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PROLONGATIONS OF CONJUGATE CONNECTIONS

0. Introduction

Let M be a manifold. We denote by $T^{p,r}M = J^r_0(R^p,M)$ the bundle of r-jets at 0 of mappings $R^p \longrightarrow M$. In Section 1 we introduce notations and we recall fundamental properties of this functor. If G is a Lie group, then $T^{p,r}G$ has a natural Lie group structure and an action of G on a manifold M can be prolongated to an action of $T^{p,r}G$ on $T^{p,r}M$. If $P(M,G)$ is a principal fibre bundle, then $T^{p,r}P(T^{p,r}M,T^{p,r}G)$ is also a principal fibre bundle. If ω is a connection in $P(M,G)$, then A. Morimoto [7], [8] defined its prolongation to $T^{p,r}P(T^{p,r}M,T^{p,r}G)$.

In Section 2 we define a natural Lie algebra isomorphism

$$\Omega_G : T^{p,r}(\mathcal{L}(G)) \longrightarrow \mathcal{L}(T^{p,r}G),$$

where $\mathcal{L}(G)$ denotes the Lie algebra of a Lie group G.

In Section 3 we modify the definition of prolongation of connections from $P(M,G)$ to $T^{p,r}P(T^{p,r}M,T^{p,r}G)$ using the natural isomorphism Ω_G. Next we prove that if ω_1 and ω_2 are φ-conjugate connections in a principal fibre bundle $P(M,G)$, where φ is a given endomorphism of the structural group G, then the prolongations of ω_1 and ω_2 are $T^{p,r}\varphi$-conjugate in $T^{p,r}P(T^{p,r}M,T^{p,r}G)$, where $T^{p,r}\varphi$ is the induced endomorphism of $T^{p,r}G$.

In this paper all manifolds are always differentiable manifolds of class C^∞ and all objects on manifolds (as functions, vector dields etc.) are also of class C^∞.

1. The functor $T^{p,r}$

Let M be a manifold. We denote by $T^{p,r}M = J^r_0(R^p,M)$ the manifold of r-jets at 0 of mappings $R^p \longrightarrow M$. If $\varphi : M \longrightarrow N$ is a differentiable mapping, then we define the induced mapping $T^{p,r}\varphi : T^{p,r}M \longrightarrow T^{p,r}N$ by the formula

$$(T^{p,r}\varphi)(j^r_0\gamma) = j^r_0(\varphi\circ\gamma) \tag{1.1}$$

$T^{p,r}M$ is a fibre bundle with the projection $\pi: T^{p,r}M \to M$ given by $\pi(j_0^r\gamma) = \gamma(0)$.

If G is a Lie group, then we define a Lie group structure on $T^{p,r}G$ by

$$j_0^r\xi \cdot j_0^r\eta = j_0^r(\xi\eta), \tag{1.2}$$

where for mappings $\xi,\eta : R^p \to G$, $\xi\eta : R^p \to G$ is given by $(\xi\eta)(u) = \xi(u)\eta(u)$ for $u \in R^p$.

If $P(M,G)$ is a principal fibre bundle, then $T^{p,r}P(T^{p,r}M,T^{p,r}G)$ is also a principal fibre bundle, where the action of $T^{p,r}G$ on $T^{p,r}P$ is defined by the formula:

$$j_0^r\gamma \cdot j_0^r\xi = j_0^r(\gamma\xi)$$

for $j_0^r\gamma \in T^{p,r}P$ and $j_0^r\xi \in T^{p,r}G$ (see A. Morimoto [7], [8]).

Let $P_0(M,H)$ be a reduced fibre bundle in a principal fibre bundle $P(M,G)$. The inclusions $i: P_0 \to P$ and $i: H \to G$ induce the inclusions $T^{p,r}i$ or $T^{p,r}P_0$ and $T^{p,r}H$ into $T^{p,r}P$ and $T^{p,r}G$ respectively. The action of $T^{p,r}H$ on $T^{p,r}P_0$ is the restriction of the action of $T^{p,r}$ on $T^{p,r}P$ to the submanifold $T^{p,r}P_0$. Thus we have

Proposition 1.1. If $P_0(M,N)$ is a reduced fibre bundle in a principal fibre bundle $P(M,G)$, then $T^{p,r}P_0(T^{p,r}M,T^{p,r}H)$ is a reduced fibre bundle in $T^{p,r}P(T^{p,r}M,T^{p,r}G)$.

Let V be a vector space. Then $T^{p,r}V$ is also a vector space with the linear combination of elements defined by

$$a\, j_0^r k + a' j_0^r k' = j_0^r(ak + a'k').$$

If A is a Lie algebra, then we define

$$[j_0^r k, j_0^r k'] = j_0^r[k,k'], \tag{1.3}$$

where for mappings $k,k' = R^p \to A$, $[k,k'] : R^p \to A$ is given by $[k,k'](u) = [k(u),k'(u)]$.

If M is a manifold, then for every p,q,r,s there exist a natural diffeomorphism

$$\alpha_M^{qs,pr} : T^{p,r}(T^{q,s}M) \to T^{q,s}(T^{p,r}M)$$

(see A. Morimoto [7],[8]). To define this diffeomorphism we need the following lemma:

Lemma 1.2. (Lemma 1.1 in [8]). If $\varphi : R^p \longrightarrow T^{q,s}M$ is a differentiable mapping, then there exist a mapping $\Psi : R^p \times R^q \longrightarrow M$ and a positive number ε such that for $u \in R^p$, $\|u\| < \varepsilon$ we have

$$\varphi(u) = j_o^s \Psi_u \tag{1.5}$$

where $\Psi_u : R^q \longrightarrow M$ is given by $\Psi_u(v) = \Psi(u,v)$.

Now we define the mapping $\alpha_M^{pr,qs}$ as follows. If $j_o^r \varphi$ is an element of $T^{p,r}(T^{q,s}M)$, then Lemma 1.2 implies that there is a mapping $\Psi : R^p \times R^q \longrightarrow M$ such that the quality (1.5) holds. We set

$$\alpha_M^{pr,qs}(j_o^r) = j_o^s \bar{\varphi} \tag{1.6}$$

where $\bar{\varphi} : R^q \longrightarrow T^{p,r}M$ is given by

$$\bar{\varphi}(v) = j_o^r \, {}^v\Psi \tag{1.7}$$

and for $u \in R^p$, $v \in R^q$

$$\Psi^v(u) = \Psi_u(v) = \Psi(u,v) \tag{1.8}$$

The mapping $\alpha_M^{pr,qs}$ is a diffeomorphism such that for any mapping $\varphi : M \longrightarrow N$ the following diagram

$$\begin{array}{ccc} T^{p,r}(T^{q,s}M) & \xrightarrow{T^{p,r}(T^{q,s}\varphi)} & T^{p,r}(T^{q,s}N) \\ \Big\downarrow \alpha_M^{pr,qs} & & \Big\downarrow \alpha_N^{pr,qs} \\ T^{q,s}(T^{p,r}M) & \xrightarrow{T^{q,s}(T^{p,r}\varphi)} & T^{q,s}(T^{p,r}N) \end{array} \tag{1.9}$$

commutes (see Proposition 1.6 in [8]).

For the inverse mapping we have the following formula (see Corollary 1.4 in [8]):

$$(\alpha_M^{pr,qs})^{-1} = \alpha_M^{qs,pr}. \tag{1.10}$$

Let N be a submanifold of M and $i : N \longrightarrow M$ be the natural inclusion. Now $T^{p,r}i : T^{p,r}N \longrightarrow T^{p,r}M$ is the natural inclusion of $T^{p,r}N$ into $T^{p,r}M$ and the commutativity of the diagram (2.9) implies

$$\alpha_N^{pr,qs} = \alpha_M^{pr,qs}|T^{p,r}N. \tag{1.11}$$

2. A natural isomorphism between $T^{p,r}(\mathcal{L}(G))$ and $\mathcal{L}(T^{p,r}G)$

Let G be a Lie group. We will construct a natural isomorphism between two Lie algebras $T^{p,r}(\mathcal{L}(G))$ and $\mathcal{L}(T^{p,r}G)$. This isomorphism is also used in [4].

Let $X = j_0^r k$ be an element of $T^{p,r}(\mathcal{L}(G))$, where $k: R^p \longrightarrow \mathcal{L}(G)$ is a mapping. It means that for every $u \in R^p$, $k(u)$ is a left invariant vector field on G. We consider the mapping

$$\bar{k}: R^p \times R \ni (u,t) \longrightarrow \exp_G t\, k(u) \in G \tag{2.1}$$

and we define $k^u: R \longrightarrow G$ and $k_t: R^p \longrightarrow G$ by the following formulae

$$\bar{k}^u(t) = \bar{k}_t(u) = \bar{k}(u,t). \tag{2.2}$$

From (2.1) we obtain

$$\left(\frac{d}{dt}\bar{k}^u\right)(0) = k_e(u), \tag{2.3}$$

where e is the neutral element of G. Since $k(0,u) = e$, thus $j_0^r k_0 = \bar{e}$ is the neutral element of the group $T^{p,r}G$. Let $\Omega_G(X)$ be the left invariant vector field on $T^{p,r}G$ such that

$$(\Omega_G(X))_{\bar{e}} = \frac{d}{dt}(j_0^r \bar{k}_t)\Big|_{t=0} \in T_{\bar{e}}(T^{p,r}G), \tag{2.4}$$

where $\bar{k}$ and $\bar{k}_t$ are given by (2.1) and (2.2).

Proposition 2.1. The mapping

$$\Omega_G: T^{p,r}(\mathcal{L}(G)) \longrightarrow \mathcal{L}(T^{p,r}G)$$

is linear.

Proof. Ω_G is a mapping of class C^∞ between two finite dimensional vector spaces. If X is an element of $T^{p,r}(\mathcal{L}(G))$ and a is a real number, then we denote by $\bar{k}$ and $\bar{\bar{k}}$ the mappings defined by (2.1) for X and aX respectively. Since $aX = j_0^r(ak)$, thus

$$\bar{\bar{k}}(u,t) = \exp_G(taX) = \bar{k}(at,u).$$

It implies that $\bar{\bar{k}}_t = \bar{k}_{at}$ and now we have

$$(\Omega_G(aX))_{\bar{e}} = \frac{d}{dt}(j_o^r \bar{\bar{k}}_t)\Big|_{t=0} = a\frac{d}{dt}(j_o^r \bar{k}_t)\Big|_{t=0} = a(\Omega_G(X))_{\bar{e}}$$

that is, $\Omega_G(aX) = a\Omega_G(X)$. Since Ω_G is of class C^∞, thus Ω_G is linear.

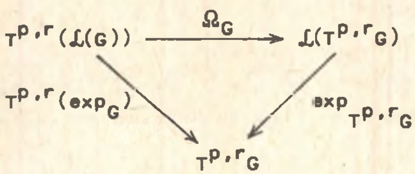
<u>Proposition 2.2</u>. Let $X = j_o^r k$ be an element of $T^{p,r}(\mathcal{L}(G))$. If $\bar{k}$ and $\bar{k}_t$ are given by formulas (2.1) and (2.2), then $A_t = j_o^r \bar{k}_t$ is the 1-parameter subgroup of $\Omega_G(X)$.

<u>Proof</u>. For a fixed element $u \in R^p$, $k_t(u)$ is the 1-parameter subgroup of $k(u) \in \mathcal{L}(G)$. It implies that $k_t(u)k_s(u) = k_{t+s}(u)$ and $k_o(u) = e$. From this we obtain

$$A_t A_s = A_{t+s}, \quad A_o = \bar{e}$$

that is, A_t is an 1-parameter subgroup of $T^{p,r}G$. According to (2.4), A_t induces $\Omega_G(X)$.

<u>Proposition 2.3</u>. The following diagram



commutes.

<u>Proof</u>. Let $X = j_o^r k \in T^{p,r}(\mathcal{L}(G))$ and $\bar{k}$, $\bar{k}_t$, A_t be such as in Proposition 2.2. Now

$$(\exp_{T^{p,r}G} \circ \Omega_G)(X) = A_1 = j_o^r(\exp_G \circ k) = T^{p,r}(\exp_G)(X).$$

<u>Proposition 2.4</u>. Ω_G is a linear isomorphism.

<u>Proof</u>. Let $X = j_o^r k$ be an element of $T^{p,r}(\mathcal{L}(G))$ such that $\Omega_G(X) = 0$ and let $\bar{k}$, $\bar{k}_t$, A_t be such as in Proposition 2.2. The condition $\Omega_G(X)=0$ implies that $A_t = \bar{e}$ for all t. According to Proposition 2.3 we obtain

$$(T^{p,r}\exp_G)(tX) = (\exp_{T^{p,r}G} \circ \Omega_G)(tX) = A_t = \bar{e}.$$

Since $T^{p,r}\exp_G$ is a diffeomorphism of some neighborhood V of zero onto a neighborhood of $\bar{e}$ and tX belongs to V for a sufficient small t, thus $tX = 0$, and hence, $X = 0$.

In this way we proved that Ω_G is injective. Ω_G is now a linear isomorphism beacause

$$\dim T^{p,r}(\mathcal{L}(G)) = \binom{p+r}{r}\dim\mathcal{L}(G) = \binom{p+r}{r}\dim G$$

$$\dim\mathcal{L}(T^{p,r}G) = \dim T^{p,r}G = \binom{p+r}{r}\dim G.$$

Proposition 2.5. If $j_o^r\xi$ is an element of $T^{p,r}G$, then the mapping $\overline{\mathrm{Ad}}_{j_o\xi} = \Omega_G^{-1} \circ \mathrm{Ad}_{j_o^r\xi} \circ \Omega_G$ is given by the formula

$$\overline{\mathrm{Ad}}_{j_o^r\xi}(X) = X'$$

where $X = j_o^r k$, $X' = j_o^r k'$ and $k'(u) = \mathrm{Ad}_{\xi(u)}(k(u))$ and Ad denotes the adjoint representation of a Lie group on its Lie algebra.

Proof. Let $\bar{k}$, $\bar{k}_t$ be defined by (2.1) and (2.2) for an element $X = j_o^r k$ of $T^{p,r}(\mathcal{L}(G))$. Now

$$\overline{\mathrm{Ad}}_{j_o^r\xi}(X) = \Omega_G^{-1} \frac{d}{dt} (j_o^r(\xi\bar{k}_t\xi^{-1})\Big|_{t=0}). \tag{2.6}$$

Let $\bar{k}^1(u,t) = (u)\bar{k}_t(u)\xi^{-1}(u)$ and $\bar{k}_t^1(u) = k^1(u,t)$. For a fixed element $u \in R^p$, $\bar{k}_t^1(u)$ is an 1-parameter subgroup of G which induces on G a left invariant vector field

$$k'(u) = \mathrm{Ad}_{\xi(u)}(k(u))$$

Now, from (2.6) and (2.4) we obtain (2.5)

Proposition 2.6. Ω_G is a Lie algebra isomorphism.

Proof. Let $X = j_o^r k$ and $Y = j_o^r l$ be two elements of $T^{p,r}(\mathcal{L}(G))$. According to Proposition 2.4 we need to show only

$$\Omega_G[X,Y] = [\Omega_G(X),\Omega_G(Y)].$$

Let $\bar{k},\bar{k}_t$ be mappings defined by (2.1) and (2.2) for $X = j_o^r k$. According to Proposition 2.2, $A_t = j_o^r\bar{k}_t$ is the 1-parameter subgroup of $\Omega_G(X)$ and now

$$[\Omega_G(X),\Omega_G(Y)] = \frac{d}{dt} (\mathrm{Ad}_{A_t}(\Omega_G(Y))\Big|_{t=0} = \Omega_G(\frac{d}{dt}(\Omega_G^{-1}\circ \mathrm{Ad}_{A_t} \circ \Omega_G)(Y)\Big|_{t=0})$$

(in the last equality we used the linerarity of Ω_G). Next using Proposition 2.5 we have

$$\left[\Omega_G(X),\Omega_G(Y)\right] = \Omega_G(\frac{d}{dt}\, j_o^r\, \bar{l}_t)\Big|_{t=0}$$

where

$$\bar{l}_t(u) = Ad_{\bar{k}_t(u)}(l(u))$$

Since $\bar{k}_t(u)$ is the 1-parameter subgroup of $k(u)$, thus

$$\frac{d}{dt}\, \bar{l}_t(u)\Big|_{t=0} = \left[k(u), l(u)\right]$$

and hence

$$\left[\Omega_G(X),\Omega_G(Y)\right] = \Omega_G(j_o^r[k,l]) = \Omega_G[X,Y],$$

that is, the proof is finished.

<u>Proposition 2.7</u>. If $f: G \longrightarrow G'$ is a Lie group homomorphism, then the diagram

$$\begin{array}{ccc}
T^{p,r}(\mathcal{L}(G)) & \xrightarrow{T^{p,r}(\mathcal{L}(f))} & T^{p,r}(\mathcal{L}(G')) \\
\Omega_G \downarrow & & \downarrow \Omega_{G'} \\
\mathcal{L}(T^{p,r}G) & \xrightarrow{\mathcal{L}(T^{p,r}f)} & \mathcal{L}(T^{p,r}G')
\end{array}$$

commutes.

<u>Proof</u>. Let $X = j_o^r k$ be an element of $T^{p,r}(\mathcal{L}(G))$ and $\bar{k}$, $\bar{k}_t$ be defined by (2.1) and (2.2). We define

$$\bar{k}'(u,t) = f(\bar{k}(u,t)), \quad \bar{k}'_t(u) = f(\bar{k}_t(u)).$$

Now $\bar{k}'_t(u)$ is the 1-parameter group of $\mathcal{L}(f)k(u)$. According to (2.4) it implies that

$$((\Omega_{G'} \circ T^{p,r}(\mathcal{L}f))(X))_{\bar{e}'} = \frac{d}{dt}\,(j_o^r \bar{k}'_t)\Big|_{t=0} = \frac{d}{dt}\,(j_o^r(f\circ \bar{k}_t)\Big|_{t=0} =$$

$$= (\mathcal{L}(T^{p,r}f)(\frac{d}{dt}\, j_o^r k_t)\Big|_{t=0}))_{\bar{e}'} = ((\mathcal{L}(T^{p,r}f)\circ \Omega_G)(X))_{\bar{e}'},$$

where $\bar{e}'$ is the neutral element of $T^{p,r}G'$.

Propositions 2.1 - 2.7 prove the following main theorem of this section

Theorem 2.8. Ω_G is a natural Lie algebra isomorphism. This theorem implies:

Proposition 2.9. If H is a Lie subgroup of a Lie group G, then

$$\Omega_H = \Omega_G | T^{p,r}(\mathcal{L}(H).$$

Proof. The inclusion $i: H \to G$ induces the inclusions $T^{p,r}(\mathcal{L}(i))$ and $\mathcal{L}(T^{p,r}i)$ of $T^{p,r}(\mathcal{L}(H))$ and $\mathcal{L}(T^{p,r}H)$ into $T^{p,r}(\mathcal{L}(G))$ and $\mathcal{L}(T^{p,r}G)$ respectively. Now, the commutativity of the diagram in Proposition 2.7 implies this proposition.

To finish this section we prove

Proposition 2.10. Let $X = j^r_o k$ nr an element of $T^{p,r}(\mathcal{L}(G))$ and $j^r_o \xi$ be an element of $T^{p,r}G$. If $k_\xi : R^p \to TG$ is given by

$$k_\xi(u) = dL_{\xi(u)}(k_e(u))$$

then $\hat{X} : T^{p,r}G \to T^{p,r}(TG)$ defined by

$$\hat{X}_{j^r_o\xi} = j^r_o(k_\xi)$$

is a section of $T^{p,r}(TG) \xrightarrow{T^{p,r}(\pi)} T^{p,r}G$, where $\pi: TG \to G$ is the tangent bundle projection, and we have

$$\Omega_G(X) = \alpha_G^{11,pr} \circ \hat{X}.$$

Proof. Since

$$(T^{p,r}\pi)(\hat{X}_{j^r_o\xi}) = j^r_o(\pi \circ k_\xi) = j^r_o\xi,$$

thus $\hat{X}$ is a section and $\alpha_G^{11,pr} \circ \hat{X}$ is a vector field on $T^{p,r}G$. For the neutral element $\bar{e}$ of $T^{p,r}G$ we have $\hat{X}_{\bar{e}} = X$ because $k_{\bar{e}} = k$. According to (2.1) we can used $\bar{k}$ given by (2.1) to calculate $\alpha_G^{11,pr}$. We obtain

$$(\Omega_G(X))_{\bar{e}} = (\alpha_G^{11,pr} \circ \hat{X})_{\bar{e}}$$

Hence, to verify (2.7) we need only to show that $\alpha_G^{11,pr} \circ \hat{X}$ is a left invariant vector field, that is, we need to show that for every $j^r_o\xi$, $j^r_o\eta \in T^{p,r}G$ we have

$$(dL_{j_o^r\xi} \circ \alpha_G^{11,pr})(\hat{X}_{j_o^r\eta}) = \alpha_G^{11,pr}(\hat{X}_{j_o^r(\xi\eta)}) \tag{2.8}$$

where $dL_{j_o^r\xi}$ is the tangent mapping.

Let $\bar{k}$, $\bar{k}_t$ be defined by (2.1) and (2.2) for $X = j_o^r k$. We denote

$$\bar{k}_\xi(u,t) = \xi(u)(\bar{k}(u,t))$$

$$(\bar{k}_\xi)_u(t) = (\bar{k}_\xi)^t(u) = \bar{k}_\xi(u,t).$$

Now

$$(\frac{d}{dt}(\bar{k}_\xi)_u)\Big|_{t=0} = dL_{\xi(u)}(k_e(u)) = k_\xi(u).$$

Thus from (1.6) - (1.7) we obtain

$$(dL_{j_o^r\xi} \circ \alpha_G^{11,pr})(\hat{X}_{j_o^r\eta}) = dL_{j_o^r\xi}(\frac{d}{dt}(j_o^r(\bar{k}_\eta)^t)\Big|_{t=0})$$

$$= \frac{d}{dt}(j_o^r(\xi k_\eta^t)\Big|_{t=0}) = \alpha_G^{11,pr}(j_o^r k_{\xi\eta}) = \hat{X}_{j_o^r(\xi\eta)}$$

because $\xi(u)\bar{k}_\eta^t(u) = \xi(u)\eta(u)\bar{k}(u,t) = k_{\xi\eta}^t(u)$. Thus the equality (2.8) holds and the proof is finished.

3. Prolongations of conjugate connections

Let $P(M,G)$ be a principal fibre bundle and ω be a connection form on $P(M,G)$. We consider ω as a mapping

$$\omega : TP \longrightarrow \mathcal{L}(G) = T_eG \subset TG$$

of TP into TG, and we define

$$\omega^{pr} = \alpha_P^{11,pr} \circ T^{p,r}\omega \circ \alpha_P^{pr,11} : T(T^{p,r}P) \longrightarrow T(T^{p,r}G) \tag{3.1}$$

ω^{pr} is a mapping such that

$$\omega^{pr}(T(T^{p,r}P)) \subset T_{\bar{e}}(T^{p,r}G),$$

where $\bar{e}$ ist the neutral element of $T^{p,r}G$, and for every point p of $T^{p,r}P$, ω^{pr} transforms zero of $T_p(T^{p,r}P)$ into zero of $T_{\bar{e}}(T^{p,r}G)$. This allows us to calculate the linear part of ω^{pr}. We denote by

$$\omega_{pr} = (\omega^{pr})^{(0)} \tag{3.2}$$

this linear part of ω^{pr}. ω_{pr} determines a connection form in $T^{p,r}P$ called prolongation of ω (for detals, see A. Morimoto [7], [8]).

Using the natural isomorphism Ω_G we can define the prolongation ω_{pr} in the following way. We consider ω as a mapping

$$\omega : TP \longrightarrow \mathcal{L}(G).$$

Now from (3.1), (3.2), Proposition 2.10 we obtain immediately

$$\omega_{pr} = \Omega_G \circ (T^{p,r}\omega \circ \alpha_P^{pr,11})^{(0)}$$

(We need to verify that $T^{p,r}\omega \circ \alpha_P^{pr,11}$ transforms zero of $T_p(T^{p,r}P)$ into zero of $T^{p,r}(\mathcal{L}(G))$.).

We recall now the definition of conjugate connections ([3], [9]):

Definition 3.1. Let $\omega, \bar{\omega} : TP \longrightarrow \mathcal{L}(G)$ be two connection forms in a principal fibre bundle $P(M,G)$ and $\varphi : G \longrightarrow G$ be an endomorphism of the structural group G. We denote

$$H_\varphi = \left\{ \xi \in G : \varphi(\xi) = \xi \right\}. \tag{3.4}$$

ω is called φ-conjugate with $\bar{\omega}$ if there exists a reduced fibre bundle $P_0(M,H_\varphi)$ in $P(M,G)$ such that for every local section $\sigma : U \longrightarrow P_0$ of $P_0(M,H_\varphi)$ we have

$$\sigma^*\bar{\omega} = \mathcal{L}\varphi \circ \sigma^*\omega,$$

where $\sigma^*\omega$ denotes the pull-back of ω by σ.

It is easy to show the following proposition (see [3]):

Proposition 3.2. ω is φ-conjugate with $\bar{\omega}$ if there exist a reduced fibre bundle $P_0(M,H_\varphi)$ and a family of local sections $\sigma_a : U_a \longrightarrow P$, $a \in A$, of $P(M,G)$ such that $\{U_a\}_{a\in A}$ is an open covering of M and for every $a \in A$ we have $\sigma_a(U_a) \subset P_0$ and

$$\sigma_a^*\bar{\omega} = \mathcal{L}\varphi \circ \sigma_a^*\omega. \tag{3.5}$$

We have the following theorem:

<u>Theorem 3.3</u>. Let ω, $\bar{\omega}$ be connections in a principal fibre bundle $P(M,G)$ and $\varphi: G \to G$ be an endomorphism of G. If ω is φ-conjugate with $\bar{\omega}$, then ω_{pr} is $T^{p,r}\varphi$-conjugate with $\bar{\omega}_{pr}$.

<u>Proof</u>. If a connection ω is considered as a mapping $\omega: TP \to \mathcal{L}(G)$, then the pull-back $\sigma^*\omega$ is defined by

$$\sigma^*\omega = \omega \circ d\sigma. \tag{3.6}$$

Since ω is φ-conjugate with $\bar{\omega}$, thus there exist a reduced fibre bundle $P_o(M,H_\varphi)$, where H_φ is given by (3.4), and a family, of local sections $\sigma_a: U_a \to P$ of $P(M,G)$ such that $\{U_a\}$ is an open covering of M and for every $a \in A$

$$\sigma_a(U_a) \subset P_o \tag{3.7}$$

$$\sigma_a^*\bar{\omega} = \mathcal{L}\varphi \circ \sigma_a^*\omega. \tag{3.8}$$

We define

$$\bar{U}_a = T^{p,r}U_a = T^{p,r}M|U_a \tag{3.9}$$

$$\bar{\sigma}_a = T^{p,r}\sigma_a : \bar{U}_a \to T^{p,r}P. \tag{3.10}$$

The family $\{\bar{U}_a\}_{a\in A}$ is an open covering of $T^{p,r}M$. According to Proposition 1.1, $T^{p,r}P_o(T^{p,r}M, T^{p,r}H_\varphi)$ is a reduced fibre bundle in $T^{p,r}P(T^{p,r}M, T^{p,r}G)$. If $j_o^r\xi \in T^{p,r}H$, then $\xi: R^p \to H_\varphi$, that is $\varphi \circ \xi = \xi$. It implies that

$$(T^{p,r}\varphi)(j_o^r\xi) = j_o^r(\varphi \circ \xi) = j_o^r\xi.$$

Thus, we have

$$T^{p,r}H_\varphi \subset \bar{H} = H_{T^{p,r}\varphi} = \{j_o^r\xi : (T^{p,r}\varphi)(j_o^r\xi) = j_o^r\xi\}$$

Now, using standard methods ([5], [6]), we obtain that there exists a reduced fibre bundle $\bar{P}_o(T^{p,r}M, \bar{H})$ in $T^{p,r}P(T^{p,r}M, T^{p,r}G)$ such that

$$T^{p,r}P_o \subset \bar{P}_o. \tag{3.11}$$

From (3.7), (3.10) and (3.11) we obtain $\bar{\sigma}_a(\bar{U}_a) \subset \bar{P}_o$ for every $a \in A$.

From (3.6) and (3.8) we have

$$\bar{\omega} \circ d\bar{\sigma}_a = \mathcal{L}\varphi \circ \omega \circ d\sigma_a$$

and it implies

$$T^{p,r}\bar{\omega} \circ T^{p,r}(d\bar{\sigma}_a) = T^{p,r}(\mathcal{L}\varphi) \circ T^{p,r}\omega \circ T^{p,r}(d\sigma_a). \qquad (3.12)$$

According to the general notations for tangent mapping we have $d\sigma_a = T^{1,1}\sigma_a$ and now from the commutativity of the diagram (1.9) and the formula (2.10) we obtain

$$T^{p,r}(d\sigma_a) = (\alpha_P^{11,pr})^{-1} \circ d\bar{\sigma}_a \alpha_{U_a}^{11,pr} = \alpha_P^{pr,11} \circ d\bar{\sigma}_a \circ \alpha_{U_a}^{11,pr} \qquad (3.13)$$

Now, from (3.12), (3.13) and Proposition 2,7 we have

$$\Omega_G \circ T^{p,r}\bar{\omega} \circ \alpha_P^{pr,11} \circ d\bar{\sigma}_a \circ \alpha_{U_a}^{11,pr} = \Omega_G \circ T^{p,r}\bar{\omega} \circ T^{p,r}(d\bar{\sigma}_a) =$$

$$= \Omega_G \circ T^{p,r}(\mathcal{L}\varphi) \circ T^{p,r}\omega \circ \alpha_P^{pr,11} d\bar{\sigma}_a \circ \alpha_{U_a}^{11,pr}$$

$$= \mathcal{L}(T^{p,r}\varphi) \circ \Omega_G \circ T^{p,r}\omega \circ \alpha_P^{pr,11} \circ d\bar{\sigma}_a \circ \alpha_{U_a}^{11,pr}$$

Since $\alpha_{U_a}^{11,pr}$ is a diffeomorphism, thus

$$\Omega_G \circ T^{p,r}\bar{\omega} \circ \alpha_P^{11,pr} \circ d\bar{\sigma}_a = \mathcal{L}(T^{p,r}\varphi) \circ \Omega_G \circ T^{p,r}\omega \circ \alpha_P^{11,pr} \circ d\bar{\sigma}_a.$$

Since Ω_G and $\mathcal{L}(T^{p,r}\varphi)$ are linear and restrictions of $d\bar{\sigma}_a$ to every fibre are linear, thus the last equality implies

$$\Omega_G \circ (T^{p,r}\bar{\omega} \circ \alpha_P^{11,pr})^{(0)} \circ d\bar{\sigma}_a =$$

$$= \mathcal{L}(T^{p,r}\varphi) \circ \Omega_G \circ (T^{p,r}\omega \circ \alpha_P^{11,pr})^{(0)} \circ d\sigma_a$$

and using (3.3) amd (3.6) we obtain

$$\bar{\sigma}_a^*\bar{\omega}_{pr} = \mathcal{L}(T^{p,r}\varphi) \circ \sigma_a^*\omega_{pr}.$$

According to Proposition 3.2, the last egality means that $\bar{\omega}_{pr}$ is $T^{p,r}\varphi$ -conjugate with ω_{pr}. The proof is finished.

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