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ON A PROBLEM OF Z. DARÓCZY

The problem concerns the functional equation

$$F(x) = F(x+1) + F(x(x+1)) \tag{1}$$

and its solutions $F: (0,+\infty) \to \mathbf{R}$ and $F: \mathbf{N} \to \mathbf{R}$ as well. Posing the problem Z. Daróczy was interested in finding a weak condition under which the solution $F: (0,+\infty) \to \mathbf{R}$ is of the form

$$F(x) = \frac{c}{x} \tag{2}$$

with a constant $c \in \mathbf{R}$ and he conjectured also that each solution $F: \mathbf{N} \to (0,+\infty)$ has to be of this form (see [2]). This conjecture has been recently disproved by M. Laczkovich and R. Redheffer (see [7; pp. 115-117] and [5; Corollary 2]). Regarding conditions under which the solution $F: (0,+\infty) \to \mathbf{R}$ or $F: \mathbf{N} \to \mathbf{R}$ has the form (2) the following has been proposed in [1].

Proposition. If $F: (0,+\infty) \to \mathbf{R}$ or $F: \mathbf{N} \to \mathbf{R}$ is a solution of (1), and there exists a real constant c such that

$$\lim_{x \to +\infty} xF(x) = c, \tag{3}$$

then F has form (2).

The proof given in [1] is based on a general theorem on functional equations. It is the aim of the present note to give an elementary and direct proof of the proposition. I was looking for such a proof being stimulated by Professor Zoltán Daróczy to whom I thank for it cordially.

We start with the following lemma.

Lemma. Suppose $F: (0,+\infty) \to \mathbf{R}$ or $F: \mathbf{N} \to \mathbf{R}$ is a solution of (1). Let c be a real number.

If there exist a positive number M such that

$$F(x) \leq \frac{c}{x} \quad \text{for} \quad x > M$$

then

$$F(x) \leqslant \frac{c}{x} \quad \text{for} \quad x > 0.$$

Proof. Replacing the solution F by $F(x) - \frac{c}{x}$ we may (and we do) assume that $c=0$. By induction and (1):

$$\text{if} \quad F(x) \leqslant 0 \quad \text{for} \quad x \geqslant n+1, \quad \text{then} \quad F(x) \leqslant 0 \quad \text{for} \quad x \geqslant n.$$

Hence

$$F(x) \leqslant 0 \quad \text{for} \quad x \geqslant 1. \tag{4}$$

This ends the proof of the lemma in the case where F is defined for positive integers only.

Assume the solution F is defined on $(0,+\infty)$. Taking (4) and (1) into account we have

$$F(x) \leqslant F(x(x+1)) \quad \text{for} \quad x \in (0,+\infty). \tag{5}$$

Define

$$a_0 = 1, \; a_n = \frac{\sqrt{4a_{n-1}+1} - 1}{2}, \; n \in \mathbf{N}, \tag{6}$$

and observe that this sequence strictly decreases,

$$\lim_{n \to +\infty} a_n = 0, \tag{7}$$

and, for every $n \in \mathbf{N}$,

$$\text{if} \quad x \geqslant a_n, \quad \text{then} \quad x(x+1) \geqslant a_{n-1}.$$

Hence and from (5) we infer that, for every $n \in \mathbf{N}$,

$$\text{if} \quad F(x) \leqslant 0 \quad \text{for} \quad x \geqslant a_{n-1}, \quad \text{then} \quad F(x) \leqslant 0 \quad \text{for} \quad x \geqslant a_n.$$

This jointly with (4) and (6) shows that

$$F(x) \leqslant 0 \quad \text{for} \quad x \geqslant a_{n-1} \quad \text{and} \quad n \in \mathbf{N},$$

which together with (7) gives

$$F(x) \leq 0 \quad \text{for} \quad x > 0$$

and ends the proof of the lemma.

Using this lemma we shall prove what follows.

Theorem. Suppose $F: (0,+\infty) \to \mathbf{R}$ or $F: \mathbf{N} \to \mathbf{R}$ is a solution of (1). If there exists the limit

$$\lim_{x\to+\infty} xF(x), \tag{8}$$

then it is necessarily finite and F has the form (2) with c being the limit (8).

Proof. Assume the limit (8) equals $-\infty$ and fix arbitrarily a real number c. Then there exists a positive number M such that

$$xF(x) \leq c \quad \text{for} \quad x > M.$$

Hence and from the lemma we obtain

$$xF(x) \leq c \quad \text{for} \quad x > 0 \tag{9}$$

which is of course impossible as c was fixed arbitrarily. The case where the limit (8) equals $+\infty$ reduces to the previous one by considering the function $-F$.

Up to now we have proved that limit (8) is finite. Denote it by c and fix arbitrarily a positive number ε. It follows from (3) that there exists a positive number M such that

$$xF(x) \leq c+\varepsilon \quad \text{for} \quad x > M.$$

Hence and from the lemma we obtain

$$xF(x) \leq c+\varepsilon \quad \text{for} \quad x > 0.$$

Consequently, as the positive number ε has been fixed arbitrarily, (9) holds. Applying it to the function $-F$ we shall obtain the reverse inequality which shows that F has form (2) and ends the proof.

The theorem, being a little more general than the proposition, allows us to state immediately the following corollary, discovered on another way by W. Jarczyk (see [3; Theorem 1]).

Corollary. Suppose $F: (0,+\infty) \to \mathbf{R}$ or $F: \mathbf{N} \to \mathbf{R}$ is a solution of (1).

If the function $xF(x)$ is monotonic in a vicinity of infinity then F has form (2).

Remark. If a real function f defined on a vicinity of infinity is convex then in a vicinity of infinity the function $\frac{f(x)}{x}$ is monotonic (see [4; Lemme 2]). Hence, if $F: (0,+\infty) \to \mathbf{R}$ is a solution of (1) such that $x^2F(x)$ is convex in a vicinity of infinity then F has the form (2). Similarly, if $F: (0,+\infty) \to \mathbf{R}$ is a solution of (1) such that the function $\frac{1}{F(x)}$ is convex in a vicinity of infinity, then $\frac{1}{xF(x)}$ has a limit at infinity and, due to the fact that F has a constant sign in a vicinity of infinity, there exists the limit (8); consequently, F has the form (2). These facts have been obtained by W. Jarczyk directly (see [3 ; Theorems 2 and 3]). In Jarczyk's paper the reader may find also further results connected with Daróczy's problem.

We end by the remark that equation (1) has many solutions $F: (0,+\infty) \to \mathbf{R}$, even continuous ones, as Z. Moszner has shown in [6].

REFERENCES

[1] K. Baron: P283R2. Aequationes Mathematicae 35 (1988), 301-303.

[2] Z. Daróczy: P283. Aequationes Mathematicae 32 (1987), 136-137.

[3] W. Jarczyk: On a problem of Z. Daróczy. Annales Mathematicae Silesianae (to appear).

[4] M. Kuczma: Un théorème d'unicité pour l'équation fonctionnelle de Böttcher. Mathematica (Cluj) 9 (32) (1967), 285-293.

[5] M. Laczkovich and R. Redheffer: Oscillating solutions of integral equations and linear recursions. Manuscript.

[6] Z. Moszner: P283R1. Aequationes Mathematicae 32(1987), 146.

[7] Report of Meeting: The Twenty-sixth International Symposium on Functional Equations, April 24 - May 3, 1988, Sant Feliu de Guixols, Catalonia, Spain. Aequationes Mathematicae 37 (1989), 57-127.