ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1990

Seria: MATEMATYKA-FIZYKA z. 64 Nr kol. 1070

DEDICATED TO PROFESSOR MIECZYSŁAW KUCHARZEWSKI
WITH BEST WISHES ON HIS 70TH BIRTHDAY

Andrzej KAMIŃSKI

REMARKS ON THE CONVOLUTION OF DISTRIBUTIONS WITH COMPATIBLE SUPPORTS

Summary. It is shown that known results about the convolution of distributions, tempered distributions and Gel'fand-Shilov distributions in $K'(M_p)$-spaces with the supports which are compatible, polynomially compatible and (M_p)-compatible, respectively, remain true if the supports are meant in the sense of Łojasiewicz. Moreover, theorems on convergence of distributions of the respective classes with the supports compatible in a suitable sense are proved.

1. Various conditions for the existence of the convolution of locally integrable functions and the convolution of Schwartz distributions defined on R^d are known in the literature. In particular, there is a condition which guarantees the existence of the convolution independently of the growth of the convolution factors (functions or distributions), viz. it is expressed in terms of the supports of the convolution factors. The condition is called (Σ)-condition ([4], p. 383) or the condition of compatibility of supports (sets) in R^d ([12], [1], p. 124).

The following results are very well known: (a) if f and g are distributions with the supports in compatible sets, then the convolution $f*g$ exists and represents a distribution, (b) if f and g are locally integrable functions with the supports contained in compatible sets, then the convolution $f*g$ exists almost everywhere and represents a locally integrable function.

It was shown in [7] that the statement (a) can be reversed as follows: (a') if X,Y are two sets in R^d such that the convolution $f*g$ exists in the distributional sense for arbitrary distributions f and g with the supports contained in X and Y, respectively, then X and Y are compatible. To reverse the statement (b) a modification of the notion of compatibility was introduced in [9].

An analogous condition for the space of tempered distributions (polynomial compatibility of supports) was introduced in [5] (see also [6]) and for the space $K'(M_p)$ of Gel'fand-Shilov type ((M_p)-compatibility) in [15] (see also [16]). The results similar to the statements (a) and (a') were proved for tempered distributions in [5] (see also [6]) and [7] and for distributions of Gel'fand-Shilov type in [15] (see also [16]).

In all the results mentioned above the support of a distribution is meant in the classical sense, i.e. as the smallest closed set outside which the distribution vanishes. In case of locally integrable functions the following modification of the notion of support is natural: this is the smallest closed set outside which a given locally integrable function vanishes almost everywhere (see [13], p. 196 and [7]).

However, the notion of support of a distribution can be sharpened by using the concept of the Łojasiewicz value of a distribution at a point ([10]). Namely, by the support of a distribution in the sense of Łojasiewicz we shall mean the set of all points in R^d at which the value of the distribution does not exist or is different from 0.

We shall show that all the results mentioned above remain true if the supports of distributions are meant in the sense of Łojasiewicz.

Moreover, we shall prove the following results on convergence of sequences of distributions of a given class. Suppose that (f_n) and (g_n) are two sequences of distributions (tempered distributions, distributions of Gel'fand-Shilov type) such that the supports (in the classical or Łojasiewicz sense) of f_n are contained in a set A, the supports of g_n are contained in a set B for $n=1,2,\ldots$ and the sets A,B are compatible (almost compatible, (M_p)-compatible). If $f_n \longrightarrow f_o$ and $g_n \longrightarrow g_o$ in the sense of distributions (tempered distributions, distributions of Gel'fand-Shilov type), then $f_n * g_n \longrightarrow f_o * g_o$ in the respective sense.

2. For arbitrary sets $A,B \subset R^d$ we use the following notation:

$$A \pm B = \{x \pm y:\ x \in A,\ y \in B\};$$

$$S^x(A,B) = \{y \in R^d:\ x - y \in A,\ y \in B\};$$

$$A^{\Delta} = \{(x,y) \in R^{2d}:\ x + y \in A\}.$$

For given $x = (\xi_1,\ldots,\xi_d)$, $y = (\eta_1,\ldots,\eta_d) \in R^d$ and $\lambda \in R^1$, we use the standard notation:

$$x \pm y = (\xi_1 \pm \eta_1,\ldots,\xi_d \pm \eta_d);$$

$$\lambda x = (\lambda\xi_1,\ldots,\lambda\xi_d);$$

$$|x| = \sqrt{\xi_1^2 + \ldots + \xi_d^2}.$$

By intervals in R^d we mean sets of the form $I = I_1 \times \ldots \times I_d$, where $I_i = [\alpha_i, \beta_i]$ are closed intervals in R^1.

We shall prove the following theorem:

Proposition 1. Let $X,Y \subset R^d$. The following conditions are equivalent:

(i) for every interval $I \subset R^d$ there is an interval $J \subset R^d$ such that $S^x(X,Y) \subset J$ for all $x \in I$, (i′) for every bounded set $K \subset R^d$ the set $(K-X) \cap Y$ is bounded in R^d;

(ii) for every interval $I \subset R^d$ there is an interval $J \subset R^d$ such that $S^x(Y,X) \subset J$ for al $x \in I$;

(ii′) for every bounded set $K \subset R^d$ the set $X \cap (K-Y)$ is bounded in R^d;

(iii) for arbitrary sequences $(x_n) \subset X$, $(y_n) \subset Y$, $|x_n| + |y_n| \to \infty$ implies $|x_n+y_n| \to \infty$;

(iii′) for every bounded set $K \subset R^d$ the set $(X\times Y) \cap K^\Delta$ is bounded in R^{2d}.

Remark 1. Conditions (i), (ii), (iii) were formulated and proved to be equivalent in [12] (see also [1], p. 125).

Remark 2. Conditions (i′), (ii′), (iii′) with the word "bounded" replaced by the word "compact" are well known and they are sometimes called (Σ)-conditions (see [4], p. 383). They are matched to supports (of distributions or functions) in the classical sense which are closed sets. Clearly, if X and Y are closed subsets of R^d, then the two formulations with the words "bounded" and "compact", respectively, of any of conditions (i), (ii′), (iii′) are equivalent. This follows from the following facts:

1° a set in R^d is compact if and only if it is bounded and closed;

2° if $A \subset R^d$ is closed, then $A^\Delta \subset R^{2d}$ is closed;

3° if $A \subset R^d$ is compact and $B \subset R^d$ is closed, then $A-B \subset R^d$ is closed.

In the case of supports in the sense of Łojasiewicz which are not closed sets, the above formulations of conditions (i′), (ii′), (iii′) are more appropriate.

Below we shall give a complete proof of the equivalence of all conditions (i) - (iii), (i′) - (iii′).

Proof of Proposition 1.

(i) $\Longrightarrow$ (i′) It is enough to notice that $S^x(X,Y) = (x-X) \cap Y$.

(i′) $\Longrightarrow$ (iii′) Let $K \subset R^d$ be bounded. Then the sets $M = (K-X) \cap Y$ and $L = K - M$ are bounded and

$$(X \times Y) \cap K^\Delta \subset L \times M,$$

i.e. the set $(X \times Y) \cap K^\Delta$ is bounded, as desired.

(iii′) $\Longrightarrow$ (iii) Suppose that (iii) does not hold, i.e. there are sequences $(x_n) \in X$ and $(y_n) \in Y$ and a positive $m > 0$ such that $|x_n| + |y_n| \to \infty$ and $|x_n+y_n| \leqslant m$ for $n \subset N$. Put $K = [-m,m]^d$. We have

$x_n + y_n \in K$ or, equivalently, $(x_n, y_n) \in (X \times Y) \cap K^{\Delta}$ for $n \in N$, which means that the set $(X \times Y) \cap K^{\Delta}$ is unbounded in R^{2d} while K is bounded in R^d. This contradicts (iii′).

(iii) $\Longrightarrow$ (i) Suppose that (i) does not hold. Then there exist an interval $I \subset R^d$ and sequences $(x_n) \subset I$ and $(y_n) \subset Y$ such that

$$\bar{x}_n = x_n - y_n \in X \quad \text{and} \quad |y_n| > n$$

for all $n \in N$. Hence $|\bar{x}_n| + |y_n| \to \infty$ and the sequence $(|\bar{x}_n + y_n|)$ is bounded. Contradiction.

The proof of the following chain of implications is similar:

$$\text{(ii)} \Longrightarrow \text{(ii}'\text{)} \Longrightarrow \text{(iii}'\text{)} \Longrightarrow \text{(ii)}.$$

Hence all the conditions in Theorem 1 are equivalent.

Definition 1. Sets $X, Y \subset R^d$ are said to be compatible if one of conditions (i) - (iii), (i′) - (iii′) holds (cf. [12], [1], p. 124).

Proposition 2 (cf. [7]). Let $X, Y \subset R^d$. The following conditions are equivalent:

(I) there exists a polynomial p of real variable such that

$$S^x(X,Y) \subset [-p(|x|),\ p(|x|)] \tag{1}$$

for all $x \in R^d$;

(II) there exists a polynomial p of real variable such that

$$|x| + |y| \leqslant p(|x+y|)$$

for arbitrary $x \in X$ and $y \in Y$;

(III) there exists a $k \in N$ and a $c > 0$ such that

$$|x| + |y| \leqslant c(1+|x+y|)^k$$

for arbitrary $x \in X$ and $y \in Y$.

Proof.

(I) $\Longrightarrow$ (II) Suppose that (I) holds for some polynomial of real variable and let $x \in X$, $y \in Y$. Put $\bar{x} = x + y$. Since $\bar{x} - y \in X$ and $y \in Y$, we get

$$|y| \leqslant p(|\bar{x}|),$$

in view of (1). Hence

$$|x| + |y| \leqslant |x+y| + |-y|+|y| \leqslant |x+y| + 2p(|\bar{x}|) \leqslant q(|x+y|),$$

where q is the plynomial defined by $q(t) = 2p(t)+t$ for $t \in R^1$. The implication is proved.

(II) $\Longrightarrow$ (III) It is enough to notice that for every polynomial p of real variable there exist a $k \in N$ and a $C > 0$ such that

$$p(t) \leqslant C(1 + 1)^k \quad \text{for} \quad t \in R^1.$$

(III) $\Longrightarrow$ (I) Suppose that (III) holds for some $k \in N$, $C > 0$ and put $p(t) = C\cdot 2^k(1+t^2)^k$. Assume that $x - y \in X$ and $y \in Y$. Then we have

$$|y| \leqslant |x-y| + |y| \leqslant C(1+|x|)^k \leqslant p(|x|),$$

which proves (1) and completes the whole proof.

Definition 2. Sets $X,Y \subset R^d$ are said to be polynomially compatible if one of conditions (I) - (III) holds (cf. [5] and [6]).

Now, consider a sequence of continuous functions $M_p : R^d \to [1,\infty)$ satisfying the following conditions:

(a) $1 \leqslant M_p(x) \leqslant M_{p+1}(x)$ for $x \in R^d$; and $p \in N$;

(b) for each $j=1,\dots,d$ and $p \in N$ there is a $C_{pj} > 1$ such that $M_p(\xi_1,\dots,\xi_j',\dots,\xi_d) \leqslant C_{pj}M_p(\xi_1,\dots,\xi_j'',\dots\xi_d)$, whenever $|\xi_j'| \leqslant |\xi_j''|$ and $\xi_j' \dots \xi_j'' \geqslant 0$;

(c) for each $p \in N$ there exists a $q \in N$ such that $M_pM_q^{-1} \in L^1$ and $\lim\limits_{|x|\to\infty} M_p(x)M_q^{-1}(x) = 0$;

(d) for each $p \in N$ there exist a $q \in N$ and a constant $C_p > 0$ such that $M_p(x+y) \leqslant C_pM_q(x)M_q(y)$ for $x,y \in R^d$;

(e) for each $p \in N$ there is a $q \in N$ and a $C_p > 0$ such that $M_p(-x) \leqslant C_pM_q(x)$ for $x \in R^d$;

(f) for each $p \in N$ there is a $q \in N$ such that $M_p^2(x) \leqslant F_pM_q(x)$ for $x \in R^d$;

(g) there exist a $p \in N$ and a $C_p > 0$ such that $|x| \leqslant C_pM_p(x)$ for $x \in R^d$;

Conditions (a) - (c) were assumed already by Gel'fand, Shilov and Vilenkin in [2], [3]. Condition (d) together with the assumption that M_p are even (i.e. a stronger condition than (e)) is usually postulated when the convolution of the distributions in $K'(M_p)$ are studied (see e.g. [11]). Conditions (f) and (g) appear in [15].

Proposition 3. Let $X, Y \subset R^d$. The following conditions are equivalent

(A) for each $p \in N$ there exist a $q > N$ and a constant $A_p > 0$ such that

$$M_p(y) \leqslant A_p M_q(x) \quad \text{for all} \quad y \in S^x(X,Y) \quad \text{and} \quad x \in R^d;$$

(B) for each $p \in N$ there are a $q \in N$ and a constant $B_p > 0$ such that

$$M_p(x)M_p(y) \leqslant B_p M_q(x+y) \quad \text{for all} \quad x \in X \quad \text{and} \quad y \in Y.$$

Proof. Assume that (A) holds and fix $p \in N$. For this p choose a suitable index q_o, according to (d). Now applying subsequently conditions (e), (f), (A) end again (f), we match and index q_1 to q_o, then an index q_3 to $q_2 = \max(p, q_1)$, then a q_4 to q_3 and finally an index q to $q_5 = \max(q_o, q_4)$, respectively.

Take arbitrary $x \in X$ and $y \in Y$. Clearly, $y \in S^{x+y}(X,Y)$. Hence, in view of the conditions just mentioned, we have

$$M_p(x)M_p(y) \leqslant D_p M_{q_o}(x+y) M_{q_o}(-y) M_p(y)$$

$$\leqslant D_p E_{q_o} M_{q_o}(x+y) M_{q_1}(y) M_p(y)$$

$$\leqslant D_p E_{q_o} F_{q_2} M_{q_o}(x+y) M_{q_3}(y)$$

$$\leqslant D_p E_{q_o} F_{q_2} A_{q_3} M_{q_o}(x+y) M_{q_4}(x+y)$$

$$\leqslant D_p E_{q_o} F_{q_2} A_{q_3} F_{q_5} M_q(x+y),$$

i.e. (B) holds.

Now, suppose that condition (B) is valid and let $p \in N$. According to (B), find a suitable $q \in N$ and a constant $B_p > 0$ such that the respective inequality holds. Let $y \in S^x(X,Y)$ for an arbitrary $x \in R^d$, i.e. $x-y \in X$ and $y \in Y$. By (a), we have

$$M_p(y) \leqslant M_p(x-y)M_p(y) \leqslant B_p M_q(x),$$

i.e. (A) holds.

The proof is completed.

Definition 3. Sets $X,Y \subset R^d$ are said to be (M_p)-compatible if one of conditions (A) - (B) holds.

Remark 3. If sets $X,Y \subset R^d$ are polynomially compatible, then they are compatible. This results, for instance, from the inclusion (1), because $\sup\{p(|x|) : x \in I\} < \infty$ for an arbitrary polynomial p and a compact set (interval) $I \subset R^d$. The converse implication does not hold (see [5],[6]).

Remark 4. The original definition of (M_p)-compatible sets was given by J. Uryga in [13] and [14] in the form of condition (B). He also proved in [13] and [14] that (M_p)-compatibility implies compatibility.

Remark 5. If $M_p(x) = (1+|x|)^k$, then (M_p)-compatibility is equivalent to polynomial compatibility of sets in R^d (see [13]).

3. We shall use the standard notation and classical properties of distributions, tempered distributions and distributions of the spaces $K'(M_p)$ of Gel'fand-Shilov type. We refer the reader to the books [14], [2] and [3] for details.

In the whole paper, we shall consider distributions defined on R^d.

Given a distribution $f \in \mathcal{D}'$, by $s(f)$ we shall denote the support of f in the classical sense, i.e. the smallest closed set in R^d outside which $f = 0$. By $s_Ł(f)$, we shall denote the support of f in the sense of Łojasiewicz, i.e. the set of all $x \in R^d$ for which the Łojasiewicz value of f at x does not exist or exists and differs from 0.

The following proposition is crucial for our considerations and it is based on results of the paper [10].

Proposition 3. For an arbitrary $f \in \mathcal{D}'$, we have

$$cl(s_Ł(f)) = s(f).$$

Proof. Evidently, if $x \notin s(f)$, then $f = 0$ in some neighbourhood of x, so the Łojasiewicz value $f(x)$ is 0, by Théorème in [10], part 2.3, i.e. $s_Ł(f) \subset s(f)$ and thus $cl(s_Ł(f)) \subset s(f)$.

Suppose, conversely, that $x \notin cl(s_Ł(f))$. Then the Łojasiewicz values $f(y)$ at all points y from some neighbourhood of x are equal 0. By Corollaire 3 in [10], part 5.2, $f = 0$ in this neighbourhood, i.e. $x \notin s(f)$ and the implication $s(f) \subset cl(s_Ł(f))$ holds as well.

We shall need the following properties of compatible, polynomially compatible and (M_p)-compatible sets:

Proposition 4. If X,Y are compatible, polynomially compatible or (M_p)-compatible sets in R^d and $U \subset X$, $V \subset Y$, then U,V are also compatible, polynomially compatible, (M_p)-compatible sets, respectively.

Proof. The proof follows directly from the definitions.

For a given non-empty set A in R^d and $\alpha > 0$ let

$$A_\alpha = \left\{x \in R^d: d(x,A) = \inf\left\{|x-y| : y \in A\right\} < \alpha\right\}.$$

In addition, we may adopt $\emptyset_\alpha = \emptyset$ for every $\alpha > 0$.

Proposition 5. If X,Y are compatible, polynomially compatible or (M_p)-compatible sets in R^d, then for arbitrary $\alpha, \beta > 0$ the sets X_α, Y_β and the sets $cl(X)$, $cl(Y)$ are compatible, polynomially compatible and (M_p)-compatible, respectively.

Proof. In case of the sets X_α, Y_β, the assertion is proved: in [1], p. 126, for compatible sets X,Y; in [5] for polynomially compatible sets X,Y; and in [15] for (M_p)-compatible sets X,Y. Since $A_\alpha \supset cl(A)$ for arbitrary $A \subset R^d$ and $\alpha > 0$, the second assertion is a consequence of Proposition 4.

From among many equivalent definitions of the convolution let us choose the definition of V.S. Vladimirov (see [18], p. 62) and modify it to the case of tempered distributions and distributions of $K'(M_p)$--spaces.

Recall that by a unit-sequence we mean a sequence of functions $\eta_n \in \mathcal{D}$ such that

1^o $\{x; \eta_n(x) = 1\} \nearrow R^d$;

2^o for each $k \in N^d$ there exists a $C_k > 0$ such that

$$\int_{R^d} |D^k \eta_n(x)|\, dx < C_k.$$

Let $f,g \in \mathcal{D}'$; $f,g \in \mathcal{S}'$; or $f,g \in K'(M_p)$, respectively. We say that the convolution $f*g$ exists in $\mathcal{D}'$; $\mathcal{S}'$; or $K'(M_p)$, respectively if the limit

$$\lim_{n\to\infty} \langle f(x) \otimes g(y),\ \eta_n(x,y)\varphi(x+y)\rangle$$

exists for every unit-sequence (η_n) and every $\varphi \in \mathcal{D}$; every $\varphi \in \mathcal{S}$; or every $\varphi \in K(M_p)$, respectively. Then the limit defines the values of the linear continuous functional $f*g$ on the respective spaces.

The following results about the convolution $f*g$ of distributions f and g with compatible supports, tempered distributions with polynomially compatible supports (cf. [7]) and distributions in $K'(M_p)$ with (M_p)-compatible supports (c.f [16]) were proved in [7] and [16] in the case of supports $s(f)$, $s(g)$. By Propositions 3 and 5, it follows immediately that these results are true also in the case of supports $s_t(f)$, $s_t(g)$.

Theorem 1. Let $f,g \in \mathcal{D}'$ and suppose that $s_{Ł}(f)$, $s_{Ł}(g)$ are compatible sets in R^d. Then $f*g$ exists in $\mathcal{D}'$.

Conversely, let X,Y be sets in R^d such that $f*g$ exists in $\mathcal{D}'$ for arbitrary $f,g \in \mathcal{D}'$ such that $s_{Ł}(f) \subset X$ and $s_{Ł}(g) \subset Y$. Then X,Y are compatible.

Theorem 2. Let $f,g \in \mathcal{S}'$ and suppose that $s_{Ł}(f)$, $s_{Ł}(g)$ are polynomially compatible sets in R^d. Then $f*g$ exists in $\mathcal{S}'$.

Conversely, let $X,Y \subset R^d$ and $f*g$ exists in $\mathcal{S}'$ for all $f,g \in \mathcal{S}'$ such that $s_{Ł}(f) \subset X$, $s_{Ł}(f) \subset Y$. Then X,Y are polynomially compatible.

Theorem 3. Let $f,g \in K'(M_p)$ and suppose that $s_{Ł}(f)$, $s_{Ł}(g)$ are (M_p)-compatible sets in R^d. Then $f*g$ exists in $K'(M_p)$.

Conversely, let $X,Y \subset R^d$ and $f*g$ exists in $K'(M_p)$ for all $f,g \in K'(M_p)$ such that $s_{Ł}(f) \subset X$, $s_{Ł}(g) \subset Y$. Then X,Y are (M_p)-compatible.

4. In [1] (p. 158), the following result on continuity of the convolution of distributions restricted by the condition of compatibility was proved in the case of supports in the classical sense.

By Proposition 3 and 5, we obtain a similar result for supports in the sense of Łojasiewicz: .

Theorem 4. Suppose that $f_n \longrightarrow f_0$ in $\mathcal{D}'$, $g_n \longrightarrow g_0$ in $\mathcal{D}'$ and $s_{Ł}(f_n) \subset X$, $s_{Ł}(g_n) \subset Y$ for $n \in N$, where X and Y are compatible sets in R^d. Then $f_n * g_n \longrightarrow f_0 * g_0$ in $\mathcal{D}'$.

Analogous results hold also for tempered distributions and distributions in $K'(M_p)$. For the proof we need a characterization of the convergence in $K'(M_p)$.

Under conditions (a), (M), (N), (P) (equivalent to our conditions (a) - (c)), I.M. Gel'fand and G.E. Shilov gave in [2] a description of distributions of the class $K'(M_p)$ which can be formulated as follows:

Proposition 6 (see [2], p. 113). $f \in K'(M_p)$ if and only if there exist an index $p \in N$ and bounded measurable functions F_i on R^d (where i is a multi-index with $|i| \leq p$) such that

$$f = \sum_{|i| \leq p} D^i(M_p F_i) \tag{2}$$

in the distributional sense.

Under the same conditions, L. Kitchens and C. Swartz gave in [11] the following characterization of the convergence in $K'(M_p)$:

Proposition 7 (see [11]). $f_n \longrightarrow 0$ in $K'(M_p)$ if and only if there exist a $p \in N$ and functions $F_{ni} \in L^2$ on R^d (where $n \in N$ and i is a multi-index with $|i| \leq p$) such that

$$f_n = \sum_{|i| \leq p} D^i(M_p F_{ni}) \quad \text{for} \quad n \in N \tag{3}$$

in the distributional sense, and

$$F_{ni} \longrightarrow 0 \quad \text{in} \quad L^2 \quad \text{as} \quad n \to \infty . \tag{4}$$

The above characterizations can be simplified if the additional condition (N′) is assumed (see [8]). Then the sums in (2) and (3) can be reduced to a single derivatives, the F's in (2) and (3) can be assumed to be continuous functions of the class L^α with an arbitrary α, $1 \leq \alpha \leq \infty$, and the convergence in (4) can be assumed in any L^α, $1 \leq \alpha \leq \infty$.

It was shown by J. Uryga in [17] that condition (N′) cannot be omitted to obtain a single derivative representation instead of (2) and (3). We shall show, however, that the following more convenient forms of Propositions 6 and 7 can be obtained even without condition (N′).

Proposition 6′. $f \in K'(M_p)$ if and only if there exist a $p \in N$ and continuous (or, equivalently, measurable) functions $F_i \in L^\alpha$ for every (or, equivalently, some) α with $1 \leq \alpha \leq \infty$ such that (2) holds.

Proposition 7′. The following conditions are equivalent:

(i) $f_n \longrightarrow 0$ in $K'(M_p)$;

(ii) there exist a $p \in N$ and continuous (or, equivalently, measurable) functions $F_{ni} \in L^\alpha$ for $n \in N$ such that (3) holds end $F_{ni} \longrightarrow 0$ in L^α as $n \to \infty$ for every (or, equivalently some) α with $1 \leq \alpha \leq \infty$,

(iii) there exist a $p \in N$ and continuous (or, equivalently, measurable) functions $F_{ni} \in L^\infty$ for $n \in N$ such that (3) holds and $F_{ni} \longrightarrow 0$ pointwise (almost everywhere) in R^d and F_{ni} are commonly bounded (almost everywhere) in R^d.

Proof of Proposition 6′. It suffices to show that if $F \in L^\alpha$ for some $1 \leq \alpha \leq \infty$, then for each $p \in N$ there exists an $r \in N$ such that the function

$$G_r(x) = M_r^{-1}(x) \int_0^x M_o(t)F(t)dt \qquad (x \in R^d)$$

belongs to all the classes L^β for $1 \leq \beta \leq \infty$.

For a given $p \in N$ select a $q \in N$ such that $M_q^{-1}M_p \in L^1$ and then select an $r \in N$ such that $M_r^{-1}M_q \in L^1$, according to condition (c). In view of (b), we have

$$\|G_q\|_\infty = \sup\left\{|G_q(x)| : x \in R^d\right\} \leq B_p \int_{R^d} |(M_q^{-1}M_pF)(x)|\,dx \leq A_\alpha B_p \|F\|_\alpha < \infty, \tag{5}$$

where $B_p = B_{p_1} \cdot \ldots \cdot B_{p_d}$ (B_{p_j} are constant from condition (b)) and

$$A_\alpha = \| M^{-1} M_p \|_{\alpha'}$$

with α' such that $1/\alpha + 1/\alpha' = 1$. Note that $A_\alpha < \infty$ for an arbitrary α, $1 \le \alpha \le \infty$, because $A_1 = \| M_q^{-1} M_p \|_\infty < \infty$ and

$$(A_\alpha)^{\alpha'} = (\| M_q^{-1} M_p \|_{\alpha'})^{\alpha'} \le (\| M_q^{-1} M_p \|_\alpha)^{\alpha'-1} \| M_q^{-1} M_p \|_1 < \infty ,$$

in view of continuity of $M_q^{-1} M_p$ and condition (c).

Now, we have

$$\| G_r \|_1 = \| (M_r^{-1} M_q) G_q \|_1 \le \| M_r^{-1} M_q \|_1 \cdot \| G_q \|_\infty < \infty, \tag{6}$$

$$(\| G_r \|_\beta)^\beta \le (\| M_r^{-1} M_q \|_\infty)^{\beta-1} \| M_r^{-1} M_q \|_1 \cdot (\| G_q \|_\infty)^\beta < \infty \tag{7}$$

for $1 < \beta < \infty$, and

$$\| G_r \|_\infty \le \| M_r^{-1} M_q \|_\infty \cdot \| G_q \|_\infty < \infty. \tag{8}$$

Consequently, $G_r \in L^\beta$ for all β such that $1 \le \beta \le \infty$.

Proof of Proposition 7$'$. The equivalence of conditions (i) - (ii) follows from inequalities (5) - (8). It remains to show that if $F_n \in L^\infty$, $F_n \to 0$ almost everywhere in R^d and F_n are commonly bounded almost everywhere, then, given $p \in N$,

$$G_n^r(x) = M_r^{-1}(x) \int_0^x M_p(t) F_n(t) dt \rightrightarrows 0 \quad \text{on} \quad R^d \tag{9}$$

as $n \to \infty$ for suitably chosen $r \in N$.

Notice that if $q \in N$ is selected to satisfy the relation $M_q^{-1} M_p \in L^1_d$, then $G_n^q(x) \to 0$ as $n \to \infty$ uniformly on every compact set in R^d, by virtue of condition (b) and the Lebesgue dominated convergence theorem. Now, select an $r \in N$ such that $M_r^{-1}(x) M_q(x) \to 0$ as $|x| \to \infty$. It is easy to see that (9) holds for r chosen in that way.

The proof is therefore completed.

As a simple consequence of Propositions 6$'$ and 7$'$, we get the following characterization of the convergence $f_n \to f$ in $K'(M_p)$ for an arbitrary $f \in K'(M_p)$.

Corollary 1. The following conditions are equivalent:

(i) $f_n \to f$ in $K'(M_p)$;

(ii) there exist a $p \in N$ and continuous (equivalently, measurable) functions $F_{ni} \in L^{\alpha}$ for $n \in N_o$ and $|i| \leq p$ such that (3) holds for $n \in N_o$ and $F_{ni} \to F_{no}$ in L^{α} as $n \to \infty$ for every (equivalently, some) α with $1 \leq \alpha \leq \infty$;

(iii) there exist a $p \in N$ and continuous (equivalently, measurable) functions $F_{ni} \in L^{\infty}$ for $n \in N_o$ and $|i| \leq p$ such that (3) holds for $n \in N_o$, $F_{ni} \to F_{no}$ pointwise (almost everywhere) and F_{ni} are commonly bounded (almost everywhere) in R^d.

The next corollary results directly from the above statement.

Corollary 2. If $f_n \to f$ in $K'(M_p)$ and $g_n \to g$ in $K'(M_p)$, then

$$\alpha D^k f_n + \beta D^m g_n \to \alpha D^k f + \beta D^m g$$

for arbitrary $\alpha, \beta \in R^1$ and $k, m \in N_o^d$.

The characterization of the convergence in $K'(M_p)$ formulated above will be useful in the proof of the following analogue of Theorem 4, true both in the case of supports in the classical and Łojasiewicz sense.

Theorem 5. Suppose that $f_n \to f_o$ in $K'(M_p)$, $g_n \to g_o$ in $K'(M_p)$ and $s(f_n) \subset X$, $s(g_n) \subset Y$ $[s_Ł(f_n) \subset X,\ s_Ł(g_n) \subset Y]$ for $n \in N$, where X and Y are (M_p)-compatible sets in R^d. Then $f_n * g_n \to f_o * g_o$ in $K'(M_p)$.

Proof. By Propositions 3 and 5, it suffices to prove our assertion for supports in the sense of Łojasiewicz.

Let ω we a smooth function on R^d vanishing for $|x| > \rho > 0$ such that

$$\int_{R^d} \omega(t)\,dt = 1.$$

Put

$$\Phi = \chi_X * \omega \quad \text{and} \quad \Psi = \chi_Y * \omega,$$

where χ_X and χ_Y are characteristic functions of the sets X and Y, respectively.

It is easy to see that $D^{\alpha}\Phi$ and $D^{\beta}\Psi$ are bounded functions for $\alpha, \beta \in N^d$. Moreover

$$\Phi(x) = 1 \quad \text{for} \quad x \in X_{\rho};\ \Phi(x) = 0 \quad \text{for} \quad x \notin X_{3\rho} \tag{10}$$

and

$$\Psi(x) = 1 \quad \text{for} \quad x \in Y_{\rho};\ \Psi(x) = 0 \quad \text{for} \quad x \notin Y_{3\rho}. \tag{11}$$

By the assumptions, it follows that there exist $p,q \in N$ and continuous functions $F_{nk} \in L^{\infty}$, $G_{nl} \in L^{\infty}$ for $|k| \leq p$, $|l| \leq q$ and $n \in N_o$ such that

$$f_n = \sum_{|k|\leq p} D^k\left[M_p F_{nk}\right], \quad g_n = \sum_{|l|\leq q} D^l\left[M_q G_{nl}\right] \tag{12}$$

for $n \in N_o$;

$$F_{nk} \longrightarrow F_o, \; G_{nl} \longrightarrow G_o \quad \text{pointwise as} \quad n \to \infty \tag{13}$$

for $|k| \leq p$, $|l| \leq q$; and

$$F_{nk}, \; G_{nl} \text{ are commonly bounded.} \tag{14}$$

Applying the Leibniz formula, we have

$$f_n = f_n\Phi = \sum_{|k|\leq p} \sum_{0\leq i\leq k} (-1)^i \binom{k}{i} D^{k-i}\left[M_p F_{nk}(D^i\Phi)\right] \tag{15}$$

and

$$g_n = g_n\Psi = \sum_{|l|\leq q} \sum_{0\leq j\leq l} (-1)^j \binom{l}{j} D^{l-j}\left[M_q G_{nl}(D^j\Psi)\right] \tag{16}$$

for $n \in N_o$, where the symbols $0 \leq i \leq k$, $(-1)^i$, $\binom{k}{i}$ have the d-dimensional interpretation (see e.g. [1], pp. 261-267).

Put

$$\bar{F}_{nki} = M_p F_{nk}(D^i\Phi), \quad \bar{G}_{nlj} = M_q G_{nl}(D^j\Psi)$$

for $k,l,i,j \in N_o^d$ with $|k| \leq p$, $|i| \leq p$, $|l| \leq q$, $|j| \leq q$ and $n \in N_o$.

By virtue of (10) and (11), the functions $\bar{F}_{nki}$ and $\bar{G}_{nlj}$ for $n \in N$ have supports contained in the sets $X_{3\varrho}$ and $Y_{3\varrho}$, which are (M_p)-compatible, owing to Proposition 5.

Note that for arbitrary k,i,l,j there exist constants $C_{kilj} > 0$ such that

$$|\bar{F}_{nki}(x-t)\bar{G}_{nlj}(t)| \leq C_{kilj} M_p(x-t) M_q(t) \chi_{X_{3\varrho}}(x-t) \chi_{Y_{3\varrho}}(t) \tag{17}$$

for all $x,t \in R^d$, according to boundedness of the functions $D^i\Phi$, $D^j\Psi$ and common boundedness of F_{nk} and G_{nl}.

On the other hand, there exist an $r \in N$ and a constant $C_{pq} > 0$ such that

$$\int_{R^d} M_p(x-t)M_q(t)\chi_{X_{3\varrho}}(x-t)\chi_{Y_{3\varrho}}(t)dt = \int_{S^x(X_{3\varrho},Y_{3\varrho})} M_p(x-t)M_q(t)dt$$

$$\leqslant C_{pq}\, M_r(x)1^d(S^x(X_{3\varrho},Y_{3\varrho})), \tag{18}$$

in view of conditions (d), (e), (f), (A) ((M_p)-compatibility of the sets $X_{3\varrho}$ and $Y_{3\varrho}$) and again (f), where 1^d is the Lebesgue measure in R^d.

Notice that, by virtue of (g) and (A), there are indices $r_1, r_2 \in N$ and constants G_{r_1}, A_{r_1} 0 such that

$$|y| \leqslant G_{r_1} M_{r_1}(y) \leqslant A_{r_1} M_{r_2}(x)$$

for $y \in S^x(X_{3\varrho},Y_{3\varrho})$, $x \in R^d$. Hence, by (18) and condition (f), we see that the integral over R^d of the right hand side of (17) is bounded by $A\ M_s(x)$ for some $s \in N$ and $A > 0$.

By the Lebesque dominated convergence theorem, we get

$$(\overline{F}_{nki} * \overline{G}_{nlj})(x) \longrightarrow (\overline{F}_{oki} * \overline{G}_{olj})(x)$$

for each $x \in R^d$. Moreover

$$\overline{F}_{nki} * \overline{G}_{nlj} = M_s M_s^{-1}(\overline{F}_{nki} * \overline{G}_{nlj})$$

for $n \in N_o$ and the functions

$$M_s^{-1}(\overline{F}_{nki} * \overline{G}_{nlj}) \quad (n \in N)$$

are commonly bounded. In view of Corollary 1 (condition (iii))

$$\overline{F}_{nki} * \overline{G}_{nlj} \longrightarrow \overline{F}_{oki} * \overline{G}_{olj} \quad \text{in} \quad K'(M_p) \tag{19}$$

as $n \to \infty$ for all k,i,l,j such that $0 \leqslant i \leqslant k$, $0 \leqslant j \leqslant l$, $|k| \leqslant p$, $|l| \leqslant q$. By the known properties of the convolution and by Corollary 2, we conclude from (15), (16) and (19) that

$$f_n * g_n \longrightarrow f_o * g_o \quad \text{in} \quad K'(M_p)$$

as $n \to \infty$, which was to be proved.

Since $\mathcal{S}'$ is a $K'(M_p)$-space, generated by the sequence $M_p(x) = (1+|x|)^p$ which satisfies all our conditions (a) - (g), and (M_p)-compatibility is equivalent to polynomial compatibility in this case, we obtain the following corollary from Theorem 5.

Theorem 6. Suppose that $f_n \longrightarrow f_o$ in $\mathcal{S}'$, $g_n \longrightarrow g_o$ in $\mathcal{S}'$ and $s(f_n) \subset X$, $s(g_n) \subset Y$ $[s_t(f_n) \subset X,\ s_t(g_n) \subset Y]$ for $n \in N$, where X and Y are polynomially compatible sets in R^d. Then $f_n * g_n \longrightarrow f_o * g_o$ in $\mathcal{S}'$.

REFERENCES

[1] P. Antosik, J. Mikusiński, R. Sikorski: Theory of distributions. The sequential approach, PWN-Elsevier, Warszawa-Amsterdam, 1973.

[2] I.M. Gel'fand, G.E. Shilov: Generalized Functions, vol. 2 and 3, Academic Press, New York - London 1964-1967.

[3] I.M. Gel'fand, N.Ya. Vilenkin: Generalized Functions, vol. 4, Academic Press, New York - London 1967.

[4] J. Horvath: Topological Vector Spaces and Distributions, vol. I, Addison-Vesley, Reading 1966.

[5] A. Kamiński, Operacje regularne i nieregularne na dystrybucjach [Regular and irregular operations on distributions], Preprint 11, Ser. B, Institute of Mathematics, Polish Academy of Sciences, Warszawa 1981.

[6] A. Kamiński: On convolutions, products and Fourier transforms of distributions, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 25(1977), 369-374.

[7] A. Kamiński: On the Rényi theory of conditional probabilities, Studia Math. 79(1984), 151-191.

[8] A. Kamiński: Remarks on $K\{M_p\}'$-spaces, Studia Math. 77(1984), 499--508.

[9] A. Kamiński: On almost compatible supports of locally integrable functions, Zesz. Nauk. Pol. Śl. Ser. Mat.-Fiz. 48(1986), 81-90.

[10] S. Łojasiewicz: Sur la valeur et la limite d'une distribution en un point, Studia Math. 16(1957), 1-36.

[11] L. Kitchens, C. Swartz: Convergence in the dual of certain $K\{M_p\}$--spaces, Colloq. Math. 30(1974), 149-155.

[12] J. Mikusiński: Sequential theory of the convolution of distributions, Studia Math. 29(1968), 151-160.

[13] J. Mikusiński: The Bochner Integral, Birkhäuser, Bassel-Stuttgart 1978.

[14] L. Schwartz: Theorie des distributions, Hermann, Paris 1966.

[15] J. Uryga: O pewnym kryterium istnienia splotu funkcji uogólnionych w przestrzeniach typu $K\{M_p\}'$. [On some criterion of the existence of the convolution of generalized function in spaces of $K\{M_p\}'$-type], Preprint 15, Ser. B, Institute of Mathematics, Polish Academy of Sciences, Warszawa 1986.

[16] J. Uryga: On compatibility of supports of generalized functions of Gelfand-Shilov type, Bull. Pol. Ac.: Math. 36 (1988), 143-150.

[17] J. Uryga: On representations of generalized functions of Gelfand--Shilov type, Bull. Acad. Pol. Ac.: Math. 36(1988), 229-238.

[18] V.S. Vladimirov: Generalized Functions in Mathematical Physics, Mir, Moscow 1979.