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DEDICATED TO PROFESSOR MIECZYSŁAW KUCHARZEWSKI
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SOME PROPERTIES OF ALMOST SYMPLECTIC PSEUDO-RIEMANNIAN MANIFOLDS

Let M be an m-dimensional smooth manifold and A be a smooth tensor field on M of type $(1,1)$ such that for every $x \in M$, $A_x : T_xM \to T_xM$ is an isomorphism. Let

$$P = \frac{1}{2}(\delta + A) \quad \text{and} \quad Q = \frac{1}{2}(\delta - A)$$

where $\delta_x = \delta|_{T_xM} = \mathrm{id}_{T_xM}$ every $x \in M$. Evidently $P + Q = \delta$ and $P - Q = A$.

<u>Lemma 1</u>. The following are true

1. $AP = PA$, $AQ = QA$.
2. $PQ = QP$.

For every $x \in M$, moreover:

3. $A_x[\mathrm{Ker}\, P_x] = \mathrm{Ker}\, P_x$, $A_x[\mathrm{Ker}\, Q_x] = \mathrm{Ker}\, Q_x$.
4. $\mathrm{Ker}\, P_x \subseteq \mathrm{Im}\, Q_x$, $\mathrm{Ker}\, Q_x \subseteq \mathrm{Im}\, P_x$.
5. $\mathrm{Ker}\, P_x \cap \mathrm{Ker}\, Q_x = \{0\}$.

<u>Proof</u>. Simple.

A is said to be an almost product structure on M if $A^2 = \delta$.

<u>Lemma 2</u>. The following are equivalent:

1. A is an almost product structure.
2. $P^2 = P$ (respectively $Q^2 = Q$).
3. $AP = PA = P$ (respectively $AQ = QA = Q$).
4. $PQ = QP = 0$.
5. For every $x \in M$, $\mathrm{Ker}\, P_x = \mathrm{Im}\, Q_x$ (respectively $\mathrm{Ker}\, Q_x = \mathrm{Im}\, P_x$).

<u>Proof</u>. Simple.

<u>Corollary</u>. If A is an almost product structure, then $T_xM = \mathrm{Ker}\, P_x \oplus \mathrm{Im}\, P_x$ for every $x \in M$.

Let us now specialize M and A. Take $m = 2n$ and assume that on M there is given an almost symplectic structure ω and moreover a pseudo-

Riemannian metric g with signature $(k,2n-k)$, where $k \in \{0,\dots,n\}$ denotes the number of minus signs in the canonical form of g. On M there exists exactly one smooth tensor field A of type $(1,1)$ such that

$$\omega(u,v) = g(A_x(u),v) \tag{1}$$

for every $x \in M$ and every $u, v \in T_xM$, where $A_x : T_xM \to T_xM$ is an isomorphism. From now on let A be this tensor field.

To get the explicite form of A, take for arbitrary $x \in M$ a canonical basis $\{e_1,\dots,e_{2n}\}$ of T_xM with respect to g such that

$$g_{ii} = g(e_i,e_i) = \begin{cases} -1 & \text{if } i \in \{1,\dots,k\} \\ 1 & \text{if } i \in \{k+1,\dots,2n\}. \end{cases}$$

For every $i \in \{1,\dots,2n\}$ then we have $A_x(e_i) = \sum_{j=1}^{2n} \alpha_{ij} e_j$ with

$$\alpha_{ij} = \begin{cases} -\omega(e_i,e_j) & \text{if } j \in \{1,\dots,k\} \\ \omega(e_i,e_j) & \text{if } j \in \{k+1,\dots,2n\}. \end{cases}$$

For every $x \in M$ obviously $A_x \neq \delta_x, -\delta_x$, hence $P_x \neq 0$ and $Q_x \neq 0$. (1) is equivalent to every of the two relations

$$\begin{cases} \omega(u,v) = 2g(P_x(u),v) - g(u,v) \\ \omega(u,v) = -2g(Q_x(u),v) + g(u,v). \end{cases} \tag{1'}$$

(1) implies

$$g(A_x(u),v) = -g(u,A_x(v)) \tag{2}$$

and

$$\omega(A_x(u),v) = -\omega(u,A_x(v)) \tag{3}$$

which respectively are equivalent to every of the two relations

$$\begin{cases} g(u,v) = p(P_x(u),v) + g(u,P_x(v)) \\ g(u,v) = g(Q_x(u),v) + g(u,Q_x(v)) \end{cases} \tag{2'}$$

or

$$\begin{cases} \omega(u,v) = \omega(P_x(u),v) + \omega(u,P_x(v)) \\ \omega(u,v) = \omega(Q_x(u),v) + \omega(u,Q_x(v)). \end{cases} \tag{3'}$$

Because of (2′) and (3′) for every $x \in M$ the linear subspaces Ker P_x and Ker Q_x of M are ω- as well as g-isotropic.

<u>Theorem</u>. For every $x \in M$

$$\dim \operatorname{Ker} P_x = \dim \operatorname{Ker} Q_x$$

and thus

$$\dim \operatorname{Im} P_x = \dim \operatorname{Im} Q_x .$$

<u>Proof</u>. For arbitrary $x \in M$, let $\{e_1,\dots,e_{2n}\}$, g_{ii} $(i \in \{1,\dots,2n\})$ and α_{ij} $(i,j \in \{1,\dots,2n\})$ be as above. (2) implies that $\alpha_{ij}g_{jj} = -\alpha_{ji}g_{ii}$ for every $i,j \in \{1,\dots,2n\}$. Hence the matrix $\mathfrak{A} = (\alpha_{ij})$ associated to A_x with respect to $\{e_1,\dots,e_{2n}\}$ can be written as

$$\mathfrak{A} = \left(\begin{array}{cccc:cccc}
0 & \alpha_{12} & \cdots & \alpha_{1k} & \alpha_{1\,k+1} & \alpha_{1\,k+2} & \cdots & \alpha_{1\,2n} \\
-\alpha_{12} & 0 & \cdots & \alpha_{2k} & \alpha_{2\,k+1} & \alpha_{2\,k+2} & & \alpha_{2\,2n} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
-\alpha_{1k} & -\alpha_{2k} & \cdots & 0 & \alpha_{k\,k+1} & \alpha_{k\,k+2} & \cdots & \alpha_{k\,2n} \\
\hdashline
\alpha_{1\,k+1} & \alpha_{2\,k+1} & \cdots & \alpha_{k\,k+1} & 0 & \alpha_{k+1\,k+2} & \cdots & \alpha_{k+1\,2n} \\
\alpha_{1\,k+2} & \alpha_{2\,k+2} & \cdots & \alpha_{k\,k+2} & -\alpha_{k+1\,k+2} & 0 & & \alpha_{k+2\,2n} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\alpha_{1\,2n} & \alpha_{2\,2n} & \cdots & \alpha_{k\,2n} & -\alpha_{k+1\,2n} & -\alpha_{k+2\,2n} & \cdots & 0
\end{array}\right)$$

ε being +1 or -1, or -1, consider $\mathfrak{A} = \mathfrak{A} + \varepsilon\mathcal{E}$. With respect to $\{e_1,\dots,e_{2n}\}$ then $\mathfrak{A}_{+1}$ is associated to $2P_x$ and $\mathfrak{A}_{-1}$ is associated to $2Q_x$. To prove our theorem we have to show that $\mathfrak{A}_{+1}$ and $\mathfrak{A}_{-1}$ have the same rank. For this we transform $\mathfrak{A}_{+1}$ into $\mathfrak{A}_{-1}$ in for steps by procedures which always are such that the rank of the matrix is not changed.

1. We multiply the elements of the first k rows of $\mathfrak{A}_{+1}$ by -1 and get a matrix of the type

$$\left(\begin{array}{cccccc}
-1 & & & & & \\
 & \ddots & & & \alpha^- & \\
 & & -1 & & & \\
 & & & 1 & & \\
 & -\alpha^T & & & \ddots & \\
 & & & & & 1
\end{array}\right) .$$

$$\underbrace{\qquad\qquad}_{k}\quad\underbrace{\qquad\qquad}_{2n-k}$$

2. Transposition of this matrix gives

$$\begin{pmatrix} -1 & & & & & & -\alpha^{-} \\ & \ddots & & & & & \\ & & -1 & & & & \\ & & & 1 & & & \\ & \alpha^{-T} & & & \ddots & & \\ & & & & & & 1 \end{pmatrix}.$$

3. By multiplication (of all elements) with -1 we get the matrix

$$\begin{pmatrix} 1 & & & & & & \alpha^{-} \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & -1 & & & \\ & -\alpha^{-T} & & & \ddots & & \\ & & & & & & -1 \end{pmatrix}.$$

4. Multiplication of the first k rows of this matrix by -1 leads to $\mathfrak{a}_{-1}$.

This shows that really $\mathfrak{a}_{+1}$ and $\mathfrak{a}_{-1}$ have the same rank. Thus the theorem is proven.

By a standard procedure it becomes obvious that the common number $d = \dim \operatorname{Ker} P_x = \dim \operatorname{Ker} Q_x$ is independent of x.

<u>Corollary 1</u>. There exist two ω- and g-isotropic d-dimensional smooth distributions D^1 and D^2 on M such that $D^1_x \cap D^2_x = \{0\}$ for every $x \in M$.

<u>Proof</u>. Define D^1 and D^2 by $D^1_x = \operatorname{Ker} P_x$ respectively $D^2_x = \operatorname{Ker} Q_x$ for every $x \in M$.

<u>Corollary 2</u>. $d \leqslant k$.

<u>Proof</u>. For arbitrary $x \in M$, let $\{e_1,\dots,e_{2n}\}$ and g_{ii} $(i \in \{1,\dots,2n\})$ be as above. Without difficulty we can find a basis $\{b_1,\dots,b_d\}$ of $\operatorname{Ker} P_x$ such that for every $i \in \{1,\dots,d\}$ we have

$$b_i = \sum_{j=1}^{2n} \beta_{ij} e_j .$$

Under the assumption $d > k$ from $g(b_d, b_d) = 0$ we get $\sum\limits_{j=d}^{2n} \beta_{dj}^2 = 0$, hence $b_d = 0$ which gives a contradiction. Thus we have $d \leqslant k$.

As a direct consequence of Corollary 2 we get

<u>Corollary 3</u>. If g is Riemannian, then $d = 0$.

<u>Corollary 4</u>. If A is an almost product structure, then $d = k = n$.

<u>Proof</u>. By Lemma 2 and the Theorem we get $\dim \operatorname{Ker} P_x = \dim \operatorname{Im} P_x$. Hence by means of $2n = \dim \operatorname{Ker} P_x + \dim \operatorname{Im} P_x$ the assertion of the corollary becomes obvious.

We now give three examples with $n=k=2$ illustrating what may happen with respect to d. For this for arbitrary $x \in M$ let $\{e_1,\dots,e_4\}$ be a basis of T_xM with

$$g(e_i,e_j) = \begin{cases} -1 \text{ if } i=j \in \{1, 2\} \\ 1 \text{ if } i=j \in \{3, 4\} \\ 0 \text{ otherwise} \end{cases}.$$

By $\mathcal{W}$ we denote the matrix $(\omega(e_i,e_j))$. It is easy to verify the following

1. If $\mathcal{W} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$, then $d = 0$.

2. If $\mathcal{W} = \begin{pmatrix} 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, then $d = 1$.

3. If $\mathcal{W} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$, then $d = 2$.

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