ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1990

Seria: MATEMATYKA-FIZYKA z. 64 Nr kol. 1070

DEDICATED TO PROFESSOR MIECZYSŁAW KUCHARZEWSKI
WITH BEST WISHES ON HIS 70TH BIRTHDAY

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STABILITY OF ADDITION FORMULAE FOR TRIGONOMETRIC MAPPINGS

§ 0. Introduction

It is known that the functional equations of d'Alembert

$$f(x+y) + f(x-y) = 2f(x)f(y) \tag{1}$$

and Wilson

$$g(x+y)\cdot g(x-y) = g^2(x) - g^2(y), \tag{2}$$

satisfied by $f = \cos$ and $g = \sin$, respectively, are both stable in the sense of Baker ("superstable"). Namely, the following theorems hold true:

Theorem A. (J.A. Baker [1]) Let $\varepsilon \geqslant 0$ be a given number and let $(G,+)$ be an Abelian group. Then any unbounded solution $f\colon G \to \mathbb{C}$ of the inequality

$$|f(x+y) + f(x,y) - 2f(x)f(y)| \leqslant \varepsilon, \quad x,y \in G,$$

satisfies d'Alembert's equation (1).

Theorem B. (P.W. Cholewa [3]) Let $\varepsilon \geqslant 0$ be a given number and let $(G,+)$ be a uniquely two-divisible Abelian group. Then any unbounded solution $g\colon G \to \mathbb{C}$ of the inequality

$$|g(x+y)\, g(x-y) - g^2(x) + g^2(y)| \leqslant \varepsilon, \quad x,y \in G,$$

satisfies Wilson's equation (2).

Since the pair $f = \cos$, $g = \sin$ yields also a solution of the following system

$$\begin{cases} f(x+y) = f(x)f(y) - g(x)g(y) \\ g(x+y) = f(x)g(y) + g(x)f(y) \end{cases} \tag{3}$$

a natural question arises whether the superstability phenomenon described above carries over system (3). In spite of the fact that nowadays stability problems are pretty vivid in the field of functional equations, it seems that considerations of that type with regard to systems of equations have not been undertaken till now.

§ 1. Exponential mappings and their stability

In the light of the classical Euler formulae linking complex trigonometric functions and the exponential one, the stability behaviour of exponential mappings will play the crucial role in the sequel. Therefore, we begin with a generalization of J. Lawrence s result from [4] (Theorem 1).

Theorem 1. Let $(S,+)$ be a semigroup and let $(A, \|\cdot\|)$ be a normed algebra. Assume that the Cauchy difference

$$C_F(x,y) := F(x+y) - F(x)\,F(y), \quad x,y \in S,$$

of a map $F: S \longrightarrow A$ has the property that all its sections $C_F(x,\cdot)$, $x \in S$, are bounded (not necessarily uniformly). Then, for every a belonging to the right ideal generated by $C_F(S \times S)$ in the subalgebra $\langle F(S) \rangle$ of A, generated by $F(S)$, the function $a \cdot F$ is bounded.

Proof. We shall suitably modify the induction procedure applied by J. Lawrence in the proof of his Theorem 1 from [4]. Observe that any element a of the right ideal spoken of in the statement of our theorem is a linear combination of elements

$$b = C_F(x,y)F(x_1)\cdot \ldots \cdot F(x_k) \tag{4}$$

where $x_1,\ldots,x_k,x,y \in S$. Obviously, it suffices to show that $b \cdot F$ is bounded. To this aim, we shall first prove that

$$C_F(x,y)\cdot F \text{ is bounded for all } (x,y) \in S^2. \tag{5}$$

Indeed, fix arbitrarily a pair $(x,y) \in S^2$ and put

$$\varphi(s) := \sup_{t \in S} \|C_F(s,t)\|, \quad s \in S.$$

By assumption, φ is finite on the whole of S. Now, for any $z \in S$, one has

$$\begin{aligned}\|C_F(x,y)F(z)\| &\leq \|F(x+y)F(z) - F(x)F(y+z)\| + \|F(x)F(y+z) - F(x)F(y)\\ &\cdot F(z)\| \leq \|F(x+y)F(z) - F(x+y+z)\| + \|F(x+y+z) - F(x)F(y+z)\|\\ &+ \|F(x)\|\,\|F(y+z) - F(y)F(z)\| = \|C_F(x+y+z)\| + \|C_F(x,y+z)\|\\ &+ \|F(x)\|\,\|C_F(y,z)\| \leq \varphi(x+y) + \varphi(x) + \|F(x)\|\,\varphi(y)\end{aligned}$$

with the latter constant being finite and independent of z. Thus, (5) has been proved.

To show that $b \cdot F$ is bounded with b having form (4), assume inductively that $C_F(x,y)F(x_1) \dots F(x_{k-1}) \cdot F$ is bounded (say, by $c(x,y,x_1,\dots,x_{k-1})$) for any choice of elements $x,y,x_1,\dots x_{k-1} \in S$ and fix arbitrarily an $x_k \in S$. Then, for any $z \in S$, one has $\|C_F(x,y)F(x_1) \dots F(x_{k-1})F(x_k)F(z)\| \leq \|C_F(x,y)F(x_1) \dots F(x_{-1}) \cdot (F(x_k)F(z) - F(x_k+z))\| + \|C_F(x,y)F(x_1) \dots$ $\dots F(x_{k-1})F(x_k+z)\| \leq \|C_F(x,y)F(x_1) \dots F(x_{k-1})\| \cdot \|C_F(x_k,z)\| + c(x,y,x_1,\dots,x_{k-1}) \leq \|C_F(x,y)F(x_1) \dots F(x_{k-1})\| \varphi(x_k) + c(x,y,x_1,\dots,x_{k-1})$, which proves that $C_F(x,y)F(x_1) \dots F(x_k) \cdot F$ is bounded and completes the proof.

Remark 1. Plainly, if the ideal considered is the left one (resp. two-sided), then the functions $F \cdot a$ (resp. both $a \cdot F$ and $F \cdot a$) are all bounded.

Remark 2. The elements $C_F(x,y)$, $(x,y) \in S^2$, are noninvertible in the algebra $\langle F(S) \rangle$ (with unit) provided that F itself is unbounded.

Remark 3. J. Lawrence [4] has assumed that the Cauchy difference C_F of F is bounded (i.e. that all the sections $C_F(x,\cdot)$, $x \in S$, are uniformly bounded).

As a corollary we obtain a generalization up to the numerical value of the bounding constant) of J.A. Baker's superstability result [1] (see also Baker-Lawrence-Zorzitto [2]).

Theorem 2. Let $(S,+)$ be a semigroup and let $\mathbf{K}$ stand for the field of all real or complex numbers or for the field of quaternions. Assume that the Cauchy difference C_f of a function $f: S \to \mathbf{K}$ has bounded sections $C_f(x,\cdot)$, $x \in S$ (not necessarily unformly). Then either f itself is bounded or

$$f(x+y) = f(x)f(y)$$

for all $x,y \in S$.

Proof. Any element $a \neq 0$ of the algebra $\mathbf{K}$ is invertible; therefore, it remains to apply Remark 2.

Remark 4. It is well-known (see J.A. Baker [1]) that the superstability of exponential functions fails to hold even in the case of mappings having values in the algebra $M_2(\mathbf{C})$ of all complex (2x2)-matrices. Baker's counter-example reads as follows: take any positive $\delta \neq 1$ and $F: \mathbf{R} \to M_2(\mathbf{C})$ given by the formula

$$F(x) := \begin{bmatrix} e^x, & 0 \\ 0, & \delta \end{bmatrix}, \quad x \in \mathbf{R}.$$

Then $\|F(x+y) - F(x)F(y)\| = \mathrm{const} > 0$. Therefore, the Cauchy difference C_F is bounded but F is neither bounded nor exponential.

§ 2. Stability of the trigonometric system; the scalar case

Formally, system (3) is not superstable. To see this, take functions $f,g\colon \mathbb{R} \to \mathbb{C}$ given by the formulas:

$$f(x) := \frac{1}{2}\left(e^x + r(x)\right), \quad g(x) := \frac{1}{2i}\left(e^x - r(x)\right), \quad x \in \mathbb{R},$$

where $r\colon \mathbb{R} \to \mathbb{C}$ is any nonexponential function such that $|r(x)| \leq 1$, $x \in \mathbb{R}$. Then both f and g are unbounded and satisfy the system of inequalities

$$\begin{cases} |f(x+y) - f(x)f(y) + g(x)g(y)| \leq 1 \\ |g(x+y) - f(x)g(y) - g(x)f(y)| \leq 1 \end{cases}, \quad (x,y) \in \mathbb{R}^2,$$

but the pair (f,g) does not yield a solution to (3).

Nevertheless, in a sense, we are close to superstability (see also Theorem 7 below); namely, we have the following

Theorem 3. Let $(S,+)$ be a semigroup and let $h,k\colon S \times S \to [0,\infty)$ be such that the sections $h(x,\cdot)$ and $k(x,\cdot)$ are bounded for all $x \in S$. Assume that a pair (f,g) of complex functions on S satisfies estimations

$$\begin{cases} |f(x+y) - f(x)f(y) + g(x)g(y)| \leq h(x,y) \\ |g(x+y) - f(x)g(y) - g(x)f(y)| \leq k(x,y) \end{cases} \tag{6}$$

for all $(x,y) \in S \times S$. Then either

(i) both f and g are bounded,

or

(ii) the pair (f,g) is a solution of (3),

or

(iii) $f = \frac{1}{2}(F + G)$, $g = \frac{1}{2i}(F - G)$,

where $F\colon S \to \mathbb{C}$ is exponential and $G\colon S \to \mathbb{C}$ is bounded, or

(iv) f and g are defined by (iii) but F is bounded and G is exponential.

In particular, there exists a pair (f_0,g_0) of complex functions on S such that (f_0,g_0) yields a solution to (3) and the differences $f - f_0$ and $g - g_0$ are bounded.

Proof. Put $F := f + ig$ and $G := f - ig$ and note that

$$|F(x+y) - F(x)F(y)| = |(f(x+y) - f(x)f(y) + g(x)g(y)) + i\cdot(g(x+y) - f(x)g(y) - g(x)f(y))| \leq h(x,y) + k(x,y) =: H(x,y),$$
$$(x,y) \in S^2.$$

Similarly,

$$|G(x+y) - G(x)G(y)| \leq H(x,y), \quad (x,y) \in S^2.$$

This means that, for any $x \in S$, the sections $C_F(x,\cdot)$ and $C_G(x,\cdot)$ of the Cauchy differences C_F and C_G of F and G, respectively, are bounded (by $H(x,\cdot) = h(x,\cdot) + k(x,\cdot)$). An appeal to Theorem 2 leads now to the following four cases:

(i′) both F and G are bounded,
(ii′) both F and G are exponential,
(iii′) F is exponential and G is bounded,
(iv′) F is bounded and G is exponential.

Since $f = \frac{1}{2}(F + G)$ and $g = \frac{1}{2i}(F - G)$ the statements (i′) - (iv′) imply the corresponding properties (i) - (iv) which were to be proved.

To show the last assertion one has to take $(f_o,g_o) = (0,0)$ in case (i), $(f_o,g_o) = (f,g)$ in case (ii) and

$$(f_o,g_o) = (\tfrac{1}{2}(f + ig), \tfrac{1}{2}(g - if)),$$

$$(f_o,g_o) = (\tfrac{1}{2}(f - ig), \tfrac{1}{2}(g + if))$$

in cases (iii) and (iv), respectively. Actually, only the latter two statements (those concerning cases (iii) and (iv) require a motivation. So, assume (iii) holds true. Then

$$f_o := \tfrac{1}{2}(f + ig) = \tfrac{1}{2}F \quad \text{and} \quad g_o := \tfrac{1}{2}(g - if) = \tfrac{1}{2i} F$$

whence

$$f_o(x+y) - f_o(x)f_o(y) + g_o(x)g_o(y) = \tfrac{1}{2}F(x+y) - \tfrac{1}{4}F(x)F(y) - \tfrac{1}{4}F(x)F(y) = 0$$

since F is exponential; moreover,

$$g_o(x+y) - f_o(x)g_o(y) - g_o(x)f_o(y) = \tfrac{1}{2i}F(x+y) - \tfrac{1}{4i}F(x)F(y) - \tfrac{1}{4i}F(x)F(y) = 0$$

Thus, the pair (f_o,g_o) yields a solution of system (3). On the other hand, the functions

$$f - f_o = f - \tfrac{1}{2}f - \tfrac{1}{2}ig = \tfrac{1}{2}(f - ig) = \tfrac{1}{2}G$$

and

$$g - g_o = g - \tfrac{1}{2}g + \tfrac{1}{2}if = \tfrac{1}{2}i(f - ig) = \tfrac{1}{2}iG$$

are both bounded, for so is G.

The proof in case (iv′) is similar.

Remark 5. Conversely, any pair (f,g) of mappings $f: S \longrightarrow \mathbb{C}$ and $g: S \longrightarrow \mathbb{C}$ fulfilling any of the conditions (i) - iv) yields a solution of system (6) with some constant functions h and k. More precisely, if $|f| \leqslant \alpha$ and $|g| \leqslant \beta$ for some constants α, β, then (6) holds true with $h := \alpha + \alpha^2 + \beta^2$, $k := \beta + 2\alpha\beta$. In case (ii) inequalities (6) are satisfied with $h = k = 0$. Finally, one has (6) with

$$h(x,y) = k(x,y) =: \frac{1}{2}|G(x+y) - G(x)G(y)|, \quad (x,y) \in S^2,$$

and

$$h(x,y) = k(x,y) =: \frac{1}{2}|F(x,y) - F(x)F(y)|, \quad (x,y) \in S^2,$$

in cases (iii) and (iv), respectively.

§ 3. Stability of the trigonometric system; the vector-valued case

We have already seen (cf. Remark 4) that the stability problem for exponential functions is much more sophisticated in the case where the mappings considered are vector-valued. Observe that for the trigonometric system superstability fails to hold even in the scalar case. To say something in the affirmative we shall try to reduce a suitable trigonometric system to exponential type vector-valued mappings by using complexifications.

Let $(A, \|\cdot\|)$ be a normed algebra (real or complex). By a complexification $c(A)$ of the algebra A we mean the product linear space $c(A) := A \times A$ endowed with addition and multiplication by scalars defined coordinatewise and with multiplication

$$(x,y)\cdot(u,v) := (xu - yv,\ xv + yu),\ (x,y),\ (u,v) \in c(A)$$

(see Żelazko [5], for instance). One of possible norms in $c(A)$ is $\|(x,y)\| := \|x\| + \|y\|$, $x,y \in A$. If the algebra A has a unit e, then the pair $(e,0)$ serves as the unit in $c(A)$.

Let us recall that an algebra is called simple provided that it does not admit any nontrivial two-sided ideals. For instance, the algebra $M_n(\mathbb{C})$ of all complex $(n \times n)$-matrices is simple.

Theorem 4. Let $(S,+)$ be a semigroup with a neutral element (monoid) and let $(A, \|\cdot\|)$ be a normed algebra. Assume that functions $h,k: S\times S \longrightarrow [0,\infty)$ are given such that all the sections $h(x,\cdot)$ and $k(x,\cdot)$, $x \in S$, are bounded (not necessarily uniformly). Suppose that a pair (f,g) of mappings from S into A satisfies the estimations

$$\begin{cases} \|f(x+y) - f(x)f(y) + g(x)g(y)\| \leqslant h(x,y) \\ \|g(x+y) - f(x)g(y) - g(x)f(y)\| \leqslant k(x,y) \end{cases} \tag{7}$$

for all $(x,y) \in S \times S$. If the subalgebra $\langle\{(f(x),g(x)): x \in S\}\rangle$ (generated in $c(A)$ by the image of S under the diagonal map $S \ni x \longrightarrow (f(x), g(x)) \in c(A)$) is simple, then either both f and g are bounded or the pair (f,g) yields a solution of system (3); in other words, system (7) is superstable in the class of functions considered.

Proof. Consider the diagonal $F: S \longrightarrow c(A)$ given by the formula $F(x) := (f(x),g(x))$, $x \in S$. Then, by means of the definition of addition and multiplication in $c(A)$, the Cauchy difference C_F of the function F assumes the form

(a) $$C_F(x,y) = F(x+y) - F(x)F(y)$$
$$= (f(x+y) - f(x)f(y)+g(x)g(y),\ g(x+y)-f(x)g(y)-g(x)f(y)),$$

$(x,y) \in S \times S$. Since

(b) $$\|(u,v)\| = \|u\| + \|v\|, \quad (u,v) \in c(A),$$

relation (a) jointly with (7) implies that

$$\|C_F(x,y)\| \leqslant H(x,y), \quad (x,y) \in S \times S,$$

where $H := k+k$. Therefore, the x-sections $C_F(x,\cdot)$ of the map C_F are bounded for all $x \in S$. An appeal to Theorem 1 (see also Remark 1) shows that all the products $a \cdot F$ remain bounded whenever $a \in \mathfrak{J}(C_F(S \times S))$ - the two-sided ideal generated in $\langle F(S)\rangle$ by $C_F(S \times S)$. Since the algebra $\langle F(S)\rangle$ is simple one has

$$C_F(S \times S) \subset \mathfrak{J}(C_F(S \times S)) = \{0\} \quad \text{or} \quad \mathfrak{J}(C_F(S \times S)) = \langle F(S)\rangle.$$

Therefore, either $C_F = 0$ (i.e. F is exponential) or the functions $F(z)\cdot F$ are bounded for all $z \in S$; in particular, in the latter case, $F(0)\cdot F$ is bounded (0 stands here for the neutral element in S) and

$$\|F(s)\| \leqslant \|F(s) - F(0)F(s)\| + \|F(0)F(s)\|$$
$$\leqslant \sup_{t \in S} H(0,t) + \sup_{t \in S} \|F(0)F(t)\|$$

for all $s \in S$, i.e. F itself is bounded. Thus F is either bounded or exponential. In view of (b) and (a), respectively, this means that either both f and g are bounded or

$$C_F(x,y) = (0,0) \quad \text{for al pairs} \quad (x,y) \in S\times S,$$

i.e. the pair (f,g) yields a solution to system (3). This completes the proof.

Remark 6. The assumption that the semigroup $(S,+)$ forms a monoid may be replaced by the requirement that the algebra A has a unit e. In fact, the boundedness of F is then derived from the fact that $e \in \langle F(S)\rangle$.

The next two theorems concern the special case where the algebra considered is the algebra $M_n(\mathbb{C})$ of all complex $(n\times n)$-matrices.

Theorem 5. Let $(S,+)$ be a semigroup and let $f,g\colon S \longrightarrow M_n(\mathbb{C})$ and $\varepsilon \geqslant 0$ be such that inequalities

$$\begin{cases}\|f(x+y) - f(x)f(y) + g(x)g(y)\| \leqslant \varepsilon \\ \|g(x+y) - f(x)g(y) - g(x)f(y)\| \leqslant \varepsilon\end{cases} \tag{8}$$

are satisfied for all $x,y \in S$. Then there exist functions $F,G\colon S \longrightarrow M_n(\mathbb{C})$ such that

$$(F(x+y) - F(x)F(y))^2 = (G(x+y) - G(x)G(y))^2 = 0, \quad x,y \in S, \tag{9}$$

and the functions $f - \frac{1}{2}(F+G)$, $g - \frac{1}{2i}(F-G)$ are bounded.

Theorem 6. Let $(S,+)$ be an Abelian group and let $f,g\colon S \longrightarrow M_2(\mathbb{C})$ and $\varepsilon \geqslant 0$ be such that system (8) holds true for all $x,y \in S$. Than there exist functions $f_0,g_0\colon S \longrightarrow M_2(\mathbb{C})$ such that the pair (f_0,g_0) yields a solution of system (3) and the differences $f-f_0$, $g-g_0$ are both bounded; in other words, system (8) is stable in the class of functions considered.

To prove the latter two theorems it suffices to put $\tilde{F} := f+ig$ and $\tilde{G} := f-ig$ getting

$$\|C_{\tilde{F}}(x,y)\| \leqslant 2\varepsilon \quad \text{and} \quad \|C_{\tilde{G}}(x,y)\| \leqslant 2\varepsilon, \quad x,y \in S.$$

Now, applying Lawrence's Theorem 5 (resp. Theorem 6) from [4] we get the existence of mappings $F,G\colon S \longrightarrow M_n(\mathbb{C})$ fulfilling (9) (resp. $F,G\colon S \to M_2(\mathbb{C})$ exponential) such that the differences $\tilde{F}-F$ and $\tilde{G}-G$ are both bounded. Consequently, setting $f_0 := \frac{1}{2}(F+G)$ and $g_0 := \frac{1}{2i}(F-G)$, we infer that $f-f_0 = \frac{1}{2}\left[(\tilde{F}-F) + (\tilde{G}-G)\right]$ and $g-g_0 = \frac{1}{2i}\left[(\tilde{F}-F) + (G-\tilde{G})\right]$ are bounded as well. Moreover, in case of Theorem 6, the pair (f_0,g_0) yields also a solution of system (3), for F and G are both exponential in that case.

§ 4. Concluding remarks

Setting $A = \mathbb{R}$ (the field of all real numebers) in Theorem 4 and bearing Remark 6 in mind we get the following

Theorem 7. Let $(S,+)$ be a semigroup and let functions $h,k: S\times S \to [0,\infty)$ be given such that all the sections $h(x,\cdot)$ and $k(x,\cdot)$, $x \in S$, are bounded (not necessarily uniformly). Suppose that functions $f,g: S \to \mathbb{R}$ satisfy system (6). It at least one of the functions f and g is unbounded then the pair (f,g) forms a solution to (3); in other words, system (6) is superstable in the class of real functions.

Proof. We have $c(\mathbb{R}) = \mathbb{C}$. Therefore, the algebra $\langle\{(f(x),g(x)) : x\in S\}\rangle$ is simple in a trivial way and it remains to apply Theorem 4.

Remark 7. Note that system (3) actually admits unbounded real solutions; the pair $f(x) = e^x\cos x$, $g(x) = e^x\sin x$, $x \in \mathbb{R}$, provides an example.

Remark 8. The simplicity assumption occuring in Theorem 4 towards the algebra generated by the set $\{(f(x),g(x)): x \in S\}$ cannot be omitted. Indeed, consider the mappings $f,g: \mathbb{C} \to M_2(\mathbb{C})$ given by the formulas

$$f(z) := \begin{bmatrix} e^z\cos z, & 0 \\ 0 & , \eta \end{bmatrix}, \quad g(z) := \begin{bmatrix} e^z\sin z, & 0 \\ 0 & , \eta \end{bmatrix}, \; z \in \mathbb{C},$$

where η is a given positive number. Then one has

$$\|f(x+y) - f(x)f(y) + g(x)g(y)\| = \left\| \begin{bmatrix} 0, & 0 \\ 0, & \eta \end{bmatrix} \right\| = \eta, \quad x,y \in \mathbb{C},$$

and

$$\|g(x+y) - f(x)g(y) - g(x)f(y)\| = \left\| \begin{bmatrix} 0, & 0 \\ 0, & \eta-2\eta^2 \end{bmatrix} \right\| = |\eta-2\eta^2|, \quad x,y \in \mathbb{C},$$

i.e. system (7) is fulfilled by the pair (f,g) with constant functions $h = \eta$ and $k = |\eta-2\eta^2|$. Nevertheless, none of the functions f and g is bounded and the pair (f,g) fails to be a solution of (3).

This happens in spite of the fact that the algebra $M_2(\mathbb{C})$ is simple. The reason is that

$$\langle\{(f(z),g(z)): z \in \mathbb{C}\}\rangle = \left\{ \left(\begin{bmatrix} x, & 0 \\ 0, & y \end{bmatrix}, \begin{bmatrix} u, & 0 \\ 0, & v \end{bmatrix} \right) : x,y,u,v \in \mathbb{C} \right\}$$

and the latter subalgebra of $c(M_2(\mathbb{C}))$ fails to be simple, for it contains the proper two-sided ideal

$$\left\{\left(\begin{bmatrix} x, & 0 \\ 0, & 0 \end{bmatrix}, \begin{bmatrix} u, & 0 \\ 0, & 0 \end{bmatrix}\right) : x, u \in \mathbf{C}\right\}.$$

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