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PURELY DIFFERENTIEL GEOMETRIC OBJECTS OF THE TYPE [3,2,1] WITH A LINEAR NON-HOMOGENOUS TRANSFORMATION FORMULA

The purpose of the present paper is to determine all purely differential geometric objects of first class with three components in a two-dimensional space, i.e. according to the terminology of J. Aczél and S. Gołąb, objects of the type [3,2,1] (cf. [2] p. 15)) of a linear non-homogenous transformation rule:

$$\begin{aligned} \omega^{1'} &= F_1^{1'}\omega^1 + F_2^{1'}\omega^2 + F_3^{1'}\omega^3 + g^{1'} \\ \omega^{2'} &= F_1^{2'}\omega^1 + F_2^{2'}\omega^2 + F_3^{2'}\omega^3 + g^{2'} \\ \omega^{3'} &= F_1^{3'}\omega^1 + F_2^{3'}\omega^2 + F_3^{3'}\omega^3 + g^{3'}. \end{aligned} \tag{o}$$

Formula (o) can be writen shortly in form:

$$\omega^{i'} = F_j^{i'}(A_\mu^{\lambda'})\omega^j + g^{i'}(A_\mu^{\lambda'}), \tag{1}$$

where $i' = 1',2',3'$, $j = 1,2,3$, $\lambda' = 1',2'$, $\mu = 1,2$, $(\omega^1,\omega^2,\omega^3) \in R^3$.

Having adopted the matrix notation

$$\omega = \begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{bmatrix} = \left[\omega^j\right] \qquad \omega' = \begin{bmatrix} \omega^{1'} \\ \omega^{2'} \\ \omega^{3'} \end{bmatrix} = \left[\omega^{i'}\right]$$

$$j = 1,2,3 \qquad i' = 1',2',3'$$

$$A = \begin{bmatrix} A_1^{1'} & A_2^{1'} \\ A_1^{2'} & A_2^{2'} \end{bmatrix} = \left[A_\mu^{\lambda'}\right] \qquad \begin{aligned} \lambda' &= 1',2' \\ \mu &= 1,2 \end{aligned}$$

$$F = \begin{bmatrix} F_1^{1'} & F_2^{1'} & F_3^{1'} \\ F_1^{2'} & F_2^{2'} & F_3^{2'} \\ F_1^{3'} & F_2^{3'} & F_3^{3'} \end{bmatrix} = \left[F_j^{i'}\right] = F(A)$$

$$g = \begin{bmatrix} g^{1'} \\ g^{2'} \\ g^{3'} \end{bmatrix} = \left[g^{i'}\right] = g(A),$$

the formula (o) or (1) can be written briefly in the matrix form:

$$\omega' = F(A) \cdot \omega + g(A). \tag{2}$$

By definition functions $F^{i'}_{j}$ and $g^{i'}$ in formula (o) as well as functions F and g in (2) depend only on the first derivatives $A^{\lambda'}_{\mu}$, where

$$A^{\lambda'}_{\mu} \overset{df}{=} \frac{\partial \xi^{\lambda'}}{\partial \xi^{\mu}} \qquad \begin{matrix} \mu = 1,2 \\ \lambda' = 1',2' \end{matrix}$$

of the new variables with respect to the old ones. The transformation $\xi^{\lambda'} = \xi^{\lambda'}(\xi^1, \xi^2)$, where ξ^{λ} and $\xi^{\lambda'}$ denote the coordinates of the same point ξ in the coordinate system (λ) and (λ') respectively is of the class C^1 with the Jacobian matrix:

$$J = \det\left[A^{\lambda'}_{\mu}\right] = \det A \neq 0.$$

Thus, $A \in L^1_2 = GL(2,R)$.

It follows from the group property of the transformation formula (2) that for all matrices $A,B \in GL(2,R) = L^1_2$ (i.e. A,B are non singular 2x2 real matrices) every pair of functions F and g in formula (2) must satisfy the system of functional equations:

$$F(A.B) = F(A) \cdot F(B) \tag{3}$$

$$g(A.B) = F(A) \cdot g(B) + g(A) \tag{4}$$

and the condition

$$F(e) = E, \tag{5}$$

where

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We do not make any assumptions concerning the regularity of the functions F and g.

The functional equation (3) does not contain the function g and therefore it can be considered independently of the functional equation

(4). The general non-singular solution F of equation (3) for all $A,B \in GL(2,R)$ is given in the authors paper [3] (cf. Theorem pp. 4-7). The method used in [3] is analogous with that used by M. Kucharzewski and M. Kuczma in [15]; however, the problem is more complicated. Applying the results of papers [3], [4], [5], [7], [8] and [9] all pairs of solutions F and g of the system of functional equations (3) and (4) was given in my paper [10](cf. Theorem 5.3 pp. 45-46, Theorem 3.1 pp. 33-35, Theorem 4.2 pp. 37-38).

The method used in papers [4] and [5] is analogous. However, the situation in [5] where non - measurable solutions exist is more complicated and more interesting than in [4] where all solutions are measurable.

All the obtained solutions of this system of functional equations (3) and (4) which have in paper [10] and applied in the present paper to determine all purely differential geometric objects of the first class with three components in a two-dimensional space, i.e., all objects of type [3,2,1] with linear non-homogeneous transformation rule.

Thus, applying the results of papers [10] and [6] we know at once all objects of type [3, 2, 1] with linear non-homogeneous transformation rule.

Now we have

<u>Theorem</u>. All purely differential geometric objects of the first class with three components in a two-dimensional space i.e., according to terminology of J. Aczél and S. Gołąb, all objects of type [3, 2, 1] with linear non-homogenous transformation rule are given by the following formulae:

$$\omega' = C \cdot F_i(A) \cdot C^{-1} \cdot \omega + \left[C \cdot F_i(A) \cdot C^{-1} - E\right] \cdot q, \tag{7.i}$$

where for $i = 1,2,3,4,5,6,7,8$ we have:

$$F_1(A) = \begin{bmatrix} \varphi(J) & 0 & 0 \\ 0 & \varphi(J) & 0 \\ 0 & 0 & \varphi_3(J) \end{bmatrix} \begin{bmatrix} A_1^{1'} & A_2^{1'} & 0 \\ A_1^{2'} & A_2^{2'} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1*}$$

$$F_2(A) = \begin{bmatrix} \varphi_1(J) & 0 & 0 \\ 0 & \varphi_2(J) & 0 \\ 0 & 0 & \varphi_3(J) \end{bmatrix} \tag{2*}$$

$$F_3(A) = \begin{bmatrix} \varphi(J) & \varphi(J)\varphi(J) & 0 \\ 0 & \varphi(J) & 0 \\ 0 & 0 & \varphi_3(J) \end{bmatrix}, \tag{3*}$$

$$F_4(A) = \begin{bmatrix} \varkappa(J) & -\sigma(J) & 0 \\ \sigma(J) & \varkappa(J) & 0 \\ 0 & 0 & \varphi_3(J) \end{bmatrix}, \tag{4*}$$

$$F_5(A) = \varphi(J) \begin{bmatrix} 1 & 0 & \alpha_1(J) \\ 0 & 1 & \alpha_2(J) \\ 0 & 0 & 1 \end{bmatrix}, \tag{5*}$$

$$F_6(A) = \varphi(J) \begin{bmatrix} 1 & \alpha_1(J) & \alpha_2(J) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6*}$$

$$F_7(A) = \varphi(J) \begin{bmatrix} 1 & \alpha_1(J) & \frac{1}{2}\alpha_1^2(J) + \alpha_2(J) \\ 0 & 1 & \alpha_1(J) \\ 0 & 0 & 1 \end{bmatrix} \tag{7*}$$

$$F_8(A) = \varphi(J) \begin{bmatrix} (A_1^{1'})^2 & 2\,A_1^{1'}A_2^{1'} & (A_2^{1'})^2 \\ A_1^{1'}A_1^{2'} & A_1^{1'}A_2^{2'} + A_2^{1'}A_1^{2'} & A_2^{1'}A_2^{2'} \\ (A_1^{2'})^2 & 2\,A_1^{2'}A_2^{2'} & (A_2^{2'})^2 \end{bmatrix} = \varphi(J)\,F^*(A), \tag{8*}$$

$\xi\varphi(\xi) \not\equiv 1$ (i.e. $\varphi(\xi) \not\equiv \frac{1}{\xi}$)

$$\omega' = C \cdot \begin{bmatrix} \varphi(J) & 0 & 0 \\ 0 & \varphi(J) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} A_1^{1'} & A_2^{1'} & 0 \\ A_1^{2'} & A_2^{2'} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot C^{-1} \cdot \omega +$$

$$+ C \cdot \begin{bmatrix} \lambda_1 \; A_1^{1'}\varphi(J) - 1 \; + \lambda_2 A_2^{1'}\varphi(J) \\ \lambda_1 \; A_1^{2'}\varphi(J) + \lambda_2\left[\varphi(J) \; A_2^{2'} - 1\right] \\ \ln\left|\Phi_3(J)\right| \end{bmatrix}, \tag{7.9}$$

$$\omega' = C \cdot \begin{bmatrix} A_1^{1'} & A_2^1 & 0 \\ A_1^{2'} & A_2^2 & 0 \\ 0 & 0 & \varphi_3(J) \end{bmatrix} \cdot C^{-1} \cdot \omega + C \cdot \begin{bmatrix} \lambda_1\left[A_1^{1'} - 1\right] + \lambda_2 A_2^{1'} \\ \lambda_1 A_1^{2'} + \lambda_2\left[A_2^{2'} - 1\right] \\ \lambda_3\left[\varphi_3(J) - 1\right] \end{bmatrix}, \quad (7.10)$$

$$\omega' = C \cdot \begin{bmatrix} A_1^{1'} & A_2^{1'} & 0 \\ A_1^{2'} & A_2^{2'} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot C^{-1} \cdot \omega + C \cdot \begin{bmatrix} \lambda_1\left[A_1^{1'} - 1\right] + \lambda_2 A_2^{1'} \\ \lambda_1 A_1^{2'} + \lambda_2\left[A_2^{2'} - 1\right] \\ \ln|\Phi_3(J)| \end{bmatrix} \quad (7.11)$$

$$\omega' = C \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \varphi_2(J) & 0 \\ 0 & 0 & \varphi_3(J) \end{bmatrix} \cdot C^{-1} \cdot \omega + C \cdot \begin{bmatrix} \ln|\Phi_1(J)| \\ \lambda_2\left[\varphi_2(J) - 1\right] \\ \lambda_3\left[\varphi_3(J) - 1\right] \end{bmatrix}, \quad (7.12)$$

$$\omega' = C \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varphi_3(J) \end{bmatrix} \cdot C^{-1} \cdot \omega + C \cdot \begin{bmatrix} \ln|\Phi_1(J)| \\ \ln|\Phi_2(J)| \\ \lambda_3\left[\varphi_3(J) - 1\right] \end{bmatrix}, \quad (7.13)$$

$$\omega' = \omega + C \cdot \begin{bmatrix} \ln|\Phi_1(J)| \\ \ln|\Phi_2(J)| \\ \ln|\Phi_3(J)| \end{bmatrix}, \text{ for } C \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot C^{-1} \cdot \omega = C\,E\,C^{-1}\omega = \omega \quad (7.14)$$

$$\omega' = C \cdot \begin{bmatrix} \varkappa(J) & -\sigma(J) & 0 \\ \sigma(J) & \varkappa(J) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot C^{-1} \cdot \omega + C \cdot \begin{bmatrix} \lambda_1\left[\varkappa(J) - 1\right] - \lambda_2\sigma(J) \\ \lambda_1\sigma(J) + \lambda_2\left[\varkappa(J) - 1\right] \\ \ln|\Phi_3(J)| \end{bmatrix}, \quad (7.15)$$

$$\omega' = C \cdot \begin{bmatrix} \varphi(J) & \varphi(J)\alpha(J) & 0 \\ 0 & \varphi(J) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot C^{-1} \cdot \omega + C \begin{bmatrix} \lambda_1\left[\varphi(J) - 1\right] + \lambda_2\varphi(J)\alpha(J) \\ \lambda_2\left[\varphi(J) - 1\right] \\ \ln|\Phi_3(J)| \end{bmatrix}, \quad (7.16)$$

$$\omega' = C \cdot \begin{bmatrix} 1 & \alpha(J) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varphi_3(J) \end{bmatrix} \cdot C^{-1} \cdot \omega + C \cdot \begin{bmatrix} \ln|\Phi_1(J)| + \tau_1\alpha^2(J) \\ 2\tau_1\alpha(J) \\ \lambda_3\left[\varphi_3(J) - 1\right] \end{bmatrix}, \quad (7.17)$$

$$\omega' = C \cdot \begin{bmatrix} 1 & \alpha(J) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot C^{-1} \cdot \omega + C \cdot \begin{bmatrix} \ln|\Phi_1(J)| + \tau_1\alpha^2(J) \\ 2\tau_1\alpha(J) \\ \ln|\Phi_3(J)| \end{bmatrix}, \quad (7.18)$$

$$\omega' = C \cdot \begin{bmatrix} 1 & 0 & \alpha_1(J) \\ 0 & 1 & \alpha_2(J) \\ 0 & 0 & 1 \end{bmatrix} \cdot C^{-1} \cdot \omega + C \cdot \begin{bmatrix} \ln|\Phi_1(J)| \\ \ln|\Phi_2(J)| \\ 0 \end{bmatrix}, \quad (7.19)$$

$$\omega' = C \cdot \begin{bmatrix} 1 & \alpha_1(J) & \alpha_2(J) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot C^{-1} \cdot \omega +$$

$$+ C \cdot \begin{bmatrix} \ln|\Phi_1(J)| + \frac{1}{2}\varepsilon_1\alpha_1^2(J) + \frac{1}{2}\bar{\varepsilon}_2\alpha_2^2(J) + \bar{\varepsilon}_1\alpha_1(J)\alpha_2(J) \\ \varepsilon_1\alpha_1(J) + \bar{\varepsilon}_1\alpha_2(J) \\ \varepsilon_1\alpha_1(J) + \bar{\varepsilon}_2\alpha_2(J) \end{bmatrix}, \quad (7.20)$$

$$\omega' = C \cdot \begin{bmatrix} 1 & \alpha_1(J) & \frac{1}{2}\alpha_1^2(J) + \alpha_2(J) \\ 0 & 1 & \alpha_1(J) \\ 0 & 0 & 1 \end{bmatrix} \cdot C^{-1} \cdot \omega +$$

$$+ C \cdot \begin{bmatrix} \ln|\Phi_1(J)| + \frac{1}{3}\tau_1\alpha_1^3(J) + \bar{\tau}\alpha_1^2(J) + 2\tau_1\alpha_1(J)\alpha_2(J) \\ \tau_1\alpha_1^2(J) + 2\bar{\tau}\alpha_1(J) + 2\tau_1\alpha_2(J) \\ 2\tau_1\alpha_1(J) \end{bmatrix}, \quad (7.21)$$

$$\omega' = C \cdot \frac{1}{J} F^*(A) \cdot C^{-1} \cdot \omega + \left\{ C \cdot \frac{1}{J} F^*(A) \cdot C^{-1} - E \right\} \cdot q +$$

$$+ C \cdot g^*(A) \quad \text{where} \quad F^*(A) \ \text{cf. } (8^*) \ \text{and} \quad (7.22)$$

$$g^*(A) = \frac{1}{J}\begin{bmatrix} \begin{vmatrix} \phi(A_1^{1'}) & \phi(A_2^{1'}) \\ A_1^{1'} & A_2^{1'} \end{vmatrix} \\ \frac{1}{2}\left\{\begin{vmatrix} \phi(A_1^{1'}) & \phi(A_2^{2'}) \\ A_1^{1'} & A_2^{2'} \end{vmatrix} + \begin{vmatrix} \phi(A_1^{2'}) & \phi(A_2^{1'}) \\ A_1^{2'} & A_2^{1'} \end{vmatrix}\right\} \\ \begin{vmatrix} \phi(A_1^{2'}) & \phi(A_2^{2'}) \\ A_1^{2'} & A_2^{2'} \end{vmatrix} \end{bmatrix}$$

In formula (7.22) ϕ is an arbitrary derivation of R. By derivation of R any function $\phi : R \longrightarrow R$ satisfying the conditions:

$$\phi(\xi+\eta) = \phi(\xi) + \phi(\eta) \tag{8}$$

for all ξ, η in R

$$\phi(\xi\eta) = \eta\phi(\xi) + \xi\phi(\eta) \tag{9}$$

is ment.

Numerous and interesting properties of derivations are given and proved in papers [5], p. 220, [6] and [26], p. 120.

Let ϕ be such a derivation. It is well known that $\phi(0) = \phi(1) = 0$, $\phi(-\xi) = -\phi(\xi)$ and $\phi(\xi^p) = p\,\xi^{p-1}\,\phi(\xi)$ for in R and p = 1,2,3,... If $\eta \neq 0$, then $\phi(\frac{1}{\eta}) = -\frac{\phi(\eta)}{\eta^2}$ and

$$\phi(\frac{\xi}{\eta}) = \frac{\eta\phi(\xi) - \xi\phi(\eta)}{\eta^2}.$$

Furthermore ϕ vanishes on the algebraic closure $\bar{Q}$ of the field Q of rationals (in R) and has a dense set of periods. It is well known that any measurable derivation of R is trivial (i.e., identically 0), but there are non-trivial derivations of R. (cf. [26]p. 124, Corollaries 1 and Corollaries 1′).

Every non-trivial derivation of R must be a non-measurable function. Every derivation of R, according to the terminology introduced by S. Gołąb, is a microperiodical function. Every rational number is period of any arbitrary derivation of R.

In formulae (7.1) - (7.22)

$$A = \begin{bmatrix} A_1^{1'} & A_2^{1'} \\ A_1^{2'} & A_2^{2'} \end{bmatrix} \in GL\,(2,R);$$

J = det. A is the determinant of A; C is an arbitrary constant non-singular 3x3 real matrix playing in formulae (7.1) - (7.22) the role of parameter; λ_i (i = 1,2,3), $\bar{\tau}$, τ_1, ε_1, $\bar{\varepsilon}_1$, ε_2 are arbitrary constants and q is an arbitrary constant vector

$$q = \begin{bmatrix} \vartheta^1 \\ \vartheta^2 \\ \vartheta^3 \end{bmatrix};$$

φ, φ_i, Φ_i (i = 1,2,3) are arbitrary multiplicative functions not vanishing identically, i.e., they are solutions of the functional equations:

$$\varphi(\xi\eta) = \varphi(\xi)\,\varphi(\eta), \quad \xi\eta \neq 0 \tag{10.1}$$

$$\varphi_i(\xi\eta) = \varphi_i(\xi)\varphi_i(\eta), \quad \xi\eta \neq 0 \tag{10.2}$$

$$\Phi_i(\xi\eta) = \Phi_i(\xi)\Phi_i(\eta) \quad i = 1,2,3, \quad \xi\eta \neq 0 \tag{10.3}$$

respectively with the restriction:

$$\varphi \not\equiv 0, \quad \varphi_i \not\equiv 0, \quad \Phi_i \not\equiv 0 \quad (i = 1,2,3). \tag{10.4}$$

Formulae (7.1) - (7.7), (7.9), (7.16), (7.10), (7.12), (7.13) and (7.17) have additional restrictions:

$$\varphi(\xi) \not\equiv 1, \quad \varphi_i(\xi) \not\equiv 1 \quad (i = 1,2,3) \quad \text{for} \quad \xi \neq 0. \tag{10.5}$$

In formula (7.8) the additional restrictions:

$$\xi\varphi(\xi) \not\equiv 1 \quad (\text{i.e.} \quad \varphi(\xi) \not\equiv \frac{1}{\xi} \quad \text{for} \quad \xi \neq 0. \tag{10.6}$$

In formulae (7.3), (7.5) - (7.7) and (7.16) - (7.21) α and α_i (i = 1,2) is an arbitrary function satisfying the functional equation:

$$\alpha(\xi\eta) = \alpha(\xi) + \alpha(\eta) \quad \xi\eta \neq 0 \tag{11.1}$$

$$\alpha_i(\xi\eta) = \alpha_i(\xi) + \alpha_i(\eta) \quad i = 1,2) \quad \xi \neq 0 \tag{11.2}$$

and the condition

$$\alpha \not\equiv 0, \quad \alpha_i \not\equiv 0 \quad i = 1,2. \tag{11.3}$$

In formulae (7.19) and (7.20) the functions

$$\alpha_1 \text{ and } \alpha_2 \text{ are lineary independent for } \xi \neq 0. \quad (11.4)$$

In formula (7.21) α_1 and α_2 are arbitrary solutions of (11.2) with the restriction

$$\alpha_1 \not\equiv 0. \quad (11.5)$$

The functions $\varkappa$ and σ are a solution of the system of functional equations:

$$\varkappa(\xi\eta) = \varkappa(\xi)\varkappa(\eta) - \sigma(\xi)\sigma(\eta) \quad (12)$$

$$\xi\eta \neq 0$$

$$\sigma(\xi\eta) = \varkappa(\xi)\sigma(\eta) + \varkappa(\eta)\sigma(\xi) \quad (13)$$

fulfilling the condition

$$\sigma \not\equiv 0. \quad (14)$$

<u>Remark 1</u>. Restrictions (11.3), (11.4), (11.5) and (14) are not essential. For, if any of inequalities (11.3), (11.4), (11.5), (14) were not fulfilled, then the corresponding cases of (7.3) - (7.7) would be reduced to case (7.2) or (7.3) and cases of (7.15) - (7.21) to case (7.12), (7.13), (7.16), (7.17) or (7.18). Detailed considerations are given in paper [10] pp. 39-41 (cf. Remark 4.1).

<u>Remark 2</u>. The cases where

$$F(A) = C \cdot \begin{bmatrix} \varphi_1(J) & 0 & 0 \\ 0 & \varphi_2(J) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot C^{-1} = C \cdot \{\varphi_1, \varphi_2, 1\} \cdot C^{-1}$$

or

$$F(A) = C \cdot \begin{bmatrix} \varphi_1(J) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varphi_3(J) \end{bmatrix} \cdot C^{-1} = C \cdot \{\varphi_1(J), 1, \varphi_3(J)\} \cdot C^{-1}$$

can easily be reduced to case (7.12), where

$$F(A) = C^* \cdot \{1, \varphi_2, \varphi_1\} \cdot (C^*)^{-1} \quad \text{or}$$

$$F(A) = C^{**} \cdot \{1, \varphi_1, \varphi_3\} \cdot (C^{**})^{-1}.$$

Analogously, the cases, where

$$F(A) = C \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \varphi_2(J) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot C^{-1} = C \cdot \{1, \varphi_2, 1\} \cdot C^{-1}$$

or

$$F(A) = C \cdot \begin{bmatrix} \varphi_1(J) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot C^{-1} = C \cdot \{\varphi_1, 1, 1\} \cdot C^{-1}$$

are reduced to (7.13). Detailed considerations are given in paper [10] pp. 35-36 (cf. Remark 3.1).

Every object ω with the transformation formula (7.22) is equivalent to an object $\Omega = C^{-1} \cdot \omega$ with the transformation formula:

$$\Omega' = \frac{1}{J} F^{*}(A) \cdot \Omega + \left\{\frac{1}{J} F^{*}(A) - E\right\} \cdot \bar{q} + g^{*}(A), \text{ where} \tag{15}$$

$F^{*}(A)$ cf. (8^{*}) and $g^{*}(A)$ cf. (7.22),

where $\bar{q} = C^{-1} \cdot q$ is an arbitrary constant 3x1 matrix whose entries are real parameters.

(Evidently the object Ω with the transformation formula (15) can be obtained substituting matrix E for matrix C in the transformation formula (7.22)).

Applying the results of my paper [5] geometric objects defined by formula (15) and the object

$$d = \begin{bmatrix} \delta^1 \\ \delta^2 \\ \delta^3 \end{bmatrix} = \left[\delta^{\gamma}\right] \quad \gamma = 1,2 \text{ with the transformation}$$

formula

$$d' = \frac{1}{J} \begin{bmatrix} (A_1^{1'})^2 & 2\, A_1^{1'} A_2^{1'} & (A_2^{1'})^2 \\ A_1^{1'} A_1^{2'} & A_1^{1'} A_2^{2'} + A_1^{2'} A_2^{1'} & A_2^{1'} A_2^{2'} \\ (A_1^{2'})^2 & 2\, A_1^{2'} A_2^{2'} & (A_2^{2'})^2 \end{bmatrix} \cdot d +$$

$$+\frac{1}{J}\begin{bmatrix} \begin{vmatrix} \phi(A_1^{1'}) & \phi(A_2^{1'}) \\ A_1^{1'} & A_2^{1'} \end{vmatrix} \\ \frac{1}{2}\left\{ \begin{vmatrix} \phi(A_1^{1'}) & \phi(A_2^{2'}) \\ A_1^{1'} & A_2^{2'} \end{vmatrix} + \begin{vmatrix} \phi(A_1^{2'}) & \phi(A_2^{1'}) \\ A_1^{2'} & A_2^{1'} \end{vmatrix} \right\} \\ \begin{vmatrix} \phi(A_1^{2'}) & \phi(A_2^{2'}) \\ A_1^{2'} & A_2^{2'} \end{vmatrix} \end{bmatrix}$$

equivalent to object (15) called a derivat (Z. Kareńska [6] p. 151) were given in paper [6] (1968).
The name "derivat" is taken from function ϕ called derivation non-trivial function which appears in the transformation formula (16). (Evidently the object d with the transformation formula (16) can be obtained substituting matrix $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ for matrix $\bar{q}$).

<u>Definition</u>. Object d with three components: $\delta^1, \delta^2, \delta^3$ with the transformation formula (16) is called a derivat if the function $g^*(A) \not\equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

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