## ZESZYTY NAUKOWE POLITECHNIKI SLASKIEJ

Seria: MATEMATYKA-FIZYKA z..64

DEDICATED TO PROFESSOR MIECZYSŁAW KUCHARZEWSKI WITH BEST WISHES ON HIS 70TH BIRTHDAY

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SOME PROPERTIES OF ALMOST SYMPLECTIC PSEUDO-RIEMANNIAN MANIFOLDS

Let M be an m-dimensional smooth manifold and A be a smooth tensor field on M of type (1,1) such that for every  $x \in M$ ,  $A_x : T_x M \longrightarrow T_x M$  is an isomorphism. Let

 $P = \frac{1}{2} (\delta + A)$  and  $Q = \frac{1}{2} (\delta - A)$ 

where  $\delta_x = \delta |_{T_x M} = id_{T_x M}$  every  $x \in M$ . Evidently  $P + Q = \delta$  and P - Q = A.

Lemma 1. The following are true

1. AP = PA, AQ = QA.

2. PQ = QP.

For every x & M, moreover:

3.  $A_x$  [Ker  $P_x$ ] = Ker  $P_x$ ,  $A_x$  [Ker  $Q_x$ ] = Ker  $Q_x$ . 4. Ker  $P_x \subseteq Im Q_x$ , Ker  $Q_x \subseteq Im P_x$ . 5. Ker  $P_y \cap$  Ker  $Q_y = \{0\}$ .

Proof, Simple.

A is said to be an almost product structure on M if  $A^2 = \delta$ .

Lemma 2. The following are equivalent:

1. A is an almost product structure.

2.  $P^2 = P$  (respectively  $Q^2 = Q$ ).

3. AP = PA = P (respectively AQ = QA = Q).

4. PQ = QP = 0.

5. For every  $x \in M$ , Ker  $P_x = Im Q_x$  (respectively Ker  $Q_x = Im P_x$ ). Proof. Simple.

<u>Corollary</u>. If A is an almost product structure, them  $T_X^M = \text{Ker P}_X \oplus$  $\oplus$  Im  $P_X$  for every  $x \in M$ .

Let us now specialize M and A. Take m = 2n and assume that on M there is given an almost symplectic structure  $\omega$  and moreover a pseudo-

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Rismannian metric g with signature (k,2n-k), where  $k \in \{0,\ldots,n\}$  denotes the number of minus signs in the canonical form of g. On M there exists exactly one smooth tensor field A of type (1,1) such that

$$\omega(u,v) = g(A_u(u),v)$$

for every  $x \in M$  and every U,  $v \in T_X^M$ , where  $A_x : T_X \longrightarrow T_M$  is an isomorphism. From now on let A be this tensor field.

To get the explicite form of A, take for arbitrary  $x \in M$  a canonical basis  $\{e_1, \ldots, e_{2n}\}$  of  $T_v M$  with respect to g such that

$$g_{ii} = g(e_i, e_i) = \begin{cases} -1 \text{ if } i \in \{1, \dots, k\} \\ 1 \text{ if } i \in \{k+1, \dots, 2n\}. \end{cases}$$

For every  $i \in \{1, \dots, 2n\}$  then we have  $A_x(e_1) = \sum_{j=1}^{2n} \alpha_{ij} e_j$  with

$$\mathbf{x}_{ij} = \begin{cases} -\omega(\mathbf{e}_{i}, \mathbf{e}_{j}) & \text{if } j \in \{1, \dots, k\} \\ \omega(\mathbf{e}_{i}, \mathbf{e}_{i}) & \text{if } j \in \{k+1, \dots, 2n\}. \end{cases}$$

For every x 6 M obviously  $A_x = \delta_x$ ,  $-\delta_x$ , hence  $P_x \neq 0$  and  $Q_x \neq 0$ . (1) is equivalent to every of the two relations

$$\begin{aligned} \omega(u,v) &= 2g(P_{\chi}(u),v) - g(u,v) \\ \omega(u,v) &= -2g(Q_{\chi}(u),v) + g(u,v). \end{aligned}$$
 (1)

(1) implies

$$g(A_{u}(u),v) = -g(u,A_{u}(v))$$
<sup>(2)</sup>

and

$$\omega(A_{u}(u),v) = -\omega(u,A_{u}(v))$$

which respectively are equivalent to every of the two relations

$$\begin{cases} g(u,v) = p(P_{x}(u),v) + g(u,P_{x}(v)) \\ g(u,v) = g(Q_{x}(u),v) + g(u,Q_{y}(v)) \end{cases}$$
(2')

or

$$\begin{cases} \omega(\mathbf{u},\mathbf{v}) = \omega(\mathsf{P}_{\mathsf{X}}(\mathbf{u}),\mathbf{v}) + \omega(\mathbf{u},\mathsf{P}_{\mathsf{X}}(\mathbf{v})) \\ \omega(\mathbf{u},\mathbf{v}) = \omega(\mathsf{Q}_{\mathsf{v}}(\mathbf{u}),\mathbf{v}) + \omega(\mathsf{u},\mathsf{Q}_{\mathsf{v}}(\mathbf{v})). \end{cases}$$

(1)

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(3)

(3')

## Some properties...

Because of (2') and (3') for every  $x \in M$  the linear subspaces Ker  $P_x$  and Ker Q, of M are  $\omega$ - as well as g-isotropic.

Theorem. For every X & M

dim Ker  $P_y = \dim Ker Q_y$ 

and thus

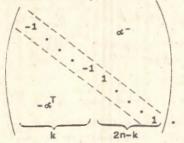
dim Im  $P_x$  = dim Im  $Q_y$ .

**Proof.** For arbitrary  $x \in M$ , let  $\{e_1, \ldots, e_{2n}\}$ ,  $g_{11}$  (i  $\in \{1, \ldots, 2n\}$ ) and  $\alpha_{1j}$  (i,j  $\in \{1, \ldots, 2n\}$ ) be as above. (2) implies that  $\alpha_{1j}g_{jj} = -\alpha_{ji}g_{1i}$ for every i,j  $\in \{1, \ldots, 2n\}$ . Hence the matrix  $O_{i} = (\alpha_{1j})$  associated to  $A_{v}$  with respect to  $\{e_{1}, \ldots, e_{2n}\}$  can be written as

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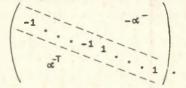
 $\mathcal{E}$  being +1 or -1, or -1, consider  $\mathcal{O}_{L} = \mathcal{O}_{L} + \mathcal{E} \mathcal{E}$ . With respect to  $\{e_{1}, \dots, e_{2n}\}$  then  $\mathcal{O}_{L_{1}}$  is associated to  $2P_{\chi}$  and  $\mathcal{O}_{L_{1}}$  is associated to  $2Q_{\chi}$ . To prove our theorem we have to show that  $\mathcal{O}_{L_{1}}$  and  $\mathcal{O}_{L_{1}}$  have the same rank. For this we transform  $\mathcal{O}_{L_{1}}$  into  $\mathcal{O}_{L_{1}}$  in for steps by procedures which always are such that the rank of the matrix is not changed.

1. We multiply the elements of the first k rows of  $\alpha_{+1}$  by -1 and get a matrix of the type

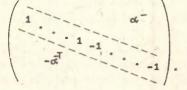


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2. Transposition of this matrix gives



3. By multiplication (of all elements) with -1 we get the matrix



4. Multiplication of the first k rows of this matrix by -1 leads to  $\operatorname{Cl}_{1^*}$ 

This shows that really  $\alpha_{+1}$  and  $\alpha_{-1}$  have the same rank. Thus the theorem is proven.

By a standard procedure it becomes obvious that the common number  $d = \dim \text{Ker P}_{v} = \dim \text{Ker Q}_{v}$  is independent of x.

<u>Corollary 1</u>. There exist two  $\omega$  - and g-isotropic d-dimensional smooth distributions  $D^1$  and  $D^2$  on M such that  $D_x^1 \cap D_x^2 = \{0\}$  for every  $x \in M$ .

<u>Proof</u>. Define D<sup>1</sup> and D<sup>2</sup> by  $D_x^1 = \text{Ker P}_x$  respectively  $D_x^2 = \text{Ker Q}_x$  for every  $x \in M$ .

Corollary 2.  $d \leq k$ .

<u>Proof</u>. For arbitrary  $x \in M$ , let  $\{e_1, \ldots, e_{2n}\}$  and  $g_{11}$  ( $1 \in \{1, \ldots, 2n\}$ ) be as above. Without difficulty we can find a basis  $\{b_1, \ldots, b_d\}$  of Ker  $P_x$  such that for every  $i \in \{1, \ldots, d\}$  we have

$$b_{i} = \sum_{j=1}^{2n} \beta_{ij} e_{j}$$

Under the assumption d > k from  $g(b_d, b_d) = 0$  we get  $\sum_{j=d}^{2\pi} \beta_{dj}^2 = 0$ , hence  $b_d = 0$  which gives a contradiction. Thus we have  $d \le k$ . As a direct consequence of Corollary 2 we get

Corollary 3. If g is Riemannian, then d = 0.

Corollary 4. If A is an almost product structure, then d = k = n.

<u>Proof</u>. By Lemma 2 and the Theorem we get dim Ker  $P_x = \dim \text{Im } P_x$ . Hence by means of 2n = dim Ker  $P_x + \dim \text{Im } P_x$  the assertion of the corollary becomes obvious.

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## Some properties....

We now give three examples with  $n \ge k=2$  illustrating what may happen with respect to d. For this for arbitrary  $x \in M$  let  $\{e_1, \ldots, e_4\}$  be a basis of  $T_{\downarrow}M$  with

 $g(e_{i},e_{j}) = \begin{cases} -1 \text{ if } i=j \in \{1,2\} \\ 1 \text{ if } i=j \in \{3,4\} \\ 0 \text{ otherwise} \end{cases}$ 

By  $\mathcal{M}$  we denote the matrix  $(\omega(e_i,e_j))$ . It is easy to verify the following

1. If 
$$\mathcal{M}_{=} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$
, then  $d = 0$ .  
2. If  $\mathcal{M}_{=} \begin{pmatrix} 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ , then  $d = 1$ .  
3. If  $\mathcal{M}_{=} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$ , then  $d = 2$ .

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