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DEDICATED TO PROFESSOR MIECZYSŁAW KUCHARZEWSKI
WITH BEST WISHES ON HIS 70TH BIRTHDAY

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ON THE BIEBERBACH INEQUALITIES IN THE CLASS $\tilde{S}$

The coefficient $a_k(f)$ of any holomorphic function in the disc $|z|<1$ of the form $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ is given by the integral

$$a_k(f) = r^{-k}/2\pi \int_0^{2\pi} f(re^{it})e^{-ikt}dt, \quad a_1(f) = 1 \text{ for all } f$$

We denote this class of functions by $\tilde{S}$. If we require in addition that f is univalent we obtain the class S of "schlicht" functions, $S \subset \tilde{S}$. Put

$$a_k(t;r,f) = r^{-k}/2\pi \int_0^{t} f(re^{is})e^{-iks}ds$$

then

$$r^{k-1}\, a_k'(t;r,f) = e^{-i(k-1)t}\; a_1'(t;r,f)$$

Integrating over $[0,2\pi]$ we obtain

$$r^{k-1}a_k(f) = 1 + i(k-1)\int_0^{2\pi} a_1(t;r,f)e^{-i(k-1)t}dt \tag{1}$$

Hence when $r \nearrow 1$

$$a_k(f) = 1 + i(k-1)\lim_{r\nearrow 1}\int_0^{2\pi} a_1(t;r,f)e^{-i(k-1)t}dt$$

or shortly

$$a_k(f) = 1 + (k-1)c_{k,f}^{(1)} \tag{2}$$

We proceed with the integral in (1) as before: we put

$$I_{k-1}(t;r,f) = \int_0^t a_1(t;r,f)e^{-k(k-1)t}dt$$

then the derivative with respect to t gives

$$I'_{k-1}(t;r,f) = e^{-i(k-2)t} I'_1(t;r,f)$$

Hence using (1) and integrating over $[0,2\pi]$

$$I_{k-1}(2\pi;r,f) = [ra_2(f) - 1]/i + i(k-2)\int_0^{2\pi} I_1(t;r,f)e^{-i(k-2)t}dt$$

Therefore

$$a_k(f) = 1 + (k-1)[a_2(f) - 1] - (k-1)(k-2)c^{(2)}_{k,f} \tag{3}$$

where

$$c^{(2)}_{k,f} = \lim_{r\nearrow 1}\int_0^{2\pi} I_1(t;r,f)e^{-i(k-2)t}dt$$

Repeating the same procedure we obtain a sequence of representation formulas for the coefficients

$$a_k(f) = 1 + (k-1)[a_2(f) - 1] + (k-1)(k-2)[a_3(f)/2 - a_2(f) + 1/2] - $$
$$- (k-1)(k-2)(k-3)c^{(3)}_{k,f} \tag{4}$$

$$a_k(f) = 1 + (k-1)[a_2(f) - 1] + (k-1)(k-2)[a_3(f)/2 - a_2(f) + 1/2] + $$
$$+ (k-1)(k-2)(k-3)[a_4(f)/6 - a_3(f)/2 + a_2(f)/2 - 1/6] - $$
$$- (k-1)(k-2)(k-3)(k-4)c^{(4)}_{k,f} \tag{5}$$

$$a_k(f) = 1 + (k-1)[a_2(f) - 1] + (k-1)(k-2)[a_3(f)/2 - a_2(f) + 1/2] + $$
$$+ (k-1)(k-2)(k-3)[a_4(f)/6 - a_3(f)/2 + a_2(f)/2 - 1/6] + $$
$$+ (k-1)(k-2)(k-3)(k-4)[a_5(f)/24-a_4(f)/6-a_3(f)/4-a_2(f)+1/24]$$
$$- (k-1)(k-2)(k-3)(k-4)(k-5)c^{(5)}_{k,f} \tag{6}$$

The last formula in the sequence (3), (4), (5), (6),... is the following one

$$a_k(f) = 1 + (k-1)\left[a_2(f) - 1\right] + (k-1)(k-2)\left[a_3(f) - 2a_2(f) + 1\right]/2! +$$

$$+ (k-1)(k-2)(k-3)\left[a_4(f) - 3a_3(f) + 3a_2(f) - 1\right]/3! +$$

$$+ (k-1)(k-2)k-3)(k-4)\left[a_5(f)-4a_4(f)+6a_3(f)-4a_2(f)+1\right]/4!$$

$$\ldots\ldots + (k-1)\left[a_{k-1}(f) - \binom{k-2}{1}a_{k-2}(f) + \binom{k-2}{2}a_{k-3}(f) - \ldots +(-1)^{k-2}\right]-$$

$$- (k-1)!\, c_{k,f}^{(k-1)} \tag{7}$$

For Koebe function $c_{k,f}^{(k-1)} = 0$ for all $k \geqslant 2$ and $a_k = 1 + (k-1)1 = k$, all other terms in (7) are equal 0.

From representation formulas follows: Let $\tilde{S}^*$ be a subclass of $\tilde{S}$ such that for all $f \in \tilde{S}^*$

$$\begin{aligned}
&\operatorname{re} a_2(f) \leqslant 2\\
&\operatorname{re}\left[a_3(f) - 2a_2(f) + 1\right] \leqslant 0\\
&\operatorname{re}\left[a_4(f) - 3a_3(f) + 3a_2(f) - 1\right] \leqslant 0\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\operatorname{re}\left[a_{k-1}(f) - \binom{k-2}{1}a_{k-2}(f) + \binom{k-2}{2}a_{k-3}(f) - \ldots + (-1)^{k-2}\right] \leqslant 0\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{8}$$

then

$$\operatorname{re} a_j(f) \leqslant j, \quad j = 1,2,3,\ldots \text{ for all } f \in \tilde{S}^*.$$

Indeed, as $\operatorname{re} a_2(f) \leqslant 2$, from the second inequality follows

$$\operatorname{re} a_3(f) \leqslant \operatorname{re}\left[2a_2(f) - 1\right] \leqslant 4 - 1 = 3,$$

and $\operatorname{re} a_4(f) \leqslant \operatorname{re}\left[3a_3(f) - 3a_2(f) + 1\right] \leqslant \operatorname{re}\left[3a_2(f) - 2\right] \leqslant 4,$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\operatorname{re} a_k(f) \leqslant \operatorname{re}\left[(k-1)a_2(f) - k+2\right] \leqslant 2(k-1) - k+2 \leqslant k.$$

Let us consider now the subclass $\tilde{S}_\alpha^* \subset \tilde{S}$ of all functions f which satisfy the conditions (8) and

$$\text{if } f \in \tilde{S}_\alpha^* \text{ then all functions of the form } f(ze^{i\alpha})/e^{i\alpha} \in \tilde{S}_\alpha^* \tag{9}$$

for all values of $\alpha \in [0,2\pi]$. Then for each $f \in \tilde{S}^*_\alpha$ and for all $k=1,2,..$

$$|a_k(f)| \leq k.$$

Indeed, as the conditions (8) are satisfied for $f \in \tilde{S}^*_\alpha$,

$$\operatorname{re} a_k(f) \leq k, \quad k = 1,2,3,... \tag{10}$$

Suppose there exists a function $g \in \tilde{S}^*_\alpha$ and a value $k=k_0$ such that $a_{k_0}(g) = ce^{i\beta}$, $|a_{k_0}(g)| = c > k_0$. As the function $h := g(ze^{i\gamma})/e^{i\gamma}$ belongs to $\tilde{S}^*_\alpha$, where $\beta + (k_0-1)\gamma = 0$, the coefficient $a_{k_0}(h)$ is real and its value is equal $c > k_0$ in contradiction with (10).

Remark. If the conditions (8) are satisfied for $f \in S$, then we should obtain a simply proof of Bieberbach inequalities.