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ON TWO RELATED TYPES OF FUNCTIONAL EQUATIONS DESCRIBING MEAN VALUE PROPERTIES

1. Introduction. The classical mean value theorem of the differential calculus states that for every differentiable function f on a real interval I and for all pairs $x_1 \neq x_2$ in I there exists an $\eta(x_1,x_2)$ between x_1 and x_2 such that

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} = f'[\eta(x_1,x_2)], \tag{1}$$

that is, the tangent at $\eta(x_1,x_2)$ is parallel to the straight line connecting the points $(x_1,f(x_1))$ and $(x_2,f(x_2))$. D. Pompeiu (1946; cf., among others, Stamate 1959) observed the following counterpart. If $0 \notin I$, then there exists also a $\zeta(x_1,x_2)$ between x_1 and x_2 such that

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f[\zeta(x_1,x_2)] - \zeta(x_1,x_2) f'[\zeta(x_1,x_2)]. \tag{2}$$

The geometric meaning of this is that the tangent at $\zeta(x_1,x_2)$ intersects on the y-axis the straight line connecting the points $(x_1,f(x_1))$ and $(x_2,f(x_2))$. For quadratic functions (parabolas)

$$\eta(x_1,x_2) = \frac{x_1 + x_2}{2}, \qquad \zeta(x_1,x_2) = (x_1 x_2)^{1/2}.$$

Equations (1) and (2) can be generalized into functional equations with two unknown functions

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} = \varphi[\eta(x_1,x_2)] \qquad (x_1 \neq x_2) \tag{3}$$

or

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = \varphi[\zeta(x_1,x_2)] \qquad (x_1 \neq x_2) \tag{4}$$

respectively. For (3) in the case $\eta(x_1,x_2) = (x_1+x_2)/2$, see e.g. Haruki 1979, Aczél 1985. (Actually, there $\eta(x_1,x_2) = x_1+x_2$, which gives equivalent equation on $\mathbb{R}$, but in the present paper we intend to deal with local situations, considering, for instance, (3) on intervals, and for these $\eta(x_1,x_2) = (x_1+x_2)/2$ is more suitable). For (4) and its special cases, in particular the case $\zeta(x_1,x_2) = (x_1+x_2)/2$, see e.g. Stamate 1959, Choczewski-Kuczma 1962. We refer the reader to these four papers also for the history of the subject.

Note that (2) need not be valid when $0 \in]x_1,x_2[$ (its is not valid, for instance, for quadratic functions, as may easily be seen from the geometric interpretation). But the generalization (4) makes perfect sense in this case too (and also when f is not supposed to be differentiable). In the peresent paper we deal with equations (3) and (4) on real intervals of positive length (proper real intervals for short) in the case when η and ζ are the arithmetic, geometric or harmonic mean. We determine the general real valued solutions of (3) and (4) in these cases. We touch also on the problem of solving (4) (with $\zeta(x_1,x_2) = x_1+x_2$) on fields. The results are stated in the next section; the details of the proofs will be published elsewhere, but we make here some informal comments on them and give a new proof of one of the results.

The result on (4) can be generalized from arithmetic means to quasiarithmetic means:

$$\zeta(x_1,x_2) = F^{-1}\left(\frac{F(x_1) + F(x_2)}{2}\right) \tag{5}$$

if F is defined at 0. The geometric and harmonic means, however, are quasiarithmetic means with $F(x) = \log x$ or $F(x) = 1/x$, respectively, which are not defined at 0. That is why we have to treat them separately.

Our examples of arithmetic, geometric and harmonic means are not so arbitrary as they may seem. The arithmetic mean averages the sum under which $\mathbb{R}$ is a group with 0 as neutral element. The geometric mean averages the product under which $\mathbb{R}_+ = \{x\in \mathbb{R}:\ x > 0\}$ forms a group with 1 as neutral element (and $0 \notin \mathbb{R}_+$ is an annihilator). Finally, the harmonic mean averages the operation $1/((1/x) + (1/y))$, for which 0 is again an annihilator in a sense and under which $\mathbb{R}_+$ forms a semigroup without neutral element (∞ would be a neutral element, if included). Of course, the arithmetic, geometric and harmonic means are also the most often used mean values.

2. <u>Results</u>. We start with a general result on fields.

<u>Proposition 1</u>. Let F be a commutative field of characteristic different from 2 and 3. The general solution of

$$\frac{xf(y) - yf(x)}{x - y} = \varphi(x + y) \quad (x, y \in F, \quad x \neq y) \tag{6}$$

is given by

$$f(x) = \alpha x + \beta, \quad \varphi(x) = \beta \quad (x \in F), \tag{7}$$

where α and β are arbitrary constants in F.

The condition about the characteristic cannot be dropped, as the following examples in $\mathbb{Z}_2$ and $\mathbb{Z}_3$ (fields of integers modulo 2 or 3) show. They satisfy (6), but are not of the form (7) (φ is not constant):

$$f(x) = \begin{cases} 1 & (x=0) \\ 0 & (x=1) \end{cases}, \quad \varphi(x) = \begin{cases} 0 & (x=0) \\ 1 & (x=1) \end{cases} \quad \text{on } \mathbb{Z}_2$$

and

$$f(x) = \begin{cases} 1 & (x=0,2) \\ 2 & (x=1) \end{cases}, \quad \varphi(x) = \begin{cases} 0 & (x=0) \\ 1 & (x=1,2) \end{cases} \quad \text{on } \mathbb{Z}_3.$$

Now we pass to the case of real intervals.

Theorem 2. Let I be a proper real interval. The general real valued solution of

$$\frac{xf(y) - yf(x)}{x - y} = \varphi\left(\frac{x + y}{2}\right) \quad (x, y \in I, \; x \neq y) \tag{8}$$

is given by

$$f(x) = \alpha x + \beta \quad (x \in I), \quad \varphi(t) = \beta \quad (t \in \operatorname{int} I), \tag{9}$$

where α and β are arbitrary real constants. ($\operatorname{int} I$ is the interior of I).

The proof of Proposition 1 and of Theorem 2 when $0 \in I$ (and of Proposition 4 below) consists of substitutions which are quite easy (if one knows what to substitute). Complications arise when $0 \notin I$ in Theorem 2.

Using a nice interaction between (3) and (4) (cf. the next section) one can deduce from the $0 \notin I$ case of Theorem 2 the following.

Corollary 3. Let I be a proper interval of positive numbers. The general real valued solution of

$$\frac{f(x) - f(y)}{x - y} = \varphi\left(\frac{2xy}{x + y}\right) \quad (x, y \in I \subset \mathbb{R}_+, \quad x \neq y) \tag{10}$$

is given by

$$f(x) = \alpha + \beta x \quad (x \in I), \quad \varphi(t) = \beta \quad (t \in \operatorname{int} I) \tag{11}$$

($\alpha, \beta \in \mathbb{R}$ arbitrary constants).

If $0 \in I$, then the argument used to prove Theorem 2 (with only slight modification) may be applied to treat equation (4) for quasiarithmetic means (5), where F is a continuous, strictly monotonic function, that is, the equation

$$\frac{xf(y) - yf(x)}{x - y} = \varphi\left(F^{-1}\left(\frac{F(x) + F(y)}{2}\right)\right) \quad (x, y \in I, \quad x \neq y). \tag{12}$$

In this way we arrive at the following result.

Proposition 4. Let I be a proper real interval containing 0 and F a continuous and strictly monotonic real valued function on I. Then the general solution of (12) is given by (9) where α and β are arbitrary real constants.

While Proposition 4 applies, for instance, to the root-mean-powers with positive exponents

$$\zeta(x_1, x_2) = \left(\frac{x_1^p + x_2^p}{2}\right)^{1/p} \quad (p > 0) \tag{13}$$

($F(x) = x^p$, $I = [0,b]$ or $[0,b[$, $0 < b \leq \infty$; or, extending (13) by taking (5) with $F(x) = |x|^p \operatorname{sgn} x$, I can also be $[a,b]$, $[a,b[$, $]a,b]$, $]a,b[$ with $-\infty \leq a \leq 0 \leq b \leq \infty$, $a \neq b$), the disadvantage of Proposition 4 is that the supposition that F is defined at 0 still severely restricts the quasiarithmetic means (5) to which it applies. In particular, it does not apply to the two quasiarithmetic means "most popular" after the arithmetic mean: the geometric mean and the harmonic mean

$$\zeta(x_1, x_2) = (x_1 x_2)^{1/2} \quad \text{and} \quad \zeta(x_1, x_2) = \frac{2x_1 x_2}{x_1 + x_2},$$

for which $F(x) = \log x$ and $F(x) = 1/x$ (or their affine transforms), respectively. And indeed, in these cases equation (12) does have solutions different from (9) (cf. Corollaries 6 and 8 below).

Equation (3) with $\eta(x_1, x_2) = (x_1 + x_2)/2$ has been solved for $x_1, x_2 \in \mathbb{R}$ ($x_1 \neq x_2$) - and even for arbitrary (commutative) fields of characteristic different from 2 - in Aczél 1985. By modifying the proof given there, we can solve the same equation on arbitrary real intervals, and also equation (3) with $\eta(x_1, x_2) = (x_1 x_2)^{1/2}$. In this way we get the following results.

Theorem 5. Let $I \subset \mathbb{R}$ be a proper interval. Then the general real valued solution of

$$\frac{f(x) - f(y)}{x - y} = \varphi\left(\frac{x + y}{2}\right) \quad (x, y \in I,\ x \neq y) \tag{14}$$

is given by

$$f(x) = \alpha x^2 + \beta x + \gamma \quad (x \in I), \quad \varphi(t) = 2\alpha t + \beta \quad (t \in \operatorname{int} I), \tag{15}$$

where α, β, γ are arbitrary real constants.

Corollary 6. Let I be a proper interval of positive numbers. The general real valued solution of the equation

$$\frac{xf(y) - yf(x)}{x - y} = \varphi\left(\frac{2xy}{x + y}\right) \quad (x, y \in I \subset \mathbb{R}_+,\ x \neq y) \tag{16}$$

is given by

$$f(x) = \alpha\frac{1}{x} + \beta + \gamma x \quad (x \in I), \quad \varphi(t) = 2\alpha\frac{1}{t} + \beta \quad (t \in \operatorname{int} I) \tag{17}$$

($\alpha, \beta, \gamma \in \mathbb{R}$ arbitrary constants).

Theorem 7. Let I be a proper interval of positive numbers. The general real valued solution of the equation

$$\frac{f(x) - f(y)}{x - y} = \varphi((xy)^{1/2}) \quad (x, y \in I \subset \mathbb{R}_+,\ x \neq y) \tag{18}$$

is given by

$$f(x) = \alpha\frac{1}{x} + \beta + \gamma x \quad (x \in I), \quad \varphi(t) = \gamma - \frac{\alpha}{t^2} \quad (t \in \operatorname{int} I), \tag{19}$$

where α, β, γ are arbitrary real constants.

Corollary 8. Let I be a proper interval of positive numbers. The general real valued solution of the equation

$$\frac{xf(y) - yf(x)}{x - y} = \varphi((xy)^{1/2}) \quad (x, y \in I \subset \mathbb{R}_+,\ x \neq y) \tag{20}$$

is given by

$$f(x) = \alpha x^2 + \beta x + \gamma \quad (x \in I), \quad \varphi(t) = \gamma - \alpha t^2 \quad (t \in \operatorname{int} I) \tag{21}$$

($\alpha, \beta, \gamma \in \mathbb{R}$ arbitrary constants).

Corollaries 6 and 8 are obtained from Theorems 5 and 7, respectively, by virtue of the duality between (3) and (4) (cf. the next section).

Corollaries 3 and 6 (but not Theorem 7 nor Corollary 8) remain valid if instead of interval $I \subset \mathbf{R}_+$ we take interval $I \subset \mathbf{R}_- = \{x \in \mathbf{R} : x < 0\}$.

We could permit $x = 0$ or $y = 0$ in (16) and (20) (that is, to take $0 \in I \subset \bar{\mathbf{R}}_+ = \{x \in \mathbf{R} : x \geqslant 0\}$). That would give only $f(0) = \varphi(0)$, and indeed, (17) or (21) for $x \in I\setminus\{0\}$ with $f(0) = \varphi(0) = \delta$ (arbitrary) actually satisfies (16) or (20), respectively, on the whole interval I containing 0.

Finally, note that in Corollary 3, Theorems 5 and 7 $\varphi(t) = f'(t)$, while in Theorem 2, Proposition 4, Corollaries 6 and 8 $\varphi(t) = f(t) - tf'(t)$, in accordance with (1) and (2).

3. About the proofs. As it was said already in the Introduction, detailed proofs of the results listed in the preceding section will be published elsewhere. They are all - more or less - based on similar ideas and are - sometimes essential - modifications of the argument used in Aczél 1985 to solve equation (14) on $\mathbf{R}$. Roughly speaking, this method consists in prescribing the values of the unknown functions at certain points (without loss of generality) and in deriving from the equation which we consider another equation containing only one unknown function φ. This equation is then solved, and once φ is known, f can be determined from the original equation. (Cf. also our remark after Theorem 2).

As an example we give below an alternative, direct proof of Corollary 8 (without appealing to Theorem 7). Then Theorem 7 can be obtained from Corollary 8 using the "nice interaction between (3) and (4)" mentioned earlier in this paper. To explain this "nice interaction" we will call equations (3) and (4) dual whenever one of $\eta(x_1,x_2)$ and $\zeta(x_1,x_2)$ is the arithmetic mean and the other is the harmonic mean, or when both are the geometric mean. Let I be a proper real interval such that $0 \notin I$, and let f, φ be real valued functions on I. Write $\hat{I} = I^{-1} = \left\{t = \frac{1}{x} : x \in I\right\}$, and define functions $\hat{f}, \hat{\varphi}$ on $\hat{I}$ by

$$\hat{f}(t) = tf(1/t), \quad \hat{\varphi}(t) = \varphi(1/t) \quad (t \in \hat{I}). \tag{22}$$

Then f, φ satisfy one of the equations (8), (10), (14), (16), (18), (20) for $x, y \in I$, $x \neq y$, if and only if the functions $\hat{f}, \hat{\varphi}$ satisfy for $x, y \in \hat{I}$, $x \neq y$, the dual equation. Thus formulas (11), (17) and (21) result from (9), (15) and (19), respectively, according to relation (22).

Observe that the operation $\wedge$ is an involution; we have

$$\hat{\hat{I}} = I, \quad \hat{\hat{f}} = f, \quad \hat{\hat{\varphi}} = \varphi.$$

In other words, the transformation (22) is its own inverse.

And now we pass to the announced direct

Proof of Corollary 8. Straightforward substitution shows that for arbitrary $\alpha, \beta, \gamma \in \mathbf{R}$ the functions (21) satisfy equation (20). To prove the converse observe that, since I is proper (of positive length), there exists a c in its interior. But, if f and φ satisfy (20) then, with $\tilde{x} = x/c$, $\tilde{y} = y/c$ and $\tilde{f}(t) = f(ct)$, $\tilde{\varphi}(t) = \varphi(ct)$ we get

$$\frac{\tilde{x}\tilde{f}(\tilde{y}) - \tilde{y}\tilde{f}(\tilde{x})}{\tilde{x} - \tilde{y}} = \varphi((\tilde{x}\tilde{y})^{1/2}) \quad \text{for all } \tilde{x}, \tilde{y} \in \tilde{I} = \frac{1}{c} I =$$
$$= \left\{\tilde{x} = \frac{x}{c} : x \in I\right\} \quad (\tilde{x} \neq \tilde{y}).$$

This equation is of the same form (20), but now $\tilde{I}$ contains 1 in its interior. So without loss of generality we may assume in the sequel that $1 \in \operatorname{int} I$ (since $x \longrightarrow cx$, $t \longrightarrow ct$ does not change the form of (21), the result is unchanged). Thus there exists a $b > 1$ such that $]b^{-2}, b^2[\subset I$.

Now take arbitrary $u, v \in]b^{-1}, b[$, $u \neq 1 \neq v$, $u \neq v$, and put in (20) in turn $x = uv$, $y = u/v$; $x = uv$, $y = v/u$; $x = u/v$, $y = v/u$ (in all these cases $x, y \in]b^{-2}, b^2[$). This yields

$$v^2 f(\tfrac{u}{v}) - f(uv) = \varphi(u)(v^2 - 1), \tag{23}$$

$$u^2 f(\tfrac{v}{u}) - f(uv) = \varphi(v)(u^2 - 1), \tag{24}$$

$$u^2 f(\tfrac{v}{u}) - v^2 f(\tfrac{u}{v}) = \varphi(1)(u^2 - v^2). \tag{25}$$

Subtracting (23) from (24) and comparing the resulting equation with (25) gives

$$\varphi(v)(u^2 - 1) - \varphi(u)(v^2 - 1) = \varphi(1)\left[(u^2 - 1) - (v^2 - 1)\right]. \tag{26}$$

This is valid for all $u, v \in]b^{-1}, b[$ (for $u = 1$ or $v = 1$ or $u = v$ equation (26) is trivially fulfilled). Fixing a $v_0 \in]b^{-1}, b[$, $v_0 \neq 1$, and writing t instead of u, we get with $\alpha = \left[\varphi(1) - \varphi(v_0)\right]/(v_0^2 - 1)$, $\gamma = \varphi(1) + \alpha$

$$\varphi(t) = \gamma - \alpha t^2 \quad (t \in]b^{-1}, b[). \tag{27}$$

With

$$\beta = f(1) - \alpha - \gamma \tag{28}$$

we obtain from (20) and (27), on setting $y = 1$,

$$f(x) = \alpha x^2 + \beta x + \gamma \qquad (x \in]b^{-2}, b^2[) \tag{29}$$

(really for $x \in]b^{-2}, b^2[\setminus \{1\}$, but for $x = 1$ (29) results from (28)). Putting this back into (20) we see that

$$\varphi(t) = \gamma - \alpha t^2 \quad (t \in]b^{-2}, b^2[). \tag{30}$$

Now we define the increasing sequence of numbers $\{b_n\}$ and the increasing sequence of intervals $\{I_n\}$ by

$$b_1 = b^2, \; b_{n+1} = b_n^2, \; I_n =]b_n^{-1}, b_n[\cap I \quad (n=1,2,\dots)$$

and prove that for $n = 1,2,\dots$

$$f(x) = \alpha x^2 + \beta x + \gamma \quad (x \in I_n), \quad \varphi(t) = \gamma - \alpha t^2 \quad (t \in \operatorname{int} I_n). \tag{31}$$

For $n=1$ (31) is true in view of (29) and (30). Assuming (31) true for an $n \geqslant 1$, we have for $x \in I_{n+1}$

$$\sqrt{x} \in \operatorname{int} I_n,$$

whence by (20) with $y = 1$ and by (28) and (31) (assumed valid for n)

$$f(x) = \alpha x^2 + \beta x + \gamma \quad (x \in I_{n+1}) \tag{32}$$

($x \neq 1$, but for $x = 1$ (32) results from (28)). Letting then $x \neq y$ run through I_{n+1} in (20), we have also $\varphi(t) = \gamma - \alpha t^2$ for all $t \in \operatorname{int} I_{n+1}$, which together with (32) yields (31) with n replaced by $n+1$. Thus (31) holds true for all $n = 1,2,\dots$, and we obtain (21) in view of the relations

$$I = \bigcup_{n=1}^{\infty} I_n, \quad \operatorname{int} I = \bigcup_{n=1}^{\infty} \operatorname{int} I_n.$$

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