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Feliks BARAŃSKI
Jan MUSIAŁEK

BIPARABOLIC PROBLEM FOR THE CURVILINE TRAPEZIUM

The subject of the paper is the construction of the solution of the biparabolic problem for the general domain with Lauricella boundary conditions. To the solution we shall apply the convenient heat potentials with unknown densities q_i. To determine q_i, the system of four Volterra integral equations is used.

1. Introduction

Let us consider the equation

$$P^2u(x,t) = f(x,t), \tag{1}$$

where

$$P = D_x^2-D_t,\ P^2 = P(P) = D_x^4-2D_x^2D_t+D_t^2,$$

or the equation

$$D_x^4u(x,t) - 2D_x^2D_tu(x,t) + D_t^2u(x,t) = f(x,t).$$

We assume that $u(x,t) \in C^{4,2}(D)$ with respect to the variables (x,t), where

$$D=\{(x,t):p_1(t)<x<p_2(t),t\in(0,T)\}.$$

we suppose that the function u satisfies the initial condition

$$D_t^iu(x,0) = f_i(x),\ i = 0,1,\ \text{for } x \in(p_1(0),\ p_2(0)), \tag{2}$$

and the boundary conditions

$$D_x^iu(p_1(t),t) = h_{i+1}(t),\ i=0,1,\ t \in(0,T), \tag{3}$$

$$D_x^i u(p_2(t),t) = h_{i+3}(t),\ i=0,1,\ t\in(0,T), \tag{4}$$

$f, f_j, j = 0,1$, $h_i, i = 1,2,3,4$, are given functions.

In [1] the similar problem for the bicaloric equation and for the half-plane and in [2], the three parabolic problem for n-dimensional half-space was treated. In the monograph [3] the similar problem for the equation $Pu=f$ with initial and boundary conditions and in [5] the similar problem for the equation (1) with limit conditions $P^i u(x,0) = f_i(x)$, $P^i u(p_1(t),t) = h_i(t)$, $P^i u(p_2(t),t) = k_i(t)$, $i = 0,1$, was treated. In the monograph [4], vol. II., p. 235, the biparabolic Cauchy problem was solved.

2. Some definitions and denotes

In the sequel C, C_i, denote a positive constants.

(a) Definition 1. Denote by (K) the class of all functions $u(x,t)\in C^{4,2}(D)\cap C^{1,1}(\bar{D})$

(b) Definition 2. Denotte by (P) the class of all functions $p_i(t), i = 1,2$, such that $p_i \in C^1([0,T])$, $D_t p_i(t) > 0$ for $t\in[0,T], i = 1,2$, $(p_2(t)-p_1(s)) > A > 0$ for $(s,t)\in[0,T]\times[0,T]$, A is a positive constants.

(c) Definition 3. Denote by (f) the class of all functions f such that $f\in C^4(D_1), D_1 = \{(x,0): x\in(p_1(0),p_2(T))\}, D_x^i f(p_j(0)) = 0,\ i=0,1, j=1,2.$

(d) Definition 4. Denote by (h) the class of all functions h, such that $h\in C^2([0,T]), D_t^i h(0) = 0,\ i = 0,1.$

(e) Definition 5. By (F) denote the class of the functions F, such that $F\in C(\bar{D})\cap C^1(D), D_x^i F(p_j(t),t) = 0,\ i=0,1;\ j=1,2,\ t\in[0,T]$, $F(y,s) = F(p_1(s),s)$ for $y < p_1(s)$, $F(y,s) = F(p_2(s),s)$ for $y > p_2(s)$, $s\in(0,T)$.

3. Reduction of the initial conditions (2) to the homogenous initial conditions

Let us consider the function

$$w(x,t) = u(x,t)-r(x,t),\ r(x,t) = f_0(x)+tf_1(x),$$

Lemma 1. If (a)-(e) are satisfied, u is the solution of the problem (1)-(4), then: $w\in(K)$,

$$P^2 w(x,t) = F(x,t),\ F(x,t) = f(x,t)-P^2 r(x,t),\ (x,t)\in D, \tag{1a}$$

$$D_t^i w(x,0) = 0,\ i = 0,1,\ x\in(p_1(0),p_2(0)), \tag{2a}$$

$$D_x^i w(p_1(t),t) = H_{i+1}(t), H_{i+1}(t) = h_{i+1}(t)-D_x^i r(p_1(t),t), i = 0,1 \tag{3a}$$

$$D_x^i w(p_2(t),t) = H_{i+3}(t), H_{i+3}(t) = h_{i+3}(t) - D_x^i r(p_2(t),t) \quad i=0,1,$$
$$t \in (0,T). \qquad (4a)$$

Conversely. If the function $w \in (K)$ and satisfies (1a)-(4a), then the function $u = (w+r) \in (K)$ and satisfies (1)-(4).

We omit the simple proof.

4. <u>The potentials w_i</u>

Let us consider the following potentials

$$w_1(x,t) = A\int_0^t q_1(s)(t-s)^{1/2}\exp(B(t,s)(x-p_1(s))^2)ds,$$

$$w_2(x,t) = \frac{1}{2}A\int_0^t q_2(s)(t-s)^{-1/2}(x-p_1(s))\exp(B(t,s)(x-p_1(s))^2)ds,$$

$$w_3(x,t) = A\int_0^t q_3(s)(t-s)^{1/2}\exp(B(t,s)(x-p_2(s))^2)ds,$$

$$w_4(x,t) = \frac{A}{2}\int_0^t q_4(s)(t-s)^{-1/2}(x-p_2(s))\exp(B(t,s)(x-p_2(s))^2ds,$$

$$w_5(x,t) = A\int_0^t\int_{p_1(s)}^{p_2(s)} F(y,s)(t-s)^{1/2}\exp(B(t,s)(x-y)^2)dyds,$$

where

$$A = (2\sqrt{\pi})^{-1}, \quad B(t,s) = (-4(t-s))^{-1}.$$

Let

$$w_5^1(t) = w_5(p_1(t),t) = A\int_0^t\int_{p_1(s)}^{p_2(s)} F(y,s)(t-s)^{1/2}\exp(B(t,s)\, x$$
$$x\,(p_1(t)-y)^2))dyds,$$

$$w_5^2(t) = D_x w_5(x,t)\Big|_{x=p_1(t)} = A\int_0^t\int_{p_1(s)}^{p_2(s)} F(y,s)(t-s)^{1/2}\, x$$
$$x\, D_x\exp(B(t,s)(x-y)^2)\Big|_{x=p_1(t)}dyds =$$

$$= 2A\int_0^t\int_{p_1(s)}^{p_2(s)} F(y,s)(t-s)^{1/2}B(t,s)\,(p_1(t)-y)\exp(B(t,s)(p_1(t)-y))^2 dyds$$

$$w_5^3(t) = w_5(p_2(t),t) = A\int_0^t\int_{p_1(s)}^{p_2(s)} F(y,s)(t-s)^{1/2} \times$$

$$\times \exp(B(t,s)(p_2(t)-y)^2)dyds,$$

$$w_5^4(t) = D_x w_5(x,t)\Big|_{x=p_2(t)} = A\int_0^t\int_{p_1(s)}^{p_2(s)} F(y,s)(t-s)^{1/2} \times$$

$$\times D_x \exp(B(t,s)(x-y)^2)\Big|_{x=p_2(t)} dyds = 2A\int_0^t\int_{p_1(s)}^{p_2(s)} F(y,s)(t-s)^{1/2}\times$$

$$\times (p_2(t)-y)\exp(B(t,s)(p_2(t)-y)^2)dyds.$$

5. <u>The properties of w_5</u>

<u>Lemma 2</u>. If $F\in(F)$, then: 1^0 the integral w_5 is locally uniformly convergent at every point $(p_i(t),t)$, $i=1,2$, $t\in[0,T]$, 2^0 $w_5^i(t)\to 0$ as $t\to 0$, $i=1,3$, 3^0 The integral

$$I(x,t) = A\int_0^t\int_{p_1(s)}^{p_2(s)} F(y,s)(t-s)^{1/2}D_x\exp(B(t,s)(x-y)^2)dyds =$$

$$= -A\int_0^t\int_{p_1(s)}^{p_2(s)} F(y,s)(t-s)^{-1/2}(x-y)\exp(B(t,s)(x-y)^2)dyds$$

is locally uniformly convergent at every point $(p_i(t),t)$, $i=1,2$, $t\in(0,T]$, 4^0 $w_5^i(t)\to 0$ as $t\to 0$, $i=2,4$, 5^0 $Pw_5(x,t) = A\iint_{D(t)} F(y,s)(t-s)^{-1/2} \times$ $\times \exp(B(t,s)(x-y)^2)dyds$ for $(x,t)\in D$, 6^0 $P^2w_5(x,t) = F(x,t)$ for (x,t) belonging to D.

Proof. Ad 1^0, 2^0. We have

$$|w_5(x,t)| \le A\iint_{D(t)} (t-s)^{1/2}\, ds \le Ct^{3/2} \le CT^{3/2}$$

Consequently the integrals w_5^i, $i = 1,3$, are locally uniformly convergent for $t \in [0,T]$ and are continuous for $(x,t) \in \bar{D}$. Moreover $w_5^i(x,0) = 0$, $i=1,3$.

Ad 3^0, 4^0. We have the estimate

$$|I(x,t)| \leq (A\sup_D |F|)(p_2(T)-p_1(0)) \int_0^t (t-s)^{-1/2}ds = Ct^{1/2} \leq CT^{1/2}.$$

Consequently the integral I is locally uniformly convergent in $\bar{D}$ and $D_x w_5(p_i(t),t) = I(p_i(t),t)$ and $w_5^i(x,0)=0$, $i=2,4$, for $x \in (p_1(0),p_2 0))$.

Ad 5^0, 6^0. By Poisson theorem we obtain

$$Pw_5(x,t) = \lim_{s\to t} A \int_{-\infty}^{\infty} F(y,s)(t-s)^{1/2}\exp(B(t,s)(x-y)^2)dy +$$

$$+ A\iint_D F(y,s)P_{xt}((t-s)^{1/2}\exp(B(t,s)(x-y)^2))dyds =$$

$$= A\iint_D F(y,s)(t-s)^{-1/2}\exp(B(t,s)(x-y)^2)dyds.$$

Hence $P^2 w_5(x,t) = F(x,t)$ for $(x,t) \in D$.

6. <u>The properties of</u> w_i, $i=1,2,3,4$

<u>Lemma 3</u>. If $q_j \in C([0,T])$, $j=1,2,3,4$, $p_i \in (P)$, $i=1,2$, $q_i(0) = 0$, $i=1,2,3,4$, then:

1^0 $P^2 w_i(x,t) = 0$, $i = 1,2,3,4$, for $(x,t) \in D$,

2^0 $D_t^i w_j(x,0) = 0$, $i = 0,1$; $j = 1,2,3,4$, for every $x \in (p_1(0),p_2(0))$,

3^0 $w_j(x,t) \longrightarrow w_j(p_i(t),t)$, $j = 1,2,3,4$

$$D_x w_1(x,t) \longrightarrow A \int_0^t q_1(s)(t-s)^{1/2} D_x \exp(B(t,s)(x-p_1(s))^2)\Big|_{x=p_i(t)} ds,$$

$$D_x w_3(x,t) \longrightarrow A \int_0^t q_3(s)(t-s)^{1/2} D_x \exp(B(t,s)(x-p_2(s))^2)\Big|_{x=p_i(t)} ds,$$

$$D_x w_2(x,t) \longrightarrow A \int_0^t q_2(s)(t-s)^{-1/2} D_x \left[(x-p_1(s))\exp(B(t,s)(x-p_1(s))^2\right]\Big|_{x=p_i(t)} ds$$

$$D_x w_4(x,t) \longrightarrow A \int_0^t q_4(s)(t-s)^{-1/2} D_x \left[(x-p_2(s))\exp(B(t,s)(x-p_2(s))^2\right]\Big|_{x=p_i(t)} ds$$

as $x \longrightarrow p_i(t)$, $i = 1,2$.

Proof. We shall give the proof only for the function w_1, because the proof for the remaining functions w_i, $i = 2,3,4$, is similar.

Ad 1^o. We have

$$Pw_1(x,t) = A\int_0^t q_1(s)(t-s)^{1/2}D_x^2\exp(B(t,s)(x-p_1(s))^2)ds -$$

$$- \lim_{s\to t} q_1(s)(t-s)^{1/2}\exp(B(t,s)(x-p_1(s))^2) -$$

$$- A\int_0^t q_1(s)D_t((t-s)^{1/2}\exp(B(t,s)(x-p_1(s))^2)ds =$$

$$= A\int_0^t q_1(s)P_{xt}((t-s)^{1/2}\exp(B(t,s)(x-p_1(s))^2)ds =$$

$$= A\int_0^t q_1(s)(t-s)^{-1/2}\exp(B(t,s)(x-p_1(s))^2)ds$$

By [4], p. 491, we obtain $P^2w_1(x,t)=0$.

Ad 2^o. We have

$$|w_1(x,t)| \leqslant A(\sup_{[0,T]}|q_1|)\int_0^t (t-s)^{1/2}ds \to 0 \quad as \quad t\to 0,$$

$$|D_tw_1(x,t)| \leqslant A(\sup_{[0,T]}|q_1|)\lim_{s\to t}(t-s)^{1/2}\exp(B(t,s)(x-p_1(s))^2) +$$

$$+ \frac{A}{2}\int_0^t |q_1(s)|(t-s)^{-1/2}\exp(B(t,s)(x-p_1(s))^2)ds +$$

$$+ \frac{A}{2}\int_0^t |q_1(s)|(t-s)^{-3/2}(x-p_1(s))^2\exp(B(t,s)(x-p_1(s))^2ds \leqslant$$

$$\leqslant C_1\int_0^t (t-s)^{-1/2}ds + C_2\int_0^t (t-s)^{-1/2}\frac{(x-p_1(s))^2}{t-s}\exp(B(t,s)(x-p_1(s))^2)ds$$

$$\leqslant C_3t^{1/2} \to 0 \quad as \quad t\to 0.$$

Since $z^k\exp(-z^2) \leqslant C$, where k is a positive constant and $z \geqslant 0$.

Ad 3^o-6^o. By [4], p. 490, the function w_1 is continuous for $(x,t)\in\overline{D}$ and D_xw_1 also.

7. The integral equations for q_i

Let q_i, i=1,2,3,4, be unknown functions and $w(x,t) = \sum_1^5 w_i(w,t)$ be the solution of the (1a)-(4a) problem. By the boundary conditions (3a), (4a) we obtain the following equations

$$(I_k) \quad \sum_1^4 \int_0^t q_i(s) L_{ki}(t,s)ds = H^k(t), \ k=1,2,3,4,$$

where $H^k(t) = H_k(t) - w_5^k(t)$, K=1,2,3,4,

$$L_{11}(t,s) = A(t-s)^{1/2}\exp(B(t,s)(p_1(t)-p_1(s))^2),$$

$$L_{12}(t,s) = A(t-s)^{-1/2}(p_1(t)-p_1(s))\exp(B(t,s)(p_1(t)-p_1(s))^2),$$

$$L_{13}(t\ s) = A(t-s)^{1/2}\exp(B(t,s)(p_1(t)-p_2(s))^2),$$

$$L_{14}(t,s) = A(t-s)^{-1/2}(p_1(t)-p_2(s))\exp(B(t,s)(p_1(t)-p_2(s))^2).$$

$$L_{21}(t,s) = A(t-s)^{-1/2}(p_1(t)-p_1(s))\exp(B(t,s)(p_1(t)-p_1(s))^2),$$

$$L_{22}(t,s) = A\left[(t-s)^{-1/2}+(t-s)^{-3/2}(p_1(t)-p_1(s))^2\right]\exp(B(t,s)(p_1(t)-p_1(s))^2)$$

$$L_{23}(t,s) = A(t-s)^{-1/2}(p_1(t)-p_2(s))\exp(B(t,s)(p_1(t)-p_2(s))^2),$$

$$L_{24}(t,s) = \frac{A}{2}\left[(t-s)^{-1/2}+(t-s)^{-3/2}(p_4(t)-p_2(s))^2\right] \times$$

$$\times \exp(B(t,s)(p_1(t)-p_2(s))^2),$$

$$L_{31}(t,s) = A(t-s)^{1/2}\exp(B(t,s)(p_2(t)-p_1(s))^2),$$

$$L_{32}(t,s) = A(t-s)^{-1/2}(p_2(t)-p_1(s))\exp(B(t,s)(p_2(t)-p_1(s))^2),$$

$$L_{33}(t,s) = A(t-s)^{1/2}\exp(B(t,s)(p_2(t)-p_2(s))^2),$$

$$L_{34}(t,s) = A(t-s)^{1/2}(p_2(t)-p_2(s))\exp(B(t,s)(p_2(t)-p_2(s))^2),$$

$$L_{41}(t,s) = \frac{A}{2}(t-s)^{-1/2}(p_2(t)-p_2(s))\exp(B(t,s)(p_2(t)-p_2(s))^2),$$

$$L_{42}(t,s) = \frac{A}{2}\left[(t-s)^{-1/2}+(t-s)^{3/2}(p_2(t)-p_2(s))^2\right] \times$$

$$\times \exp(B(t,s)(p_2(t)-p_2(s))^2),$$

$$L_{43}(t,s) = A(t-s)^{-1/2}(p_2(t)-p_2(s))\exp(B(t,s)(p_2(t)-p_2(s))^2),$$

$$L_{44}(t,s) = \frac{A}{2}\left[(t-s)^{-1/2}+(t-s)^{-3/2}(p_2(t)-p_2(s))^2\right] \times$$

$$\times \exp(B(t,s)(p_2(t)-p_2(s))^2).$$

8. Transformation of the equation

Differentiating both sides of the equation (I_k), assuming that $D_tH^k(0) = 0$, $k = 1,2,3,4$, we obtain the system

$$(I_ka) \quad \sum_{i=1}^{4} \int_0^t q_i(s)D_tI_{ki}(t,s)ds = D_tH^k(t), \ k=1,2,3,4.$$

Lemma 4. If $p_i \in (P)$, $i=1,2$, $H^i \in C^1([0,T])$, $D_tH^i(0) = 0$, $i=1,2,3,4$, then the systems (I_k), (I_ka), $k=1,2,3,4$, are equivalent.

Proof. We have $L_{1i}(t,s) \to 0$, $i=1,2,3,4$, as $s \to t$. Moreover the integrals $\int_0^t q_i(s)D_tI_{1i}(t,s)ds$, $i=1,2,3,4$, are locally uniformly convergent for $t \in (0,T]$. Consequently from (I_1) follows (I_1a). Integrating (I_1a) we obtain (I_1).

Similarly we can transform (I_k) to (I_ka) and conversely (I_ka) to (I_k), $k=2,3,4$.

9. Transformation of the system (I_ka) to (I_kb)

Applying to both sides of the system (I_ka), $k=1,2,3,4$, Abel transformation [3], p. 82, we obtain the system (I_kb), $k=1,\dots,4$, of form

$$(I_kb) \quad \sum_{i=1}^{4} \int_0^t q_i(s)\left[\int_s^t (t-y)^{-1/2}D_yL_{ki}(y,s)dy\right]ds = \int_0^t (t-y)^{-1/2}D_yH^k(y)dy$$

$$k=1,2,3,4.$$

Lemma 5. If $q_i \in C([0,T])$, $i=1,2,3,4$, than the systems (I_ka) and (I_kb) are equivalent.

Proof. If q_i, $i=1,2,3,4$, satisfy (I_ka), then after Abel transformation they satisfy also the system (I_kb).

Conversely. We shall prove that the solution q_i, $i=1,2,3,4$, of the system (I_kb) is the solution of (I_ka).

Let

$$R_k(x) = D_xH^k(x) - \sum_{i=1}^{4} \int_0^x Q_i(s)D_xL_{ki}(x,s)ds, \ k=1,2,3,4. \quad (5)$$

Let q_i, $i=1,2,3,4$, be the solution of $(I_k b)$. Substituting in (5) Q_i by q_i we obtain by Abel transformation

$$\int_0^z (z-x)^{-1/2} R_k(x)dx = 0, \quad k = 1,2,3,4. \tag{6}$$

We shall determine $R_k(x)$, $k = 1,2,3,4$. By (6) we obtain

$$\int_0^y (y-z)^{-1/2}\Big[\int_0^z (z-x)^{-1/2} R_k(x)dx\Big]dz = 0, \quad k = 1,2,3,4. \tag{7}$$

Interchanging in (7) the order of integration we obtain

$$\int_0^y \Big[\int_x^y ((y-z)(z-x))^{-1/2}dz\Big] R_k(x)dx = 0, \quad k = 1,2,3,4. \tag{8}$$

By transformation $z = x+(y-x)u$ we obtain

$$\int_x^y ((y-z)(z-x))^{-1/2}dz = \beta(1/2,1/2), \quad \beta \text{ being beta Euler function} \tag{9}$$

By (8), (9) we obtain

$$\int_0^y R_k(x)dx = 0 \quad \text{for} \quad y \in (0,T), \; k = 1,2,3,4.$$

Differentiating the last identity we obtain

$$R_k(t) = 0 \quad \text{for} \quad t \in (0,T), \; k = 1,2,3,4.$$

Consequently the system q_i, $I = 1,2,3,4$, satisfies the system $(I_k a)$.

Lemma 6. If $H^k \in C^2([0,T])$, $D_y H^k(0) = 0$, $k = 1,2,3,4$, then

$$\bar{H}^k(t) = \int_0^t (t-y)^{-1/2} D_y H^k(y)dy = 2\int_0^t (t-y)^{1/2} D_y^2 H^k(y)dy, \quad k = 1,2,3,4.$$

We omit the simple proof.

10. The equations $(I_k c)$ and $(I_k d)$

Applying in the system $(I_k b)$ the change of the integral variable $y = s+(t-s)u$, we obtain the system

$$(I_k c) \qquad \sum_{i=1}^{4} \int_0^t q_i(s) H_{ki}(t,s)ds = \bar{H}^k(t),\ k = 1,2,3,4,$$

where

$$H_{ki}(t,s) = \int_s^t (t-y)^{-1/2} D_y L_{ki}(y,s)dy$$

Let

$$K_{11}(t,s,u) = p_1(s+(t-s)u)-p_1(s),\ K_{22}(t,s,u) = p_2(s+(t-s)u)-p_2(s),$$

$$K_{12}(t,s,u) = p_1(s+(t-s)u)-p_2(s),\ K_{21}(t,s,u) = p_2(s+(t-s)u)-p_1(s).$$

By the last formulas we obtain

$$H_{11}(t,s) = \int_0^1 \Big[(1-u)^{-1/2}u^{-1/2} + (t-s)^{-1}(1-u)^{-1/2}u^{-3/2}K_{11}^2(t,s,u) +$$

$$+(1-u)^{-1/2}u^{-1/2}K_{11}(t,s,u)p_1'(s+(t-s)u\Big] \exp(B(s+(t-s)u,s)K_{11}^2(t,s,u)du,$$

$$H_{12}(t,s) = \int_0^1 \Big[(1-u)^{-1/2}u^{-1/2}K_{11}(t,s,u)+(1-u)^{-1/2}u^{-1/2}p_1'(s+(t-s)u)+$$

$$+(t-s)(1-u)^{-1/2}u^{-1/2}K_{11}^3(t,s,u)+(t-s)^{-1}(1-u)^{-1/2}u^{-3/2}K_{11}^2(t,s,u) \times$$

$$\times p_1'(s+(t-s)u)\Big] \exp(B(s+(t-s)u,s)K_{11}^2(t,s,u),$$

$$H_{13}(t,s) = \int_0^1 \Big[(1-u)^{-1/2}u^{-1/2}+(t-s)^{-1}(1-u)^{-1/2}u^{-3/2}K_{12}^2(t,s,u) +$$

$$+(1-u)^{-1/2}u^{-1/2} K_{12}(t,s,u)p_1'(s+(t-s)u)\Big] \exp(B(s+(t-s)u,s)K_{12}^2(t,s,u)du,$$

$$H_{14}(t,s) = \int_0^1 \Big[(t-s)^{-1}(1-u)^{-1/2}u^{-3/2}K_{12}(t,s,u)+(1-u)^{-1/2}u^{-1/2} +$$

$$+(t-s)^{-2}(1-u)^{-1/2}u^{-5/2}K_{12}^2(t,s,u)+(t-s)^{-1}(1-u)^{-1/2}u^{-3/2}K_{12}^2(t,s,u)\Big] \times$$
$$\times \exp(B(s+(t-s)u,s)K_{12}^2(t,s,u)),$$

$$H_{21}(t,s) = \int_0^1 \left[(1-u)^{-1/2}u^{-1/2}K_{11}(t,s,u)\exp(B(s+(t-s)u,s)K_{11}^2(t,s,u)\right]du,$$

$$H_{22}(t,s) = \int_0^1 \left[(1-u)^{-1/2}u^{-3/2}(t-s)^{-1}K_{11}(t,s,u)\exp(B(s+(t-s)u,s) \times\right.$$

$$\left.\times K_{11}^2(t,s,u)\right]du,$$

$$H_{23}(t,s) = \int_0^1 \left[(1-u)^{-1/2}u^{-1/2}K_{12}(t,s,u)\exp(B(s+(t-s)u,s)K_{12}^2(t,s,u)\right]du,$$

$$H_{24}(t,s) = \int_0^1 \left[(1-u)^{-1/2}u^{-1/2}+(1-u)^{-1/2}u^{-3/2}(t-s)^{-1}K_{12}(t,s,u)\right] \times$$

$$\times \exp(B(s+(t-s)u,s)K_{12}^2(t,s,u))du.$$

We obtain the kernels H_{33} replacing in H_{11}, K_{11} by K_{22} and the kernels $H_{3,k}$, $k = 1,2,4$, replacing in $H_{1,k}$, $k = 1,2,4$, p_1 by p_2, p_2 by p_1, K_{12} by K_{21} and K_{11} by K_{22}.

Similarly we obtain the kernel H_{44}, replacing in H_{22}, K_{11} by K_{22} and the kernels $H_{4,k}$, $k = 1,2,3$, replacing in H_{2k}, $k = 1,2,3$, p_1 by p_2, p_2 by p_1 and K_{21} by K_{12}, K_{11} by K_{22}.

Applying the mean value theore we prove that the kernels $H_{ik}(t,s)$ are weak singular of form $(t-s)^{-1/2}M_{ik}(t,s)$, M_{ik} being continuous.

Differentiating both sides of the system (I_kc) we obtain the system (I_kd) of the form

$$(I_1d) \quad q_1(t)+2q_2(t)p_1'(t)+\sum_{i=1}^{4}\int_0^t q_i(s)D_tH_{1i}(t,s)ds = \overline{\overline{H}}^1(t),$$

$$(I_2d) \quad q_2(t)+\sum_{i=1}^{4}\int_0^t q_i(s)D_tH_{2i}(t,s)ds = \overline{\overline{H}}^2(t),$$

$$(I_3d) \quad q_3(t)+2q_4(t)p_2'(t)+\sum_{i=1}^{4}\int_0^t q_i(s)D_tH_{3i}(t,s)ds = \overline{\overline{H}}^3(t),$$

$$(I_4d) \quad q_4(t)+\sum_{i=1}^{4}\int_0^t q_i(s)D_tH_{4i}(t,s)ds = \overline{\overline{H}}^4(t),$$

$\overline{\overline{H}}^i(t) = D_t\overline{H}^i(t)$, $i = 1,2,3,4$.

or

$$(I_1d) \quad q_1(t) = F^1(t) + \sum_{i=1}^{4}\int_0^t q_i(s)N_{1i}(t,s)ds,$$

$$F^1(t) = \overline{\overline{H}}^1(t)-2p_1'(t)\overline{\overline{H}}^2(t), N_{1i}(t,s)=2p_1'(t)D_tH_{2i}(t,s)-D_tH_{1i}(t,s),$$

$$(I_2D) \quad q_2(t) = F^2(t)+ \sum_{i=1}^{4}\int_0^t q_i(s)N_{2i}(t,s)ds,$$

$$F^2(t) = \overline{\overline{H}}^2(t), \ N_{2i}(t,s) = -H_{2i}(t,s),$$

$$(I_3d) \quad q_3(t) = F^3(t) + \sum_{i=1}^{4}\int_0^t q_i(s)N_{3i}(t,s)ds.$$

$$F^3(t) = \overline{\overline{H}}^3(t)-2p_2'(t)\overline{\overline{H}}^4(t), \ N_{3i}(t,s) = 2p_2'(t)D_tH_{4i}(t,s)-D_tH_{3i}(t,s),$$

$$(I_4d) \quad q_4(t) = F^4(t) + \sum_{i=1}^{4}\int_0^t q_i(s)N_{4i}(t,s)ds,$$

$$F^4(t) = \overline{\overline{H}}^4(t), \ N_{4i}(t,s) = -D_tH_{4i}(t,s).$$

Lemma 7. If $p_i \in (P)$, $i = 1,2$, $q_i = C([0,T])$, $i = 1,2,3,4$, $H^i \in (h)$, $i = 1,3$, $H^i \in (h_1)$, $i = 2,4$, then the systems (I_kc) and (I_kd) are equivalent.

We omit the simple proof

11. <u>Solution of the system</u> (I_kd)

Let us consider the system

$$(I_k\lambda) \quad q_i(t) = F^i(t) + \lambda\sum_{j=1}^{j=4}\int_0^t q_i(s)H_{ij}(t,s)ds, \ i = 1,2,3,4$$

Let

$$N_{ij}^{(1)}(t,s) = \sum_{k=1}^{4}\int_s^t H_{ik}(t,s_1)H_{kj}(s_1,s)ds_1,$$

$$N_{i,j}^{(n+1)}(t,s) = \sum_{k=1}^{4}\int_s^t N_{i,k}(t,s_1)N_{k,j}^{(n)}(s_1,s)ds_1, \ i,j=1,2,3,4; \ n=2,\dots,$$

$$R_{i,j}(t,s,\lambda) = N_{i,j}(t,s) + \lambda N_{i,j}^{(1)}(t,s) + \ldots + \lambda^m N_{i,j}(t,s) + \ldots$$

Theorem 1. If $F^i \in C([0,T])$, $i = 1,2,3,4$, then the functions

$$q_i(t) = F^i(t) + \int_0^t \sum_{j=1}^{4} R_{i,j}(t,s,1)F^j(s)ds, \quad i = 1,2,3,4,$$

are the solution of the system $(I_k d)$, and the solution is unique.

Proof. See [3], p. 97, [6], p. 4.

12. Fundamental theorem

Theorem 2. If the assumptions of Lemmas 1-7, and theorem 1 are satisfied, then the function $w = \sum_1^5 w_i$ is the solution of the problem (1a)-(4a) and the function $u=w+r$ is the solution of the (1)-(4) problem.

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