, Lecture Notes in Mathematics, Springer-Verlag, V. 1113, 1985.

[2] Burzyk, J.: On K-sequences: Czechoslovak Math. (to appear).