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Ivan KOLÁŘ

NATURAL OPERATORS TRANSFORMING VECTOR FIELDS AND EXTERIOR FORMS INTO EXTERIOR FORMS

Summary. We determine all natural bilinear operators transforming vector fields and exterior p-forms into exterior q-forms.

The classical theory of differential geometric objects, several important contributions to which have been presented by the eminent Polish geometric school, was revisited by A. Nijenhuis in the form of the natural bundles, [5]. His general approach to this fundamental concept of differential geometry iniciated a new wave of research. In particular, such a point of view underlined the role of the natural operators in several differential geometric problems.

In the present paper we study the natural operators transforming vector fields and exterior p-forms into exterior q-forms. In order to get the results interesting geometrically, we restrict ourselves to the bilinear operators. Using our general method from [3], we determine all of them. We find it remarkable that our Proposition 1 gives a new look to the well-known relation between the Lie derivatives and the exterior derivatives of exterior forms. - All manifolds and maps are assumed to be infinitely differentiable.

1. Let TM or $\wedge^p T^*M$ be the tangent bundle or the p-th exterior power of the cotangent bundle of an m-dimensional manifold M, respectively. Hence both T and $\wedge^p T^*$ are natural bundles over m-manifolds in the sense of A. Nijenhuis, [5]. Let $C^\infty TM$ or $C^\infty \wedge^p T M$ denote the space of all smooth sections of TM or $\wedge^p T^*M$.

Definition. A natural operator $A: T \oplus \wedge^p T^* \to \wedge^q T^*$ is a system of maps

$$A_M: C^\infty TM \times C^\infty \wedge^p T^*M \longrightarrow C^\infty \wedge^q T^*M$$

for every m-manifold M such that

a) for every diffeomorphism $f: M \longrightarrow N$, it holds

$$A_N(Tf \circ X \circ f^{-1}, \wedge^p T^*f \circ \omega \circ f^{-1}) = \wedge^q T^*f \circ A_M(X, \omega) \circ f^{-1}$$

for all $X \in C^\infty TM$ and all $\omega \in C^\infty \wedge^p T^*M$,

b) for every open subset $U \subset M$, it hold

$$A_U(X|U,\omega|U) = A_M(X,\omega)|U$$

Our problem is to find all natural bilinear operators $T \oplus \wedge^p T^* \to \wedge^q T^*$.

2. The bilinear Peetre theorem reads that every bilinear support non-increasing operator has locally finite order, [1]. This implies that every natural bilinear operator $T \oplus \wedge^p T^* \to \wedge^q T^*$ has globally a finite order r. According to the general theory, there is a canonical action of the group G_m^{r+1} of all invertible $(r+1)$-jets of R^m into R^m with source 0 and target 0 on the standard fibres $S^r := J_0^r TR^m$, $Z^r := J_0^r \wedge^p T^* R^m$ and $\wedge^q R^{m*} = \wedge^q T_0^* R^m$. (The latter action factorizes through the standard actional of $G_m^1 = GL(m,R)$ on $\wedge^q R^{m*}$.) Further, there is a canonical bijection between the r-th order natural bilinear operators $T \oplus \wedge^p T^* \to \wedge^q T^*$ and the G_m^{r+1}-equivariant bilinear maps $S^x \times Z^r \to \wedge^q R^{m*}$, [3].

Let α, β be multiindices of range m. Denote by

$$x^i_\alpha \qquad 0 \le |\alpha| \le r$$

the canonical coordinates on S^r, by

$$b_{i_1 \dots i_p, \beta} \qquad 0 \le |\beta| \le r$$

the canonical coordinates on Z^r and by

$$c_{i_1 \dots i_q}$$

the canonical coordinates on $\wedge^q R^{m*}$. In the main part of this paper we shall need the explicit expression of the above-mentioned actions in the case $r = 1$ only. Let a^i_j, a^i_{jk} be the usual coordinates of an element $a \in G_m^2$ and let $\tilde{a}^i_j$, $\tilde{a}^i_{jk}$ denote the coordinates of the inverse element a^{-1}. A standard evaluation yields

$$\bar{x}^i = a^i_j x^j$$

$$\bar{x}^i_j = a^i_{kl} \tilde{a}^l_j x^k + a^i_k x^k_l \tilde{a}^l_j$$

$$\bar{b}_{i_1 \dots i_p} = b_{j_1 \dots j_p} \tilde{a}^{j_1}_{i_1} \dots \tilde{a}^{j_p}_{i_p}$$

$$\bar{b}_{i_1 \dots i_p, i} = b_{j_1 \dots j_p, j} \tilde{a}^{j_1}_{i_1} \dots \tilde{a}^{j_p}_{i_p} \tilde{a}^j_i + b_{j_1 \dots j_p} \left[\tilde{a}^{j_1}_{i_1 i} \tilde{a}^{j_2}_{i_2} \dots \tilde{a}^{j_p}_{i_p} + \dots + \tilde{a}^{j_1}_{i_1} \dots \tilde{a}^{j_p}_{i_p i} \right]$$

$$\bar{c}_{i_1 \dots i_q} = c_{j_1 \dots j_q} \tilde{a}^{j_1}_{i_1} \dots \tilde{a}^{j_q}_{i_q}$$

3. Let $f: S^r \times Z^r \longrightarrow \wedge^q R^m$ be an G^{r+1}_m-equivariant ilinear map. Consider the canonical inclusion $i: GL(m,R) \longrightarrow G^{r+1}_m$ trasforming every matrix into the, (r+1)-jet of the corresponding linear trasformation. One verifies easily that the transformation laws of X^i_α, $b_{i_1 \dots i_p,\beta}$ with respect to $i(GL(m,R))$ are tensorial. Then the equivariany of f with respect to the homotheties $a^i_j = k^{-1}\delta^i_j$, $k \in R$, $k > 0$ givs the homogeneity condition

$$k^q f(X^i_\alpha, b_{i_1 \dots i_p,\beta}) = f(k^{|\alpha|-1} X^i_\alpha, k^{|\beta|+p} b_{i_1 \dots i_p,\beta})$$

Hence f is a linear combination of those products of X and $b_{i_1 \dots i_p,\beta}$ that satisfy the relation

$$q = p - 1 + |\alpha| + |\beta| \tag{1}$$

4. Consider the case $p = q$. Then the only two possibiities for (1) are $|\alpha| = 0$, $|\beta| = 1$ and $|\alpha| = 1$, $|\beta| = 0$. We first etermine all $GL(m,R)$-equivariant bilinear maps f_1: $R^m \times (\wedge^p R^{m*} \otimes \wedge R^{m*} \longrightarrow \wedge^p R^{m*}$. Analogously to [4], consider the following diagram

$$\begin{array}{ccc} R^m \times \wedge^p R^{m*} \otimes R^{m*} & \xrightarrow{\ f_1\ } & \wedge^p R^{m*} \\ \downarrow \Big\uparrow \ \mathrm{id} \times \mathrm{Alt}_p \otimes \mathrm{id} & & \downarrow \Big\uparrow \ \mathrm{Alt}_p \\ R^m \times \overset{p+1}{\otimes} R^{m*} & \longrightarrow & \overset{p}{\otimes} R^m \end{array} \tag{2}$$

where Alt denote the alternator of the indicated degree. n this diagram the vertical maps are also $GL(m,R)$-equivariant and the $L(m,R)$-equivariant map in the bottom row can be determined by the invaiant tensor theorem, [2], [3]. This implies that f_1 is a linear comination of the contraction of X^i with the derivation entry in $b_{i_1 \dots i,j}$ and of the contraction of X^i with a non-derivation entry in $b_{i_1 \dots i_p,j}$ followed by the alternation.

Next we determine all $GL(m,R)$-equivariant bilinear map f_2: $(R^m \otimes R^{m*}) \times \wedge^p R^{m*} \longrightarrow \wedge^p R^{m*}$. Consider the diagram

$$\begin{array}{ccc} R^m \otimes R^{m*} \otimes \wedge^p R^{m*} & \xrightarrow{\ \tilde{f}_2\ } & \wedge^p R^{m*} \\ \downarrow \Big\uparrow \ \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{Alt}_p & & \downarrow \Big\uparrow \ \mathrm{Alt}_p \\ R^m \otimes \overset{p+1}{\otimes} R^m & \longrightarrow & \overset{p}{\otimes} R^{m*} \end{array} \tag{3}$$

where $\tilde{f}_2$ is the linearization of f_2. Taking into account that the map in the bottom row is determined by the invariant tensor theorem, we conclude similarly as above that f_2 is a linear combination of the inner contraction X^j_j multiplied by $b_{i_1\dots i_p}$ and of the contraction $X^j_{i_1}b_{i_2\dots i_p j}$ followed by the alternation.

Thus, the equivariancy of f with respect to $i(GL(m,R))$ leads to the following 4-parameter family

$$f_{i_1\dots i_p} = aX^j b_{i_1\dots i_p,j} + bX^j b_{j[i_2\dots i_p,i_1]} + cX^j_j b_{i_1\dots i_p} + eX^j_{[i_1} b_{i_2\dots i_p]j} \tag{4}$$

$a, b, c, e \in R$, where the square bracket denotes alternation. Further we express the equivariancy of f with respect to the kernel $a^i_j = \delta^i_j$ of the jet projection $G^2_m \longrightarrow G^1_m$. It is characterized by

$$0 = -aX^j(b_{ki_2\dots i_p}a^k_{i_1j} + \dots + b_{i_1\dots i_{p-1}k}a^k_{i_pj}) + bX^j b_{k[i_2\dots i_p}a^k_{i_1]j} + ca^k_{kj}X^j b_{i_1\dots i_p} + eX^k a^j_{k[i_1}b_{i_2\dots i_p]j} \tag{5}$$

This implies $c = 0$ and one linear relation among a, b, e. Interpreting the result geometrically, we obtain (provided $\lrcorner$ denotes the inner product of a vector field and of an exterior form).

<u>Proposition 1</u>. All natural vilinear operators $T \oplus \Lambda^p T^* \longrightarrow \Lambda^p T^*$ from the following 2-parameter family

$$k_1 d(X \lrcorner \omega) + k_2 (X \lrcorner d\omega) \qquad k_1, k_2 \in R$$

There is a third well-known natural bilinear operator $T \oplus \Lambda^p T^* \longrightarrow \Lambda^p T^*$, namely the Lie derivative $L_X\omega$ of ω with respect to X. Hence Proposition 1 implies that $L_X\omega$ must be a linear combination of $d(X \lrcorner \omega)$ and $X \lrcorner d\omega$. If we evaluate $k_1 = 1 = k_2$ in two suitable cases, we obtain an interesting proof of the classical formula.

5. In the case $q = p - 1$ relation (1) can be satisfied for $|\alpha| = |\beta| = 0$ only. Analogously to (2) or (3), we then deduce that all $GL(m,R)$-equivariant bilinear maps $R^m \times \Lambda^p R^{m*} \longrightarrow \Lambda^{p-1} R^{m*}$ are the constant multiples of the tensor contraction. This proves.

<u>Proposition 2</u>. The only bilinear natural operators $T \oplus \Lambda^p T^* \longrightarrow \Lambda^{p-1} T^*$ are the constant multiples of $X \lrcorner \omega$.

6. In the case $q = p+1$ the homogeneity condition (1) and the invariant tensor theorem yield the following 4-parameter family

$$aX^j_{[i}b_{i_1 \dots i_p,j]} + bX^j_{[i}b_{j i_1 \dots i_{p-1},i_p]} + cX^j_j b_{[i_1 \dots i_p,i]} + eX^j b_{[i_1 \dots i_p,i]j} \qquad (6)$$

Considering its equivariancy with respect to the kernel of the jet projection $G^3_m \longrightarrow G^1_m$, we deduce $c = 0$ and two linear independent relations among a, b, e. Interpreting this result geometrically, we obtain

Proposition 3. The only natural bilinear operators $T \oplus \Lambda^p T^* \longrightarrow \Lambda^{p+1} T^*$ are the constant multiples of $d(X \lrcorner d\omega)$.

7. In the case $q < p-1$ relation (1) cannot be satisfied for any $|\alpha|$, $|\beta|$. This implies that the only natural bilinear operator is the zero operator.

In the case $q = p+2$ we proceed similarly to item 6. Having applied the invariant tensor theorem, we see that the only term which does not vanish after alternation is

$$aX^k_{[i}b_{i_1 \dots i_p,j]k} \qquad (7)$$

However, considering the equivariancy of (7) with respect to the subgroup $a^i_j = \delta^i_j$ in G^2_m, we find $a = 0$. In the case $q \geqslant p+3$, the invariant tensor theorem yields the zero map. Thus, we have proved

Proposition 4. In the case $q \neq p-1, p, p+1$ the only natural bilinear operator $T \oplus \Lambda^p T \longrightarrow \Lambda^q T^*$ is the zero operator.

REFERENCES

[1] Cahen M., de Wilde M., Gutt S.: Local cohomology of the algebra of C^∞-functions on a connected manifold, Lett. Math. Phys., 4(1980), 157-167.

[2] Dieudonne J.A., Carrell J.B.: Invariant Theory, Odl and New, Academic Press, New York - London 1971.

[3] Kolář I.: Some natural operators in differential geometry, Proc. Conf. Differential Geometry and its Applications, Brno 1986, D. Reidel 1987, 91-110.

[4] Kolář I., Michor P.E.: All natural concomitants of vector valued differential forms, Rendiconti del Circolo Matematico di Palermo, Supplemento, Serie II, numero 16(1987), 101-108.

[5] Nijenhuis A.: Natural bundles and their general properties, Differential Geometry, Kinokuniya, Tokyo 1972, 317-334.