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ON HOMOMORPHISM CONNECTED WITH DERIVATION

§ 1. Homomorphisms of modules of tensor fields

Let X_n be a n-dimensional space. By F, B, B^*, T^p_q we shall denote the ring of functions on X_n, the module of vector fields on X_n, the module of covector fields on X_n and the modules of tensor fields of type (p,q) on X_n respectively. Now let A, T, U be a given tensor fields of type $(q+r+t,\ p+s+w)$, (p,q), $(t+r, w+s)$ respectively, that is:

$$A \in T^{t+r+q}_{w+s+p},\quad T \in T^p_q,\quad U \in T^{t+r}_{w+s}.$$

Then A acts as homomorphism on the module T^p_q as following:

$$T^p_q \longrightarrow T^{t+r}_{w+s},\qquad T^p_q \ni T = (T^{\lambda_1\ldots\lambda_p}_{\mu_1\ldots\mu_q}) \longrightarrow$$

$$\longrightarrow AT = U = (U^{\nu_1\ldots\nu_t\,\lambda_1\ \ \lambda_r}_{\omega_1\ldots\omega_w\,\mu_1\ldots\mu_s}) \in T^{t+r}_{w+s}, \tag{1.1}$$

where for

$$A = (A^{\nu_1\ldots\nu_t\,\lambda_1\ldots\lambda_r\,\beta_1\ldots\beta_q}_{\omega_1\ldots\omega_w\,\mu_1\ldots\mu_s\,\alpha_1\ldots\alpha_p})$$

we have

$$A^{\nu_1\ldots\nu_t\,\lambda_1\ldots\lambda_r\,\beta_1\ldots\beta_q}_{\omega_1\ldots\omega_w\,\mu_1\ldots\mu_s\,\alpha_1\ldots\alpha_p} \cdot T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} = U^{\nu_1\ldots\nu_t\,\lambda_1\ldots\lambda_r}_{\omega_1\ldots\omega_w\,\mu_1\ldots\mu_s}. \tag{1.2}$$

Dor $t = w = 0$, $r = p$, $s = q$ we have in (1.1) endomorphism of module of tensor fields T^p_q of type (p,q). Tensor field A of type $(t+r+q,\ w+s+p)$ in (1.1) is the metrix of homomorphism in (1.1), where in (1.2) the indices $(\lambda_1,\ldots,\lambda_r,\mu_1,\ldots,\mu_s)$ and $(\alpha_1,\ldots,\alpha_p,\ \beta_1,\ldots,\beta_q)$ are rows and columns respectively for each square blocks of $(\nu_1,\ldots,\nu_t)$, $\omega_1,\ldots\omega_w)$.

For endomorphism, therefore for $t = w = 0$, $r = p$, $s = q$ we have only one square matrix.

When $\dim \operatorname{Der} A = 0$, then homomorphism (1.1) is iniection. For $t=w=0$, $r = p$, $s = q$ for endomorphism for $\dim \operatorname{Ker} A = 0 \iff \det A \neq 0$ we have automorphism.

With homomorphism (1.1) is associated for $t = w = 0$ following homomorphism

$$\Gamma : T^{q+r}_{p+s} \longrightarrow T^r_s, \quad T^{q+r}_{p+s} \ni A = \left(A^{\lambda_1 \dots \lambda_r \, \beta_1 \dots \beta_q}_{\mu_1 \dots \mu_s \, \alpha_1 \dots \alpha_p}\right) \longrightarrow \Gamma(A) = U =$$

$$= \left(U^{\lambda_1 \dots \lambda_r}_{\mu_1 \dots \mu_s}\right) \in T^r_s, \tag{1.3}$$

where $U = AT$ in (1.1) for $t = w = 0$.

To be made a problem to give the necessary and sufficient conditions for existence of the solution of system of linear equations $AT = U$, in which T, U are given and A is unknown.

§ 2. Derivation and homomorphism connected with derivation

A linear endomorphism D is called a derivation if it satisfies the following conditions [1]:

$$\begin{cases} \text{a) } D \text{ is type preserving, i.e.} \\ \quad D \in \operatorname{End}(T^p_q), \text{ where } T^p_q \text{ is a module of tensor fields} \\ \quad\quad \text{of type } (p,q) \\ \text{b) } D(K \times L) = DK \times L + K \times DL \quad \text{for all tensor fields } K \text{ and } L \\ \text{c) } D \text{ commutes with every contraction.} \end{cases} \tag{2.1}$$

Let D be a tensor field of type (1.1): $D \in T^1_1$. Then D acts as derivation on the module T^p_q as following:

$$DK = \sum_{i=1}^{p} D^{\lambda_i}_{\alpha_i} \cdot K^{\lambda_1 \dots \lambda_{i-1} \alpha_i \lambda_{i+1} \dots \lambda_p}_{\mu_1 \dots \mu_q} -$$

$$- \sum_{j=1}^{q} D^{\beta_j}_{\mu_j} \cdot K^{\lambda_1 \dots \lambda_p}_{\mu_1 \dots \mu_{j-1} \beta_j \mu_{j+1} \dots \mu_q} =$$

$$= D^{\lambda_1 \dots \lambda_p \beta_1 \dots \beta_q}_{\mu_1 \dots \mu_q \alpha_1 \dots \alpha_q} \cdot K^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q} \tag{2.2}$$

where $K = (K^{\lambda_1\ldots\lambda_p}_{\mu_1\ldots\mu_q}) \in T^p_q$ and

$$D^{\lambda_1\ldots\lambda_p\beta_1\ldots\beta_q}_{\mu_1\ldots\mu_q\alpha_q\ldots\alpha_p} := \sum_{i=1}^{p}\delta^{\lambda_1}_{\alpha_1}\ldots\delta^{\lambda_{i-1}}_{\alpha_{i-1}}.D^{\lambda_i}_{\alpha_i}.\delta^{\lambda_{i+1}}_{\alpha_{i+1}}\ldots\delta^{\lambda_p}_{\alpha_p}.\delta^{\beta_1}_{\mu_1}\ldots\delta^{\beta_q}_{\mu_q} -$$

$$-\sum_{j=1}^{q}\delta^{\lambda_1}_{\alpha_1}\ldots\delta^{\lambda_p}_{\alpha_p}.\delta^{\beta_1}_{\mu_1}\ldots\delta^{\beta_{j-1}}_{\mu_{j-1}}.D^{\beta_j}_{\mu_j}.\delta^{\beta_{j+1}}_{\mu_{j+1}}\ldots\delta^{\beta_q}_{\mu_q}. \tag{2.3}$$

$M_D = D^{\lambda_1\ldots\lambda_p\beta_1\ldots\beta_q}_{\mu_1\ldots\mu_q\alpha_1\ldots\alpha_q}$ is the tensor

field of type $(p+q,\ p+q)$: $M_D \in T^{p+q}_{p+q}$; M_D is the matrix too, where $(\lambda_1,\ldots,\lambda_p,\ \mu_1,\ldots,\mu_q)$ and $(\alpha_1,\ldots,\alpha_p,\ \beta_1,\ldots,\beta_q)$ are rows and columns respectively; it is the matrix of derivation by means of a tensor field D of type (1,1) as endomorphism of module of tensor fields T^p_q in (3.2).

To be made a problem to give a necessary and sufficient conditions for the derivation acting in a module of tensor fields T^p_q by means of a tensor field D ot type (1,1) be the automorphism. We shall give them in a effective form by giving the traces and the determinant of the matrix of tensor field $D = (D^{\lambda}_{\mu})$:

$$\sigma_1 := D^{\alpha}_{\alpha},\ \sigma_i := i!\, D^{\alpha_1}_{[\alpha_1}.D^{\alpha_2}_{\alpha_2}\ldots D^{\alpha_i}_{\alpha_i]}$$

$(i = 1,\ldots,n-1)$; not sum over $\qquad$ (2.4)

$$\alpha_1 < \alpha_2 < \ldots < \alpha_i;\quad \sigma_n := \det(D^{\lambda}_{\mu}).$$

Theorem ([2]). The necessary and sufficient condition that the derivation acting in the module of tensor fields $T^p_q (p \neq q)$ by means of a tensor field D of type (1.1) be the automorphism, is

$$\det M_D = \oint(\sigma_1,\sigma_2,\ldots,\sigma_n) \neq 0, \tag{2.5}$$

where $\oint$ is functions of variables $\sigma_1,\sigma_2,\ldots,\sigma_n$ and M_D is the matrix of derivation.

Effective form of the function $\oint$ for any p,q $(p \neq q)$ and any $n (n \geq 2)$ is not found. For T^p_o (or T^o_p) over 2-dimensional space X_2 the function $\oint$ is following [2]:

$$\det M_D = \oint = \prod_{k=0}^{s} \left(k(p-k)\sigma^2 + (2k-p)^2 \,.\, \Omega\right) \tag{2.6}$$

$$s = \tfrac{1}{2}(p-1) \text{ for } p \text{ odd}$$

$$s = \tfrac{1}{2}p \text{ for } p \text{ even;}$$

$$\sigma := D^{\alpha}_{\alpha}, \qquad \Omega := \det(D^{\lambda}_{\mu}).$$

<u>Theorem ([2])</u>. A derivation acting in the module T^p_p of type (p,p) by means of a tensor field D of type $(1,1)$ can never be the automorphism.

Homomorphism acting by help of tensor field $H \in T^{t+1}_{w+1}$ on the module T^p_q as derivation is following:

$$T^p_q \longrightarrow T^{t+p}_{w+q}, \quad T^p_q \ni T \longrightarrow HT = U \in T^{t+p}_{w+q},$$

$$HT = \sum_{i=1}^{p} H^{\nu_1\ldots\nu_t\lambda_i}_{\omega_1\ldots\omega_w\alpha_i} \,.\, T^{\lambda_1\ldots\lambda_{i-1}\alpha_i\lambda_{i+1}\ldots\lambda_p}_{\mu_1\ldots\mu_q} -$$

$$- \sum_{j=1}^{q} H^{\nu_1\ldots\nu_t\beta_j}_{\omega_1\ldots\omega_w\mu_j} \,.\, T^{\lambda_1\ldots\lambda_p}_{\mu_1\ldots\mu_{j-1}\beta_j\mu_{j+1}\ldots\mu_q}; \tag{2.7}$$

$$U = \left(U^{\nu_1\ldots\nu_t\lambda_1\ldots\lambda_p}_{\omega_1\ldots\omega_w\mu_1\ldots\mu_q}\right) \in T^{t+p}_{w+q},$$

where for A in (1.1) and in (1.2) we have $r = p$, $s = q$ and

$$A^{\nu_1\ldots\nu_t\lambda_1\ldots\lambda_p\beta_1\ldots\beta_q}_{\omega_1\ldots\omega_w\mu_1\ldots\mu_q\alpha_1\ldots\alpha_p} :=$$

$$:= \sum_{i=1}^{p} \delta^{\lambda_1}_{\alpha_1}\ldots\delta^{\lambda_{i-1}}_{\alpha_{i-1}} \,.\, H^{\nu_1\ldots\nu_t\lambda_i}_{\omega_1\ldots\omega_w\alpha_i} \,.\, \delta^{\lambda_{i+1}}_{\alpha_{i+1}} \ldots \delta^{\lambda_p}_{\alpha_p} \,.\, \delta^{\beta_1}_{\mu_1} \ldots \delta^{\beta_q}_{\mu_q} \tag{2.8}$$

$$- \sum_{j=1}^{q} \delta^{\lambda_1}_{\alpha_1} \ldots \delta^{\lambda_p}_{\mu_p} \,.\, \delta^{\beta_1}_{\mu_1} \ldots \delta^{\beta_{j-1}}_{\mu_{j-1}} \,.\, H^{\nu_1\ldots\nu_t\beta_j}_{\omega_1\ldots\omega_w\mu_j} \,.\, \delta^{\beta_{j+1}}_{\mu_{j+1}} \ldots \delta^{\beta_q}_{\mu_q}.$$

For $t = w = 0$ we have endomorphism as derivation.

For $t = 0$, $w = 1$, for $U^{\lambda_1 \dots \lambda_p}_{\sigma \mu_1 \dots \mu_q} = \nabla_\sigma T^{\lambda_1 \dots \lambda_p}_{\mu_1 \dots \mu_q} - \partial_\sigma T^{\lambda_1 \dots \lambda_p}_{\mu_1 \dots \mu_q}$ and for $H^\alpha_{\sigma\beta} = \Gamma^\alpha_{\sigma\beta}$ we have covariant derivative $\nabla_\sigma T$ of tensor field T by help connection $(\Gamma^\alpha_{\sigma\beta})$.

To be made a problem to give a necessary and sufficient conditions for homomorphism acting by help of tensor field $H \in T^{t+1}_{w+1}$ on the module T^0_{2p} as derivation for

$$\operatorname{Ker} m \neq 0, \quad \det(T) \neq 0, \tag{2.9}$$

where

$$m: T^0_{2p} \longrightarrow T^t_{w+2p}, \; T^0_{2p} \ni T = (T_{\lambda_1 \dots \lambda_p \mu_1 \dots \mu_p}) \longrightarrow HT = U =$$

$$= (U^{\nu_1 \dots \nu_1}_{\omega_1 \dots \omega_w \lambda_1 \dots \lambda_p \mu_1 \dots \mu_p}) \in T^t_{w2+p},$$

where

$$HT = -\sum_{i=1}^{p} H^{\nu_1 \dots \nu_t \alpha_i}_{\omega_1 \dots \omega_w \lambda_i} \cdot T_{\lambda_1 \dots \lambda_{i-1} \alpha_i \lambda_{i+1} \dots \lambda_p \mu_q \dots \mu_p} \tag{2.10}$$

$$- \sum_{j=1}^{p} H^{\nu_1 \dots \nu_t \beta_j}_{\omega_1 \dots \omega_w \mu_j} \cdot T_{\lambda_1 \dots \lambda_p \mu_1 \dots \mu_{j-1} \beta_j \mu_{j+1} \dots \mu_p},$$

$$\det(T) = \det(T_{\lambda_1 \dots \lambda_p \mu_1 \dots \mu_p}), \tag{2.11}$$

where in the matrix $(T_{\lambda_1 \dots \lambda_p \mu_1 \dots \mu_p})$ the indices $(\lambda_1, \dots, \lambda_p)$, $(\mu_1, \dots, \mu_p)$ are rows and columns respectively.

Analogously problem is for the module T^p_p of tensor fields of type (p,p).

§ 3. Homomorphism associated with derivation

With derivation (2.7) is associated for $t = w = 0$ the following homomorphism:

$$\Gamma: T^1_1 \longrightarrow T^p_q, \; T^1_1, \; T^1_1 \ni D \longrightarrow \Gamma(D) = DT = U \in T^p_q, \tag{4.1}$$

where DT is defined in (2.7) for $D = H$ and for $T = w = 0$. The matrix of this homomorphism is following:

$$A^{\lambda_1\ldots\lambda_p\alpha}_{\mu_1\ldots\mu_q\beta} = \sum_{i=1}^{p} \delta^{\lambda_i}_{\beta} \cdot T^{\lambda_1\ldots\lambda_{i-1}\alpha\lambda_{i+1}\ldots\lambda_p}_{\mu_1\ldots\mu_q}$$

$$- \sum_{j=1}^{q} \delta^{\alpha}_{\mu_j} \cdot T^{\lambda_1\ldots\lambda_p}_{\mu_1\ldots\mu_{j-1}\beta\mu_{j+1}\ldots\mu_q}, \tag{3.2}$$

where $(\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q)$ and (α,β) are rows and columns respectively.

To be made a problem to give the necessary and sufficient conditions for existence of the solution of system of linear equations $DT = U$, in which T,U are given and D are unknown. The general solution for all tensor fields T,U is in [2].

With derivation (2.7) is associate for $t = 0$, $w = 1$ following symmetrical homomorphism:

$$\Gamma : T_2^{(s)1} \longrightarrow T^p_{p+1},\ T_2^{(s)1} \ni S = (S^{\alpha}_{\beta\gamma}) \longrightarrow \Gamma(S) = ST =$$

$$= (U^{\lambda_1\ldots\lambda_p}_{\sigma\mu_1\ldots\mu_q}) = U \in T^p_{q+1}, \tag{3.3}$$

where ST is defined in (2.7) for $S = H$ and for $t = 0$, $w = 1$ and where S is symmetrical tensor field of type (1.2), i.e. $S^{\alpha}_{\beta\gamma} = S^{\alpha}_{\gamma\beta}$. The matrix of this homomorphism is following:

$$2A^{\lambda_1\ldots\lambda_p(\beta\gamma)}_{\mu_1\ldots\mu_q\sigma\alpha} = 2\sum_{i=1}^{p} \delta^{(\gamma}_{\sigma}\delta^{|\lambda_i}_{\alpha} \cdot T^{\lambda_1\ldots\lambda_{i-1}|\beta)\lambda_{i+1}\ldots\lambda_p}_{\mu_1\ldots\mu_q}$$

$$-2\sum_{j=1}^{q} \delta^{(\gamma}_{\sigma}\delta^{\beta)}_{\mu} T^{\lambda_1\ldots\lambda_p}_{\mu_1\ldots\mu_{j-1}\alpha\mu_{j+1}\ldots\mu_q} \tag{3.4}$$

where $(\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q,\sigma)$ and $(\alpha,\beta,\gamma;\ \beta \leq \gamma)$ are rows and columns respectively.

Tensor field

$$U = (U^{\lambda_1\ldots\lambda_p}_{\sigma\lambda_1\ldots\mu_q}) \in T^p_{q+1}. \quad \text{For} \quad U^{\lambda_1\ldots\lambda_p}_{\sigma\lambda_1\ldots\mu_q} = \nabla_\sigma T^{\lambda_1\ldots\lambda_p}_{\mu_1\ldots\mu_q} - \partial T^{\lambda_1\ldots\lambda_p}_{\sigma\mu_1\ldots\mu_q}$$

and for $S^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma}$ $(S^{\alpha}_{\beta\gamma} = S^{\alpha}_{\gamma\beta}, \Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta})$ we have covariant derivative $\nabla_\sigma T$ of tensor field T by help of symmetrical connection $\Gamma^{\mu}_{\sigma\lambda}(\Gamma^{\mu}_{\sigma\lambda} = \Gamma^{\mu}_{\lambda\sigma})$.

To be made a problem to give the necessary and sufficient conditions for existence of the solution of system of linear equations $ST = U$, in which T, U are given and S are unknown. The genral solution for all tensor fields T, U is not known.

With derivation (2.7) is associated for $t = 0$, $w = 1$ and for T^0_3 following non-symmetrical homomorphism:

$$A : T^1_2 \longrightarrow T^0_3,\ T^1_2 \ni H = (H^{\alpha}_{\beta\gamma}) \longrightarrow A(H) = A(H^{\alpha}_{\beta\gamma}) = U = (U_{\sigma\lambda\mu}) \in T^0_3 \tag{3.5}$$

where H is non-symmetrical : $H^{\alpha}_{\beta\gamma} \neq H^{\alpha}_{\gamma\beta}$ and where $A(H^{\alpha}_{\beta\gamma})$ is defined as following:

$$A_{\sigma\lambda\mu} := -H^{\alpha}_{\sigma\lambda} \cdot g_{\alpha\mu} - H^{\alpha}_{\mu\sigma} \cdot g_{\lambda\alpha}. \tag{3.6}$$

For $A_{\sigma\lambda\mu} = E_{\sigma} g_{\lambda\mu} - \partial_{\sigma} g_{\lambda\mu}$ and for $H^{\mu}_{\sigma\lambda} = \Gamma^{\mu}_{\sigma\lambda}$ $(H^{\mu}_{\sigma\lambda} \neq H^{\mu}_{\lambda\sigma}, \Gamma^{\mu}_{\sigma\lambda} \neq \Gamma^{\mu}_{\lambda\sigma})$ we have Einstein derivative $E_{\sigma} g_{\lambda\mu}$ of tensor field $g = (g_{\lambda\mu})$ by help of non-symmetrical connection $\Gamma^{\mu}_{\sigma\lambda}$.

To be made a problem to give the necessary and sufficient conditions for existence of the solution of system of linear equations $A(H) = U$, in which A, H are given and H is unknown. The general solution for all tensor fields T, U is not known, as well as for Einstein derivative for $\det(g_{\lambda\mu}) \neq 0$ or for rank $(g_{\lambda\mu}) = 1, 2, \dots, n-1$ and for arbitrary dimension n.

§ 4. Norden-, Otsuki-, and spinor - homomorphism connected with derivation

Norned-homomorphism acting by help of $p + q$ tensor fields $\overset{(h)}{N} \in T^{t+1}_{w+1}$ $(h = 1, 2, \dots, p+q)$ on the module of tensor fields T^p_q of type (p,q) as derivation is following:

$$T^p_q \longrightarrow T^{t+p}_{w+q},\ T^p_q \ni T \longrightarrow AT = U \quad T^{t+p}_{w+q}, \tag{4.1}$$

where for A in (1.1) and in (1.2) we have $r = p$, $s = q$ and

$$A^{\nu_1 \dots \nu_t \lambda_1 \dots \lambda_p \beta_1 \dots \beta_q}_{\omega_1 \dots \omega_w \mu_1 \dots \mu_q \alpha_1 \dots \alpha_p}$$

$$\sum_{i=1}^{p} \delta^{\lambda_1}_{\alpha_1} \dots \delta^{\lambda_{i-1}}_{\alpha_{i-1}} \cdot \overset{(i)}{N}{}^{\nu_1 \dots \nu_t \lambda_i}_{\omega_1 \dots \omega_w \alpha_i} \cdot \delta^{\lambda_{i+1}}_{\alpha_{i+1}} \dots \delta^{\lambda_p}_{\alpha_p} \cdot \delta^{\beta_1}_{\mu_1} \dots \delta^{\beta_q}_{\mu_q} -$$

$$-\sum_{j=1}^{q}\delta^{\lambda_1}_{\alpha_1}\dots\delta^{\lambda_p}_{\alpha_p}\cdot\delta^{\beta_1}_{\mu_1}\dots\delta^{\beta_{j-1}}_{\mu_{j-1}}\cdot\underset{(j)}{N}{}^{\nu_1\dots\nu_t\beta_j}_{\omega_1\dots\omega_w\mu_j}\cdot\delta^{\beta_{j+1}}_{\mu_{j+1}}\dots\delta^{\beta_q}_{\mu_q}. \tag{4.2}$$

For $t = w = 0$ we have Nornde-endomorphism.

For $\overset{(1)}{N} = \overset{(2)}{N} = \dots = \overset{(p)}{N} = P$, $\underset{(1)}{N} = \underset{(2)}{N} = \dots = \underset{(q)}{N} = Q$ we have Otsuki-homomorphism and for $t = w = 0$ we have Otsuki-endomorphism.

For $\overset{(1)}{N} = \overset{(1)}{\underset{(1)}{N}}{}_2 = S$, $\overset{(2)}{N} = \overset{(2)}{\underset{(2)}{N}} = S, \dots, \overset{(q)}{N} = \overset{(q)}{\underset{(q)}{N}} = S$, $\overset{(q+1)}{N} = \overset{q+1}{S}, \dots, \overset{(q+r)}{N} = \overset{(q+r)}{S}$ $(p = q + r)$ or for $\overset{(1)}{N} = \underset{(1)}{N} = S$, $\overset{(2)}{N} = \underset{(2)}{N} = S, \dots, \overset{(p)}{N} = \underset{(p)}{N} = S$, $\underset{(p+1)}{N} = \overset{(p+1)}{S}, \dots, \underset{(p+r)}{N} = \overset{(p+r)}{S}$ $(q = p + r)$ we have spinor-homomorphism. For $t = w = 0$ we have spinor-endomorphism.

We investigate the problem to find necessary and sufficient conditions by means of the tensor fields of type (1.1) for $t = w = 0$ for the derivation, or O-endomorphism or S-endomorphism or N-endomorphism acting in T^p_q to be an automorphism. We shall give them in an effective form by giving the traces, determinants and mixed invariants of tensor fields of type (1,1).

In case of a derivation the traces and the determinante of a tensor field of type (1,1) are needed only. In case of an O-endomorphism for tensor fields of type (p,p) the traces and determinante of two tensor fields of type (1,1) are only needed as well. For others cases the problem is open.

<u>Example</u>. For O-endomorphism for $n = 2$ and for $K \in T^1_2$ the funktion of two traces and two determinante is following:

$$\det M_{o\text{-}end} = (\beta + \tau^2 - \sigma\tau)((\alpha + 4\beta)^2 - 5\beta(\alpha + 4\beta) +$$
$$+ 4\alpha\tau^2 + 4\beta\sigma^2 - 16\alpha^2),$$

where $\sigma = P^{\omega}_{\omega}$, $\tau = Q^{\omega}_{\omega}$, $\alpha = \det(P^{\lambda}_{\mu})$, $\beta = \det(Q^{\lambda}_{\mu})$ and where $M_{o\text{-}end}$ is the matrix of O-endomorphism.

For derivation, i.e. $P = Q = D$ (so $\tau = \sigma, \beta = \alpha$) we obtain:

$$\det M_D = \alpha^2 \cdot (\sigma^2 - 2\alpha).$$

<u>Example</u>. For O-endomorphism for $n=2$ and for $K \in T^1_1$ we have:

$$\det M_{o\text{-}end} = \beta\sigma^2 + \alpha\tau^2 - (\alpha+\beta)\sigma\tau + (\alpha-\beta)^2;$$

for derivation D we get $\det M_D = 0$ and a derivation acting in the module of tensor fields of type (1,1) can never be an automorphism, whet is true with theorem in § 3.

With Norden-, Otsuki-, spinor-endomorphism in (4.1), (4.2) for t=w=0 is associated following homomorphism:

$$\overset{(T)}{\Gamma} : \mathop{\mathsf{X}}_{h=1}^{p+q} T^1_1 \longrightarrow T^p_q, \left(\mathop{\mathsf{X}}_{h=1}^{p+q} T^1_1\right) \ni N = \overset{(1)}{N}, \overset{(2)}{N}, \ldots, \overset{(p)}{N}, \underset{(1)}{N}, \underset{(2)}{N}, \ldots, \tag{4.3}$$

$$\underset{(q)}{N} \longrightarrow \overset{(T)}{\Gamma}(N) = U \in T^p_p, \quad \text{where } U \text{ is defined in (5.2).}$$

For $p+q = 2$ and for $\overset{(1)}{N} = \overset{(2)}{N} = \ldots = \overset{(p)}{N} = P$, $\underset{(1)}{N} = \underset{(2)}{N} = \ldots = Q$ we have Otsuki-homomorphism. For $p = q+r$ or $q = p+r$ we have spinor-homomorphism.

To be made a problem to give the necessary and sufficient conditions for existence of the solution of system of linear equations $\overset{(T)}{\Gamma}(N) = U$, in which T, U are given and N are unknown. The general solution for all tensor fields T, U is not known.

§ 5. Homomorphism acting by help of curvature tensor as derivation

Let $R = (R^{\mu}_{\alpha\beta\lambda})$ be the curvature tensor field and let $(g_{\lambda\mu}) \in \overset{s}{T}{}^0_2$, $(G_{\alpha\beta\lambda\mu}) \in T^0_4$, where $\overset{s}{T}{}^0_2$ is the module of tensor fields of type (0,2) of symmetrical tensor fields. Then R acts as derivation on the module $\overset{s}{T}{}^0_2$ as following:

$$m : \overset{s}{T}{}^0_2 \longrightarrow T^0_4, \quad \overset{s}{T}{}^0_2 \ni (g_{\lambda\mu}) \longrightarrow m(g_{\lambda\mu}) = (G_{\alpha\beta\lambda\mu}) \in T^0_4, \tag{5.1}$$

where

$$G_{\alpha\beta\lambda\mu} := R^{\sigma}_{\alpha\beta\lambda} g_{\sigma\mu} + R^{\sigma}_{\alpha\beta\mu} g_{\lambda\sigma}. \tag{5.2}$$

Theorem (L.P. Eisenhart and D. Veblen).

The necessary condition for the metrizability of space with symmetric affine connection A_n is following:

$$(g_{\lambda\mu}) \in \mathrm{Ker}(m) \text{ and } \det(g_{\lambda\mu}) \neq 0,$$

where m is homomorphism in (5.1), (5.2).

The condition (5.3) depends on invariants a,b,c, the are following:

a := n-dim Ker R, where R is following homomorphism [3] R: $T^1_o \longrightarrow \overset{(a)}{T}{}^1_2$,
$T^1_o \ni (v^\lambda) \longrightarrow R(v^\lambda) = R_{\alpha\beta\alpha}{}^{\lambda} v^\alpha \in \overset{(a)}{T}{}^1_2$,

where T^1_o is the module of contravariant vector fields and $\overset{(a)}{T}{}^1_2$ is the module of tensor fields of type (1,2) antisymmetrical.

b := $\binom{n}{2}$ - dim Ker $\overset{2}{R}$, where $\overset{2}{R}$ is following homomorphism [3]
$\overset{2}{R} : \overset{(a)}{T}{}^2_o \longrightarrow \overset{(a)}{T}{}^2_2$. $\overset{(a)}{T}{}^2_o \ni (w^{\lambda\mu}) \longrightarrow \overset{2}{R}(w^{\lambda\mu}) = \overset{2}{R}{}_{\alpha\beta\varrho\sigma}{}^{\lambda\mu} w^{\varrho\sigma} \in \overset{(a)}{T}{}^2_2$,

where $\overset{2}{R} = \overset{2}{R}{}_{\alpha\beta\varrho\sigma}{}^{\lambda\mu} := 4R_{\alpha\beta}{}^{[\lambda}{}_{[\varrho}{}^{\mu]}{}_{\sigma]}$, $\overset{(a)}{T}{}^2_o$ is the module of skew-symmetrical tensor fields of type (2,0) : $(w^{\lambda\mu}) \in \overset{(a)}{T}{}^2_o \Longleftrightarrow w^{\mu\lambda} = -w^{\lambda\mu}$; $\overset{(a)}{T}{}^2_2$ is the module of skew-symmetrical tensor fields of type (2,2).

c := $\binom{n}{2}$ - dim Ker $\tilde{R}$, where $\tilde{R}$ is following coupled endomorfism [3]
$\tilde{R} : \overset{(a)}{T}{}^2_o \longrightarrow \overset{(a)}{T}{}^o_2$, $\overset{(c)}{T}{}^2_o \ni (w^{\varrho\sigma}) \longrightarrow \tilde{R}(w^{\varrho\sigma}) = R_{\alpha\beta\varrho\sigma} w^{\varrho\sigma} \in \overset{(a)}{T}{}^o_2$,

where $\tilde{R} = (R_{\alpha\beta\lambda\mu})$, $R_{\alpha\beta\lambda\mu} := R_{\alpha\beta\lambda}{}^{\sigma} g_{\sigma\mu}$; $\overset{(a)}{T}{}^o_2$ is the module of skew-symmetrical tensor fields of type (0,2).

For A_3 and A_4, i.e. for 3-and 4-dimensional space with symmetrical affine connection, the condition (5.3) depend only on invariants a,b,c and the cases of values of invariants a,b,c in the metrizability of A_3 and A_4 gives for (5.3) the classification of Riemann spaces V_3 and V_4 with respect to the invariants a,b,c. This classification gives exact solutions of field of Riemann spaces V_3 and V_4 or Riemann space-times.

The classification of Riemann space-times is following:

$$(V_{466}, V_{465}, V_{464}, V_{463}, V_{462}, V_{442}, V_{363}, V_{362}, V_{343}, V_{241}, V_{000}), \tag{5.4}$$

where V_{abc} denotes the Riemann space-times, for which the curvature tensor has the values of the invariants equal a,b,c,

The classivication of Einstein space-times with respect to the invariants a,b,c and Petrov-Penrose types is following

G_{abc} \ $\overset{*}{T}_i, T_i$	$\overset{*}{T}_1$	T_2	$\overset{*}{T}_2$	T_2	$\overset{*}{T}_3$	T_3
G_{466}	I, D, O	I, D	II, N	II	III	
G_{464}	I, D	I	II			III
G_{442}	D					
G_{342}				N		
G_{000}		M				

(5.5)

where $\overset{*}{T}_i$, $T_i(i = 1,2,3)$ are the Petrow types; I, D, O, II, III, N, M are the Penrose types and G_{abc} are the types with respect to the algebraical invariants a,b,c.

Example. Let us take Petrov space-time from his monography: $H = G_4\text{III}$, providet with the following metric tensor:

$$\left.\begin{aligned} &g_{11} = a(t),\ g_{22} = b(t),\ g_{23} = x \,.\, b(t) \\ &g_{33} = a + x^2 \,.\, b,\ g_{44} = c = \text{const.} \neq 0 \\ &\text{remaining}\ \ g_{\lambda\mu} = 0;\ (x,y,z,t) = (x^1,x^2,x^3,x^4). \end{aligned}\right\} \tag{5.6}$$

Exact solutions of the field (5.6) with respect to the invariants a,b,c are following

$$\left.\begin{aligned} &1)\ \underline{\text{for } H_{466}} \\ &\quad a \neq K_2(t + K_1)^2,\quad b \neq K_3(t + K_1)^2 \\ &\qquad K_1, K_2, K_3 = \text{const}; \\ &2)\ \underline{\text{for } H_{363}} \\ &\quad a = K_2(t + K_1)^2,\quad b = K_3(t + K_1)^2;\ K_3 \neq \frac{4K_2^2}{C} \\ &3)\ \underline{\text{for } H_{241}} \\ &\quad a = K_2(t + K_1)^2,\quad b = \frac{4K^2}{C}(t + K_1)^2 \end{aligned}\right\} \tag{5.7}$$

Robertson-Walker space-times. The Robertson Walker space-times has a following metric tensor:

$$
\left.\begin{aligned}
&(g_{\lambda\mu}) = \operatorname{diag}\left(1 \quad - \frac{R^2}{1-kr^2}\right) - r^2R^2 - r^2R^2\sin^2\Theta), \\
&\text{where} \quad R = R(t), \ (t,r,\Theta,\varphi) = (x^1,x^2,x^3,x^4).
\end{aligned}\right\} \qquad (5.8)
$$

Theorem. Exact solutions of the field (5.8) with respect to the invariants a,b,c are following:

for k = -1

$$
\left.\begin{aligned}
&1)\ \text{for } RN_{466} : R \neq Ct + D; \ C,D = \text{const} \\
&2)\ \text{for } RN_{363} : R = Ct + D, \ C \neq \pm 1 \\
&3)\ \text{for } M = RN_{000}\ (M = \text{Minkowski space-time}): R = \pm t + D
\end{aligned}\right\} \qquad (5.9)
$$

for k = 0

$$
\left.\begin{aligned}
&1)\ \text{for } RN_{466} : R \neq Ct + D \\
&2)\ \text{for } RN_{363} : R = Ct + D, \ C \neq D \\
&3)\ \text{for } M : R = D
\end{aligned}\right\} \qquad (5.10)
$$

for k = 1

$$
\left.\begin{aligned}
&1)\ \text{for } RN_{466} : R \neq Ct + D \\
&2)\ \text{for } RN_{363} : R = Ct + D.
\end{aligned}\right\} \qquad (5.11)
$$

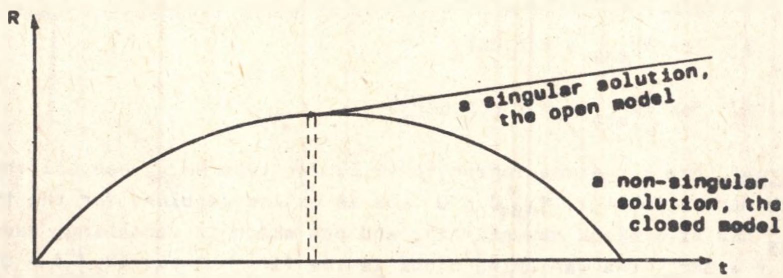
For Robertson-Walker space-time of type RN_{466} the Friedmann solutions are non-singular and they are the following [9], [10]:

$$
\left.\begin{aligned}
&1)\ \text{for } K = -1 : R = \tfrac{1}{2} A^2(\cosh\Psi - 1), \quad t = \tfrac{1}{2} A(\sinh\Psi - \Psi) \\
&2)\ \text{for } K = 0 : \left(\tfrac{3A}{2}\right)^{\frac{2}{3}} \cdot t^{\frac{2}{3}} \\
&3)\ \text{for } K = 1 : R = \tfrac{1}{2} A^2(1 - \cos\Psi), \quad t = \tfrac{1}{2} A^2(\Psi - \sin\Psi).
\end{aligned}\right\} \qquad (5.12)
$$

For the type RN_{363} the Friedmann solutions are singular and they are on the base of (5.9), (5.10), (5.11) the following

$$
\left.\begin{aligned}
&1)\ \text{for } k = -1 : R = Ct + D, \ C \neq \pm 1 \\
&2)\ \text{for } k = \ \ 0 : R = Ct + D, \ C \neq 0 \\
&3)\ \text{for } k = 1 \ \ : R = Ct + D, \ C,D \ \text{arbitrary}
\end{aligned}\right\} \qquad (5.13)
$$

Corollary. In each Friedmann model there exists a singular model, hence a closed model (for k=1) can become an open one:



§ 6. Reissner-Nordstrom space times

The Reissner-Nordstrom space-times has a following metric tensor:

$$(g_{\lambda\mu}) = \mathrm{diag}(-A,\ \frac{1}{A},\ r^2,\ r^2\sin^2\Theta), \tag{6.1}$$

where $A = 1 - \frac{\tilde{r}_o}{r} + \frac{H}{r^2}$; $r_o, H = \mathrm{const}$; $\tilde{r}_o = ar_o$. $(t,r,\Theta,\varphi)=(x^1,x^2,x^3,x^4)$. Calculating the coordinates of curvature tensor for metric tensor field in (6.1) we obtain following non vanishing components for $K = \frac{H}{r_o^2}$:

$$\left.\begin{aligned}
&R^2_{121} = \frac{A\tilde{r}_o}{r^4}(r - 3K\tilde{r}_o), \quad R^1_{122} = -\frac{\tilde{r}_o}{Ar^4}(r - 3K\tilde{r}_o)\\
&R^3_{131} = -\frac{A\tilde{r}_o}{2r^4}(r - 2K\tilde{r}_o), \quad R^1_{133} = -\frac{\tilde{r}_o}{2r^2}(r - 2K\tilde{r}_o)\\
&R^4_{141} = R^3_{131}, \quad R^1_{144} = -\frac{\tilde{r}_o\sin^2\Theta}{2r^2}(r - 2K\tilde{r}_o)\\
&R^3_{232} = \frac{\tilde{r}_o}{2Ar^4}(r - 2K\tilde{r}_o), \quad R^2_{233} = -\frac{\tilde{r}_o}{2r^2}(r - 2K\tilde{r}_o)\\
&R^4_{242} = R^3_{232}, \quad R^2_{244} = R^1_{144}\\
&R^4_{343} = -\frac{\tilde{r}_o}{r^2}(r - K\tilde{r}_o), \quad R^3_{344} = \frac{\tilde{r}_o\sin^2\Theta}{r^2}(r - K\tilde{r}_o).
\end{aligned}\right\} \tag{6.2}$$

Theorem (on classification of RN space-times, main result). We have the following exact solutions of field of Reissner-Nordstrom RN space-times:

$$\left.\begin{array}{ll}
1) & \text{for } RN_{466}: r \neq 3K\tilde{r}_0 \text{ and } r \neq 2K\tilde{r}_0 \text{ and } r \neq K\tilde{r}_0 \\
2) & \text{for } RN_{465}: r = 3K\tilde{r}_0 \text{ or } r = K\tilde{r}_0 \\
3) & \text{for } RN_{442}: r = 2K\tilde{r}_0 \\
4) & \text{for } RN_{000} \cong M: r > 5K\tilde{r}_0.
\end{array}\right\} \quad (6.3)$$

Proof. The curvature tensor in (6.2) for type RN_{466} has all the six block non-vauishing: $(R_{\alpha\beta\gamma}^{\mu}) \neq 0$ and is called regular; for the type RN_{465} has five block nonvanishing and one which is vanishing, namely for $r = 3K\tilde{r}_0$ the vanishing block is the first block: $(R_{12\lambda}^{\mu}) = 0$ and for $r = K\tilde{r}_0$ the vanishing block is the sixte block: $(R_{34\lambda}^{\mu}) = 0$; for the type RN_{442} for $r = 2K\tilde{r}_0$ we have two blocks nonvanishing, namely the first and the sixth one: $(R_{12\lambda}^{\mu}) \neq 0$, $(R_{34\lambda}^{\mu}) \neq 0$ and four blocks are vanishing: $(R_{13\lambda}^{\mu}) = (R_{14\lambda}^{\mu}) = (R_{23\lambda}^{\mu}) = (R_{24\lambda}^{\mu}) = 0$; for the type $RN_{000} \cong M$ for $r > 5K\tilde{r}_0$ the curvature tensor tends to the null curvature tensor, i.e. Minkowski space-time M.

As we observe in (6.3), the type of RN space-times changes for $r = K\tilde{r}_0$, $2K\tilde{r}_0$, $3K\tilde{r}_0$, $r > 5K\tilde{r}_0$ and it has for these values of the radins r the following properties:

Corollary. For RN_{465} for $r = 3K\tilde{r}_0$ the space-time is flat in bidirections $(v^{[1}w^{2]}, 0, 0, 0, 0, 0)$ and for $r = K\tilde{r}_0$ in bidirections $(0, 0, 0, 0, 0, v^{[3}w^{4]})$; for RN_{442} for $r = 2K\tilde{r}_0$ the space-time is flat in bidirections $(0, v^{[1}w^{3]}, v^{[1}w^{4]}, v^{[2}w^{3]}, v^{2}w^{4}, 0;)$ for $r > 5K\tilde{r}_0$ we have $RN \approx M$ and the space-time becomes the Minkowski space-time.

For RN_{422} we have following formulas: $\dim \operatorname{Ker} \overset{2}{R} = 2 \Longleftrightarrow$

$$\Longleftrightarrow \begin{cases} \overset{2}{R}{}^{14}_{\alpha\beta\varrho\sigma} \underset{1}{t}^{[\varrho} \underset{2}{t}^{\sigma]} = 0, \quad \overset{2}{R}{}^{23}_{\alpha\beta\varrho\sigma} \underset{1}{t}^{[\varrho} \underset{2}{t}^{\sigma]} = 0 \\[2ex] \overset{2}{R}{}^{\varrho\sigma}_{\alpha\beta 14} \underset{1}{c}^{[\varrho} \underset{2}{c}^{\sigma]} = 0, \quad \overset{2}{R}{}^{\varrho\sigma}_{\alpha\beta 23} \underset{1}{c}^{[\varrho} \underset{2}{c}^{\sigma]} = 0; \end{cases}$$

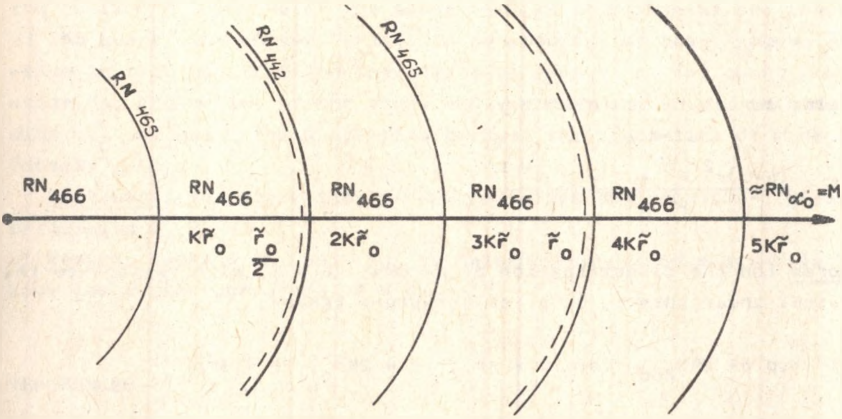
$$v^{[1}w^{2]},\ v^{[3}w^{4]},\ v^{[1}w^{3]},\ v^{[1}w^{4]},\ v^{[2}w^{3]},\ v^{[2}w^{4]},$$

$\underset{1}{t}^{[\varrho} \underset{2}{t}^{\sigma]}$, $\underset{1}{c}^{[\varrho} \underset{2}{c}^{\sigma]}$ any.

Proof. Proof results immediately from proof of theorem on the classification of RN space times.

Let us take $K = \frac{P}{4}$, $P > 1$ $(P = 1 + \varepsilon,\ \varepsilon$ small); therefore $K > \frac{1}{4}$, then $A \neq 0$ and $\frac{1}{A} \neq \infty$.

On the base of classification in (6.3) we have following diagram:



(6.4)

In monography [11] Professor Wichmann writes and reasons as follows: a treatment of the photon (quant) as a point particle is unfounded. From this it follows that it is reasonable to treat the photon as a non-point particle. Further, from this it follows that the photon (quant) has a radius $r_0 \neq 0$.

In the present paper I have based on the above statement of Professor Wichmann and I have introduced a theory of the photon (quant) arising from the wave in view of the space-time of Generalized Reissner-Nordstrom.

In monography [12] Professor Piekara writes as follows: how from the wave arises quant (photon)?? It is question, on which we expect the answer. When we this receive, then the precipice between classical physics and quantum physics will have covered.

In the paper [13] I have introduced a theory of the quant arising from the wave in view of the space-time of Reissner-Nordstrom. I have this delivered in Seminar on Differential Geometry at Institute of Matematics of Technical University of Szczecin 16th December 1988. During the discussion Professor Kucharzewski and Professor Żekanowski have said, that very important are properties of scalar curvature. Because scalar curvature for the space-time of Reissner-Nordstrom vanishes identically, therefore in the present paper I have introduced the space-time of Generalized Reissner-Nordstrom and I have given application of the space-time of p-Generalized Reissner-Nordstrom to the theory of the quant arising, as example of the application in the microscopic gravitational theory with gravitational field of p-GRN for microscopic particle (planckeon).

The metric of p- Generalized Reissner-Nordstrom space-times (in short p- GRN) is following:

$$d\sigma^2 = -A^p dt^2 + A^{p-2} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2); \; A = 1 - \frac{r_o}{r} + \frac{K\tilde{r}_o^2}{r^2},$$

$$\tilde{r}_o = d\tilde{r}_o.$$

The scalar curvature is following:

$$R = \frac{(p-1)r^2 \, A'' + 2 - 2\, A^{p-1}}{r^2 \, A^{p-1}}.$$

<u>Theorem</u> (on the classification of p- GRN, $p \neq 2$, with respect to the algebraical invariants a, b, c of curvature tensor)

1) typ p- GRN_{466} for $r \neq K\tilde{r}_o$, $r \neq 2K\tilde{r}_o$, $r \neq 3K\tilde{r}_o$

2) typ p- GRN_{465} for $r = K\tilde{r}_o$ or $r = 3K\tilde{r}_o$

3) typ p- GRN_{442} for $r = 2K\tilde{r}_o$

4) typ p- $GRN_{ooo} = M$ for $r > 5K\tilde{r}_o$ (p- GRN_{abc} are these p- GRN, which have the invariants a, b, c).

<u>Theorem</u>

$$\lim_{\varepsilon \to 0^+} R\left(\frac{\tilde{r}_o}{2}, K = \frac{1+\varepsilon}{4}\right) = +\infty, \quad \lim_{\substack{\alpha \to 0^+ \\ \alpha = p-1}} \left(\lim_{\substack{\varepsilon \to 0^+ \\ \alpha \neq 0}} R\left(K = \frac{1+\varepsilon}{4}\right)\right) = \delta\left(\frac{\tilde{r}_o}{2}\right),$$

where $\delta(\frac{\tilde{r}_o}{2})$ is Schwartz distribution of Dirac delta in $\frac{\tilde{r}_o}{2}$:

$$\delta\left(\frac{\tilde{r}_o}{2}\right) = \begin{cases} 0 & \text{for} \quad r \neq \frac{\tilde{r}_o}{2} \\ +\infty & \text{for} \quad r = \frac{\tilde{r}_o}{2}. \end{cases}$$

My proposition of application of p- GRN in the theory of the quant arising, as example in the microscopic gravitational theory with gravitational field of p- GRN is following: (on the base of the above two theorems and principle 1 of Prof. Wichmann):

Principles. 1) The quant (the photon) has the radius $r_0 \neq 0$, $\tilde{r}_0 = a \cdot r_0$. 2) The neighbourhood of the wave is the space-time provided with suitable metric tensor for which the curvature tensor is of two types at least (in the classification with respect to the invariants a, b, c of curvature tensor), this space-time tends to the space-time of Reissner-Nordstrom and in the neighbourhood inside the radius $\tilde{r}_0$ of photon the space-time has properties of the above two theorems and in the neighbourhood outside the radius $\tilde{r}_0$ of photon the space-time is of Minkowski one (nearly). 3) The quant (the photon) arises in consequency of the change of the space-time in this way, that on the inside of radius r_0 of quant the space-time has properties of the above two theorems and on the outside of radius $\tilde{r}_0$ of quant the space-time becomes the space-time of Minkowski (nearly).

Statement. The space-time p- GRN satisfies the conditions given in principles 1, 2, 3.

Remark. 1-GRN = RN and RN is Reissner-Nordstrom space-time for which have for scalar curvature R = 0.

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