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THE LIMITING PROPERTIES OF THE SOLUTIONS OF SYSTEMS OF PARABOLIC FUNCTIONAL - DIFFERENTIAL EQUATIONS

Summary. The subject paper is the continuation of the paper [6]. Let u be the solution of the system

$$u^i_t = f^i(t,x,u,u^i_x,u^i_{xx},u(t,\cdot)) \quad i = 1,\dots,m \tag{1}$$

with linear boundary conditions (see (3)). We will establish certain sufficient propositions by which the solution u of (1) has the limit s, as $t \to \infty$, where the function s is the solution of certain system which depends on $f = (f^1,\dots,f^m)$. Then after introduction of a more strict assumption on f, than the parabolicity in the sense given by J. Szarski in [7], there will be proved a analogous condition for the solution u. These results are essential generalisation of theorem proved in [4]. Moreover using the same assumption about f, we finally get also certain theorems for which the results of [1], [2] and [3], [5] are particular cases. These problems cannot be solved without the assumption of the strong parabolicity of f, (see def. 8).

1. Definitions and notations

Since the notations and definitions presented in [6], are still obligatory, we will repeat them only in a short way.

Let D be a domain in the space R^{n+1} of the variables $(t,x) = (t,x_1,\dots,x_n)$. The projection of D onto the t - axis is $(0,\infty)$ and onto R^n is S^1_o. Notice that S^1_o may be bounded or not.

Let us denote by $E_T = E \cap \{(t,x): t > T,\ x \in R^n\}$ for every $E \subset R^{n+1}$ and $T > 0$.

D_p is a subset of such points $(\tilde{t},\tilde{x}) \in \bar{D}$, which have a lower half - neighbourhood (for $t < \tilde{t}$) containing in the domain D. $S_{\tilde{t}}$ stands for the projection onto R^n of $D_p \cap \{t: t = \tilde{t}\}$, for $\tilde{t} > o$. Let Σ denote the subset of ∂D lying in the layer $o < t < \infty$, for which $\Sigma \cap D_p = \emptyset$, $S_o = \partial D \cap \{t: t = 0\}$.

Assumption A. S_0 and $S_{\tilde{t}}$ are bounded sets for any $\tilde{t} > 0$.

Let $g^i : \Sigma^i \to R_+$ and $h^i : \Sigma^i \to R_+$ be given on $\Sigma^i \subset \Sigma$ for $i=1,\ldots,m$. We define at every $(t,x) \in \Sigma^i$ the direction $l^i(t,x)$, orthogonal to t-axis, $l^i(t,x)$ penetrate in D_p, (comp. [6]).

If $S \subset R^n$ then we denote $\bar{C}(S,R^m)$ the space of bounded and continuous functions $z : S \to R^m$.

Let $u = (u^1,\ldots,u^m) \in R^m$, $q = (q_1,\ldots,q_n) \in R^n$, $r = (r_{11},r_{12},\ldots,r_{nn}) \in R^{n^2}$, where $r_{ij} = r_{ji}$, and $f = (f^1,\ldots,f^m)$: $\{(t,x,u,q,r,z)\} \to R^m$ where $z \in \bar{C}(S_0^1,R^m)$ and $(t,x) \in D_p$.

Function $u : \bar{D} \to R^m$ we call Σ-regular in D if for $i = 1,\ldots,m$, u^i are continuous in $\bar{D}$, u^i_x, u^i_{xx} and u^i_t are continuous in D_p, and there exists $\frac{du^i}{dl^i}$ on Σ^i.

We say that u is Σ-regular solution of the system (1) if u is a Σ-regular function and if it is a solution of (1) in D_p, where $u(t,\ ) : S_t \to R^m$, and $u(t,\cdot)(x) = u(t,x)$.

The parabolicity of f (see [7] and [8]) is defined as follow:

We say that f is parabolic in D with respect to the Σ-regular function u, if for every pair of arguments $r,\tilde{r} \in R^{n^2}$ such that

$$\tilde{r} \geq r \Leftrightarrow \forall \alpha = (\alpha_1,\ldots,\alpha_n) \in R^n, \text{ it is } \sum_{j,k=1}^{n} (\tilde{r}_{jk} - r_{jk})\,\alpha_j\alpha_k \geq 0,$$

the inequality

$$f^i(t,x,u(t,x),\ u^i_x(t,x),\tilde{r},u(t,\cdot)) - f^i(t,x,u(t,x),u^i_x(t,x),r,u(t,\cdot)) \geq 0 \tag{2}$$

for $(t,x) \in D_p$, $i = 1,\ldots,m$, holds.

We shall consider the system (1) with following boundary conditions: for $i = 1,\ldots,m$

$$\begin{cases} u^i(t,x) = \varphi^i_1(t,x) & \text{for } (t,x) \in (\Sigma \setminus \Sigma^i) \\ M^i(u)(t,x) = h^i(t,x)u^i(t,x) - g^i(t,x)\dfrac{d}{dl^i}u^i(t,x) = \varphi^i_2(t,x) & \text{for } (t,x) \in \Sigma^i \\ u^i(0,x) = \varphi^i_0(x) & \text{for } x \in S_0 \end{cases} \tag{3}$$

Definition 1: We say that u, Σ-regular in D, satisfies strong boundary inequalities if, for $i = 1,\ldots,m$,

$\varphi_0^i(x) < 0$ on S_0, $\varphi_1^i(t,x) < 0$ on $\Sigma \setminus \Sigma^i$

$\varphi_2^i(t,x) < 0$ on Σ^i.

2. Somme lemmas

Using the Theorems 4 and 5 given in Remark 5 in 6, we will formulate Lemma 1 under the Assumption B_1 and Lemma 2 under the Assumption B_2.

Lemma 1. Let u be the Σ-regular solution of the system (1) in D_p, satisfying the boundary condition (3) and such that f is parabolic with respect to the u. Let us assume, that there exists $T_0 \geqslant 0$ and the Σ-regular function $V : (D_p)_{T_0} \longrightarrow R^m$, which satisfies for $(t,x) \in (D_p)_{T_0}$ and $i = 1,\dots,m$ inequality

$$V_t^i(t,x) > f^i t,x,V(t,x),V_x^i(t,x),V_{xx}^i(t,x),V(t,\cdot)) \tag{4}$$

where $V(t,\cdot) \in \overline{C}(S_t,R^m)$. If the difference u-V satisfies the strong boundary inequalities according to Definition 1, on $(\Sigma)_{T_0} \cup S_{T_0}$ then $u \leqslant V$ in $(D_p)_{T_0}$.

Lemma 2. Let $T_0 \geqslant 0$, and the Σ-regular function $v:(D_p)_{T_0} \longrightarrow R^m$ satisfies the inequality

$$v_t^i(t,x) < f^i(t,x,v(t,x),v_x^i(t,x),v_{xx}^i(t,x),v(t,\cdot)) \tag{5}$$

where $v(t,\cdot) \in \overline{C}(S_t,R^m)$. If u satisfies the assumptions of Lemma 1 and the difference $v - u$ satisfies the strong boundary inequalities on $(\Sigma)_{T_0} \cup S_{T_0}$ $v \leqslant u$ in $(D_p)_{T_0}$.

Corollary 1: If we create two families of functions V_ε and v_ε such that:

1^0 $\forall \varepsilon > 0 \ \exists T_0$ such that all assumptions of Lemmas 1, 2 hold in $(D_p)_{T_0}$

2^0 $\forall \varepsilon > 0, V_\varepsilon(t,x) > 0$, $v_\varepsilon(t,x) < 0$ in $(D_p)_{T_0}$

3^0 $\forall \varepsilon > 0, \exists T_1 \geqslant T_0$ such that

$$V_\varepsilon(t,x) < \varepsilon \text{ and } v_\varepsilon(t,x) > -\varepsilon \text{ in } (D_p)_{T_1}$$

then $\lim\limits_{t \to \infty} \| u(t,\cdot)\|_t = 0$,

where $\|z(\cdot)\|_t = \max\limits_{1 \leqslant i \leqslant m} \sup\limits_{x \in S_t} |z(x)|$ (see [6] corollary 2)

3. The main theorem

Now we will prove two theorems concerning the limiting properties of the solution u. They will depend on properties of function f.

We will need some additional definitions and assumptions.

Definition 2. Let us denote:

$$\Sigma_\infty = \left\{x \in R^n ; \exists \{(t_\nu, x_\nu)\} : (t_\nu, x_\nu) \in \bar{\Sigma}, \nu = 1,2,\ldots, \lim_{\nu\to\infty} t_\nu = \infty, \lim_{\nu\to\infty} x_\nu = x\right\}$$

$$S_\infty = \left\{x \in R^n ; x \notin \Sigma_\infty, \exists \{(t_\nu, x_\nu)\} : (t_\nu, x_\nu) \in D_p, \nu = 1,2,\ldots, \lim_{\nu\to\infty} t_\nu = \infty, \lim_{\nu\to\infty} x_\nu = x\right\}$$

$$\Sigma^1_\infty = \left\{x \in R^n ; \exists \{(t_\nu, x_\nu)\} : (t_\nu, x_\nu) \in \Sigma^1, \nu = 1,2,\ldots, \lim_{\nu\to\infty} t_\nu = \infty, \lim_{\nu\to\infty} x_\nu = x\right\}$$

$$\tilde{\Sigma}_\infty = \Sigma_\infty \setminus \Sigma^1_\infty .$$

Assumption C. $S_\infty = S^1_0$ and this set can be bounded or not.

Assumption D. There exists a function $\tilde{f}$ of the argument (x,u,q,r,z) where $x \in S_\infty$, u, q, r are arbitrary, and $z \in \bar{C}(S_\infty, R^m)$, such that:

$$\forall \varepsilon > 0 \; \exists T > 0, \quad \forall (t,x) \in (D_p)_T, \; u,q,r \text{ are arbitrary } \; z \in \bar{C}(S_\infty)$$

for $i=1,\ldots,m$ $\quad |f^i(t,x,u,q,r,z) - \tilde{f}^i(x,u,q,r,z)| < \varepsilon$.

Definition 3. Let $s \in \bar{C}(\bar{S}_\infty, R^m) \cap C^2(S_\infty, R^m)$, which for every $i=1,\ldots,m$ has the first derivatives of s^i bounded on $S_\infty \cup \Sigma^1_\infty$, and which fulfills the system

$$0 = \tilde{f}(x, s(x), s^i_x(x), s^i_{xx}(x), s(\cdot)) \quad i=1,\ldots,m \tag{6}$$

and also boundary conditions on Σ_∞, which will be defined further (see Def 7)

Definition 4. Let for every $\xi \in (0, \xi_0)$, $p_\xi : [-\delta, \infty) \to R$, be define in the following way

$$\begin{cases} p_\xi(t) = \max\limits_{1\le i\le m} \sup\limits_{\substack{\tau \ge t \\ x\in S_\tau}} |f^i(\tau, x, s(x)+\xi, s^i_x(x), s^i_{xx}(x), s(\cdot)+\xi)| \text{ for } t \ge 0 \\ p_\xi(t) = p_\xi(0) \quad \text{for } -\delta \le t < 0, \quad \text{where} \quad \delta > 0 \text{ arbitrary.} \end{cases} \tag{7}$$

<u>Assumption E</u>. Assume that for every $\zeta \in C^2(S\ )$ the functions $f^i(t,x,\zeta(x),\ \zeta^i_x(x),\ \zeta^i_{xx}(x),\zeta(\cdot))$ are continuous in $\bar{D}$ and $\tilde{f}^i(x,\zeta(x),\ \zeta^i_x(x),\ \zeta^i_{xx}(x),\ \zeta(\cdot))$ are continuous in $\bar{S}_\infty$, for $i=1,\dots,m$.

<u>Remark 1</u>. The function p_ξ defined in (7) in virtue of Assumption E satisfies the following conditions: 1^o $p_\xi(t) \geq 0$ for $t \in [-\delta,\infty)$, 2^o $\lim_{t\to\infty} p_\xi(t) = 0$ (what follows from the Assumptions D,E and (6)), 3^o p_ξ is continuous and non-increasing in $[-\delta,\infty)$, 4^o $p_\xi(t) \geq |f^i(t,x,s(x)+\xi\ s^i_x(x), s^i_{xx}(x), s(\cdot)+\xi)|$ if $s(x) \in C^2(\bar{S}_\infty)$, for every $t \geq 0$, $x \in S_t$, $i = 1,\dots,m$, $\xi \in (0,\xi_o)$.
If Assumption E does not hold, then we can take an arbitrary p requiring only the conditions 1^o - 4^o to be satisfied. Therefore we can replace the Assumption E by a weaker one:

<u>Assumption E_1</u>. For every $\xi \in (0,\xi_o)$ there exists $p_\xi : [-\delta,\infty) \longrightarrow R$, which satisfies all the conditions 1^o - 4^o of the Remark 1.

<u>Remark 2</u>. If $S_\infty \neq S^1_o$, thus $S_\infty \subset S^1_o$, it should be necessary to extend s defined on $\bar{S}_\infty$ on the whole S^1_o. This would be possible if the boundary of S_∞ should be sufficiently regular, what generally does not hold.

<u>Definition 5</u>. Let $\mathcal{L}$ be the function of argument (t,x,u,q,s) where $(t,x) \in (D_p)_{T_o}$ for certain $T_o \geq 0$, $u \in R^m$, $q \in R^n$, $s \in \bar{C}(S^1_o)$, and its values belong to R^m.

<u>Assumption F</u>. For $\bar{T}_o \geq 0$ there exist four functions: $\mathcal{L}_k$ $k=1,2$ (see Def 5) and $\overset{1}{W} : \bar{S}^1_o \longrightarrow R^m_+$ and $\overset{2}{W} : \bar{S}^1_o \longrightarrow R^m_-$ continuous in $\bar{S}^1_o$, of class C^2 in S^1_o, which have the derivative $\frac{d}{dl^i(t,x)}\ \overset{k}{w}{}^i(x)$, $k=1,2$ if $(t,x) \in \sum^i$ $i=1,\dots,m$. We assume that:

1^o For every continuous function $\varphi: [0,\infty) \to R_+$ the functions $\mathcal{L}_k, \overset{k}{W}$, $k = 1,2$ satisfy for $i=1,\dots,m$ and arbitrary $\xi \in (0,\xi_o)$ the inequalities:

$$\operatorname{sgn} \overset{k}{W}\Big[f^i(t,x,\varphi(t)\ \overset{k}{W}(x)+\xi+s(x),\ \varphi(t)\ \overset{k}{W}{}^i_x(x)+s^i_x,\ s^i_{xx},\ \varphi(t)\ \overset{k}{W}(\cdot)\ +$$

$$+\ \xi + s(\cdot)) - f^i(t,x,s(x)+\xi,\ s^i_x(x),\ s^i_{xx}(x),\ s(\cdot)+\xi) \leq$$

$$\leq -\varphi(t)\mathcal{L}^i_k(t,x,\overset{k}{W}(x),\ \overset{k}{W}{}^i_x(x)\ \overset{k}{W}(\cdot)) \quad \text{in} \quad (D_p)_{T_o}. \tag{8}$$

2^0 the functions $\overset{1}{W}, \overset{2}{W}$, satisfy the conditions (for $i=1,\dots,m$, $k=1,2$)

a) $1 \leq (-1)^{k+1} \overset{k}{W}{}^{i}(x) \leq K$ for $x \in \bar{S}_0^1$

b) $(-1)^{k+1} \sum_{j,l=1}^{n} \overset{k}{W}{}^{i}_{x_j x_l}(x)\alpha_j \alpha_l \leq 0$ for $x \in S_0^1$

and every $\alpha = (\alpha_1,\dots,\alpha_n)$

c) there exists $\lambda > 0$ such that

$$\mathcal{L}^i_k(t,x,\overset{k}{W}(x), \overset{k}{W}{}^i_x(x), \overset{k}{W}(\cdot)) + (-1)^k \lambda \overset{k}{W}{}^i(x) > 0 \quad \text{in} \quad (D_p)_{T_0}$$

d) $(-1)^{k+1} M^i(\overset{k}{W})(t,x) > 0$ on $(\Sigma^i)_{T_0}$

Assumption G. Let us denote:

$$H_0 = \{\Phi : \bar{D} \to R^m,\ \Phi^i(t,x) = \varphi(t)W^i(x) \quad i=1,\dots,m\}$$

where $\varphi \in C^1([0,\infty),R_+)$ and $W^i(x) = \overset{k}{W}{}^i(x)$ $k=1,2$, as regular as it was assumed in Assumption F, and satisfying the conditions a), b). We assume that f is parabolic with respect to every function $\Phi \in H_0$.

Definition 6. Let $\phi : A \to R$ where $(t,x) \in A \subset R^{n+1}$ and $\tilde{\phi} : B \to R$ where $x \in B \subset R^n$, B is the projection of A onto R^n. Let us denote $\lim\limits_{(t,x)\in A,\, t\to\infty} \phi(t,x) = \tilde{\phi}(x)$ if:

$$\forall \varepsilon>0,\ \exists T(\varepsilon) \geq 0,\ \forall (t,x) \in A_T,\ |\phi(t,x) - \tilde{\phi}(x)| < \varepsilon$$

We introduce the following assumption concerning boundary conditions:

Assumption H. $\exists\, h_0 \in (0,1)$ such that $h^i(t,x) > h_0$ on Σ^i for $i=1,\dots,m$, and there exist functions: $\tilde{\varphi}^i_1 : \tilde{\Sigma}^i_\infty \cup S_\infty \to R$, $\tilde{\varphi}^i_2 : \Sigma^i_\infty \cup S_\infty \to R$, $\tilde{g}^i : \Sigma^i_\infty \cup S_\infty \to R$, $\tilde{h}^i : \Sigma^i_\infty \cup S_\infty \to R$, $\tilde{l}^i : \Sigma^i_\infty \cup S_\infty \to R^n$, moreover $\tilde{g}^i(x)$ are bounded functions on $\Sigma^i_\infty \cup S_\infty$. We assume that:

$$\lim_{(t,x)\in\Sigma^i,\ t\to\infty} \varphi^i_2(t,x) = \tilde{\varphi}^i_2(x),$$

$$\lim_{(t,x)\in\Sigma^i,\, t\to\infty} g^i(t,x) = \tilde{g}^i(x), \quad \lim_{(t,x)\in\Sigma^i,\, t\to\infty} h^i(t,x) = \tilde{h}^i(x)$$

$$\lim_{(t,x)\in\Sigma_j^i,\, t\to\infty} l_j^i(t,x) = \tilde{l}_j^i(x), \quad j=1,\dots,n, \quad \lim_{(t,x)\in\Sigma\setminus\Sigma^i,\, t\to\infty} \varphi_1^i(t,x) = \tilde{\varphi}_1^i(x)$$

for $i=1,\dots,m$ (see Def 6).

Now we can formulate the boundary conditions for the function s:

<u>Definition 7</u>. For $i=1,\dots,m$ we have

$$s^i(x) = \tilde{\varphi}_1^i(x) \quad \text{on} \quad \tilde{\Sigma}_\infty^i,$$

$$\tilde{h}^i(x)s^i(x) - \tilde{g}^i(x)\frac{d}{d\tilde{l}^i(x)}s^i(x) = \tilde{\varphi}_2^i(x) \quad \text{on } \Sigma_\infty^i.$$

<u>Theorem 1</u>. Let us suppose that the Assumptions C, B_1, B_2, hold, and that there exists $T_o \geqslant 0$ for which Assumptions F and G hold in $(D_p)_{T_o}$. There hold also the Assumptions D, E_1 and H. If there exists the solutions s of the system (6), as regular as it was assumed in Definition 3, and which satisfies boundary conditions according to Definition 7, then for Σ-regular solution u of the system (1) in $(D_p)_{T_o}$, satisfying the boundary conditions (3) on $(\Sigma)_{T_o}$, and such that f^o is parabolic with respect to the u, the condition

$$\lim_{t\to\infty}\|u(t,\cdot) - s(\cdot)\|_t = 0 \quad \text{is held (see Corollary 1).}$$

<u>Proof</u>: For arbitrary $\varepsilon > 0$, we put $\varepsilon_1 = h_o\varepsilon$ and let $T_1 \geqslant T_o$ be so large that from Assumption H we get

$$|\varphi_1^i(t,x) - \tilde{\varphi}_1^i(x)| < \frac{1}{2}\varepsilon_1 \quad \text{for} \quad (t,x)\in(\Sigma\setminus\Sigma^i)_{T_1}$$

and

$$|\varphi_2^i(t,x) - \tilde{\varphi}_2^i(x)| < \frac{\varepsilon_1}{2} \quad \text{for} \quad (t,x)\in(\Sigma^i)_{T_1}$$

We take, according with the Assumption F, the functions $\overset{k}{W}_i(x)$ for suitable $\lambda_1 > 0$ and we create two functions:

$$v^i(t,x) = J(t)\overset{1}{W}{}^i(x) + \varepsilon \quad v^i(t,x) = J(t)\overset{2}{W}{}^i(x) - \varepsilon \text{ where } J(t) > 0$$

for $t \geqslant 0$ we will establisch later. We have chosen T_1 in such a way that for $\tilde{u} = u - s$:

$$|\tilde{u}^i(t,x)| = |u^i(t,x) - s^i(x)| \leqslant |\varphi_1^i(t,x) - \tilde{\varphi}_1^i(x)| + |\tilde{\varphi}^i(x) - s^i(x)| < \varepsilon_1 \tag{9}$$

on $(\Sigma\setminus\Sigma^i)_{T_1}$ for $i=1,\dots,m$. We have also

$$M^i(\tilde{u}^i)(t,x) = \varphi_2^i(t,x) - \left[h^i(t,x)s^i(x) - g^i(t,x)\frac{d}{dl^i(t,x)}s^i(x)\right].$$

It is easy to see that in virtue of the Assumption H, for $\frac{\varepsilon_1}{2}$ there exists $T_2 \geq T_1$ such that

$$\left|h^i(t,x)s^i(x) - g^i(t,x)\frac{d}{dl^i(t,x)}s^i(x) - \tilde{\varphi}_2^i(x)\right| < \frac{\varepsilon_1}{2}$$

for every $(t,x) \in \Sigma^i_{T_2}$, and therefore

$$|M^i(\tilde{u}^i)(t,x)| < \varepsilon_1 \quad \text{for} \quad (t,x) \in (\Sigma^i)_{T_2},\ i=1,\dots,m. \tag{10}$$

Farther on $(\Sigma\setminus\Sigma^i)_{T_1}$ we have

$v^i(t,x) = J(t)\overset{1}{w}{}^i(x) + \varepsilon > \varepsilon > h_0\varepsilon = \varepsilon_1$ and

$v^i(t,x) < -h_0\varepsilon = -\varepsilon_1$ $i=1,\dots,m$. In virtue of (9) there is

$v^i(t,x) > \tilde{u}^i(t,x) > v^i(t,x)$ on $(\Sigma\setminus\Sigma^i)_{T_1}$ for $i=1,\dots,m$.

Because $M^i(v^i)(t,x) = J(t)\left[h^i(t,x)\overset{1}{w}{}^i(x) - g^i(t,x)\frac{d}{dl^i(t,x)}\overset{1}{w}{}^i(x)\right] + h^i(t,x)\varepsilon$, therefore $M^i(v^i)(t,x) > h_0\varepsilon = \varepsilon_1$ and similarly $M^i(v^i)(t,x) < -\varepsilon_1$ on $(\Sigma^i)_{T_1}$. It results from the, above statements and from (10) that for $i=1,\dots,m$

$$M^i(v^i-\tilde{u}^i)(t,x) > 0,\ M^i(u^i-v^i)(t,x) > 0 \quad \text{on} \quad (\Sigma^i)_{T_2}$$

We denote by $K_0 = \sup_{x\in S_{T_2}} |\tilde{u}(T_2,x)|$. Now we construct the function J.

According to Assumption E_1 we take, for the ε fixed above the function p_ε. We construct the function $\bar{p}:[-\delta,\infty) \to R$, which satisfies the conditions $1^0 - 3^0$ of E_1, and such that:

$\bar{p}(t) > p_\varepsilon(t)$ for $t > 0$,

$$\bar{p}(0) > \max\left(p_\varepsilon(0), \frac{\lambda_1 K_0}{(1-e^{-1})\exp(-\lambda_1 T_2)}\right),$$

$\bar{p}(t) = \bar{p}(0)$ for $t \in \left[-\frac{1}{\lambda_1}, 0\right)$. For this $\bar{p}(t)$ we set

$$J(t) = \int_{-\frac{1}{\lambda_1}}^{t} \bar{p}(\tau) \exp\lambda_1(\tau - t)d\tau.$$

This function has all the properties as it was formulated in the paper [6], particulary $\lim_{t\to\infty} J(t) = 0$. For our purpose we have now:

$$v^i(T_2,x) = J(T_2) \overset{1}{w}{}^i(x) + \varepsilon > \int_{-\frac{1}{\lambda_1}}^{0} \bar{p}(\tau)\exp[\lambda_1(\tau - T_2)]\, d\tau \geq$$

$$\geq \bar{p}(0) \frac{1}{\lambda_1}\left[1-e^{-1}\right] \exp(-\lambda_1 T_2) > K_0 \geq \tilde{u}^i(T_2,x) \quad \text{on} \quad S_{T_2}, \text{ for } i=1,\dots,m$$

and analogously

$$v^i(T_2,x) < -K_0 \leq \tilde{u}^i(T_2,x) \quad \text{on} \quad S_{T_2}.$$

Now we will prove that the functions V and v satisfy inequalities (4), (5) if on the right-hand side we will set the new function ζ defined as follows:

$$\zeta^i(t,x,u,q,r,z) = f^i(t,x,u+s,q+s^i_x, r+s^i_{xx}, z+s(\cdot)) \text{ for } (t,x)\in D_p \; u,q,r,z$$

arbitrary. Setting $\tilde{u} = u - s$ we see that

$$\tilde{u}^i_t = u^i_t = f^i(t,x,u(t,x)u^i_x(t,x), u^i_{xx}(t,x)u(t,\cdot) =$$

$$= f^i(t,x,\tilde{u}+s, \tilde{u}^i_s+s^i_x, \tilde{u}^i_{xx}+s^i_{xx},\tilde{u}(r,\cdot)+s(\cdot)) =$$

$$= \zeta^i(t,x,\tilde{u}, \tilde{u}^i_x, \tilde{u}^i_{xx}, \tilde{u}(t,\cdot)).$$

Now we will shaw that ζ is parabolic with respect to $\tilde{u}$. If $\tilde{r} \geq r$ then $\tilde{r} + s^i_{xx} \geq r + s^i_{xx}$ and since

$$\zeta^i(t,x,\tilde{u},\tilde{u}^i_x,\tilde{r},\tilde{u}(t,\cdot)) - \zeta^i(t,x,\tilde{u},\tilde{u}^i_x,r,\tilde{u}(t,\cdot)) =$$

$$= f^i(t,x,\tilde{u}+s,\tilde{u}^i_x+s^i_x,\tilde{r} + s^i_{xx}, \tilde{u}(t,\cdot) + s(\cdot)) -$$

$$- f^i(t,x,\tilde{u}+s,\ \tilde{u}^i_x + s^i_x, r + s^i_{xx},\ \tilde{u}(t,\cdot) + s(\cdot)) =$$

$$= f^i(t,x,u,u^i_x,\tilde{r} + s^i_{xx},\ u(t,\cdot)) -$$

$$- f^i(t,x,u,u^i_x, r + s^i_{xx},\ u(t,\cdot)) \geqslant 0$$ in virtue of parabolicity of f^i with respect to u.

Now applying succesively the properties of J and p, the condition a) of the Assumption F, after that the inequality (8), the conditions c) and b) of the Assumption F, and finally the Assumption G, we get

$$V^i_t(t,x) = \overset{1}{W}{}^i(x)\,\frac{dJ}{dt} \geqslant \bar{p}(t) - \lambda_1 \overset{1}{W}{}^i(x)J(t) \geqslant$$

$$\geqslant -\lambda_1 \overset{1}{W}{}^i(x)J(t) + f^i(t,x,s(x)+\varepsilon, s^i_x(x),\ s^i_{xx}(x),\ s(\cdot)+\varepsilon) \geqslant$$

$$\geqslant \Big[-\lambda_1 \overset{1}{W}{}^i(x) + \mathcal{L}^i_1(t,x,\overset{1}{W}(x),\overset{1}{W}{}^i_x(x),\overset{1}{W}(\cdot))\Big]J(t) +$$

$$+ f^i(t,x,V(t,x) + s(x),\ V^i_x(t,x)+s^i_x,\ s^i_{xx},\ V(t,\cdot) + s(\cdot)) >$$

$$> f^i(t,x,V(t,x) + s(x),\ V^i_x(t,x) + s^i_x, V^i_{xx} + s^i_{xx}, V(t,\cdot) + s(\cdot)) =$$

$$= \zeta^i(t,x,V(t,x),\ V^i_x(t,x),\ V^i_{xx}(t,x),\ V(t,\cdot)) \quad \text{in} \quad (D_p)_{T_1}.$$

Analogously we obtain:

$$v^i_t(t,x) < \zeta^i(t,x,v(t,x),\ v^i_x(t,x),\ v^i_{xx}(t,x),\ v(t,\cdot)) \quad \text{in} \quad (D_p)_{T_1}.$$

We have proved that for every $\varepsilon > 0$ there exists $T_2(\varepsilon) \geqslant T_0$ such that in $(D_p)_{T_2}$ the function V_ε and v_ε satisfy all the assumptions of the Lemmas 1 and 2. Therefore $v_\varepsilon(t,x) < \tilde{u}(t,x) < V_\varepsilon(t,x)$ in $(D_p)_{T_2}$.

Now we set $\varepsilon = \frac{\eta}{2} > 0$ and for $\frac{\eta}{2}$ we find $T_3 \geqslant T_2(\frac{\eta}{2}) \geqslant T_0$ such that for $i=1,\dots,m$

$$V^i_\eta(t,x) = J(t)\,\overset{1}{W}{}^i(x) + \frac{\eta}{2} < \eta,\quad v^i_\eta(t,x) = J(t)\,\overset{2}{W}{}^i(x) - \frac{\eta}{2} > -\eta \ \text{in} \ (D_p)_{T_3}.$$

Since in such a way constructed two families V_η and v_η satisfy all the assumptions of corollary 1, our proof is closed.

Remark 3. In the case of a homogeneous boundary problem we can put $\xi = 0$ in (8) of the Assumption F. The proof of Theorem 1 is then much simplier one (comp. [6], Theorem 1).

4. Some examples

Example 1. The function $u(t,x) = (\sin \frac{x}{\sqrt{a}}) \frac{t}{1+t^2}$ is the solution of the equation

$$u_t' = au_{xx} + u + (\sin \frac{x}{\sqrt{a}}) \frac{2t}{(1+t^2)^2}, \text{ where the domain } D_p = (0 < x < \pi\sqrt{a}) \times (0,\infty).$$

This solution satisfies homogeneous boundary conditions (comp. example 2 in [6]).

We have also $\lim_{t\to\infty} u(t,x) = \sin \frac{x}{\sqrt{a}} = s(x)$.

This function s satisfies the equation $as_{xx} + s = 0$ and is equal to zero for $x_1 = 0$ and $x_2 = \pi\sqrt{a}$. All the assumptions of the theorem 1 are fulfilled except the Assumption F, that will be shown later. That means F is the sufficient condition but it is not necessary, obviously. In the case of one equation we can put $V(t,x) = J(t)w(x)$ and $v(t,x) = -V(t,x)$, having $\varepsilon = 0$. (Comp, also the example 1 in [6]). Assume that there exist $\mathcal{L}$ and w fulfilling F with a), b), c), d). The inequality (8) obtains the form

$$as_{xx} + V + s + (\sin \frac{x}{\sqrt{a}}) \frac{2t}{(1+t^2)^2} - \left[as_{xx} + s + (\sin \frac{x}{\sqrt{a}}) \frac{2t}{(1+t^2)^2}\right] \equiv$$

$$\equiv V(t,x) = J(t)w(x) \leqslant -J(t)\,\mathcal{L}(w)(x)$$

in virtue of c) we have $J(t)w(x) < -\lambda J(t)w(x)$ hence $(1+\lambda)w(x) < 0$. This contradicts the condition a) in D.

Example 2. In the paper [4] there was proved the theorem 4 concerning convergence as $t \to \infty$ of the solution u of the following almost-linear equation

$$F[u] = \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j} + \sum_{k=1}^{n} b_k(x)u_{x_k} - u_t = f(t,x,u) \qquad (11)$$

for every $(t,x) \in G \times (0,\infty)$, where $G \subset R^n$ is bounded. Under the assumptions: $\lim_{t\to\infty} f(t,x,z) = f_1(x,z)$ uniformly with respect to $x \in G$, where f is non decreasing with respect to z, it is proved that $\lim_{t\to\infty} u(t,x) = s(x)$, where s is the solution of the equation

$$\sum_{i,j=1}^{n} a_{ij}(x)s_{x_ix_j} + \sum_{k=1}^{n} b_k(x)s_{x_k} - f_1(x,s(x)) = 0$$

for every $x \in G$. The proof of theorem 4 was carried out owing to the following assumption: there exists V such that: 1^o $V(t,x) > 0$ in $\bar{D}$. 2^o $\lim_{t\to\infty} V(t,x) = 0$, $3^o \exists$ $T \geq 0$ such that $F(V) \leq -|f(t,x,s(x))-f_1(x,s(x))|$ in D_T. It is easy to prove that from the condition 3^o follows that V and $v = -V$ satisfy inequalities (4) and (5) respectively, but not inversely. But we have another argument because of which the theorem 1, proved above, is weaker than the theorem 4 in [4]. We will shaw that the following condition results from the Assumption F.

$$F(w) \leq -\beta, \quad \beta > 0 \tag{12}$$

(which was assumed in [4]) but not inversely. The inequality (8) has now the form:

$$\sum_{k=1}^{n} b_k V_{x_k} - f(t,x,V+s) + f(t,x,s) \leq -J(t)\mathcal{L}(t,x,w,w_x^1).$$

Since $f(t,x,s) - f(t,x,V+s) < 0$ we can put $\mathcal{L}(w) = -\sum_{k=1}^{n} b_k w_{x_k}$ and the inequality c) obtains the form:

$$-\sum_{k=1}^{n} b_{x_k}(x)w_{x_k}(x) - \lambda w(x) > 0 \quad \text{hence} \quad \sum_{k=1}^{n} b_k(x)w_{x_k} < -\lambda w(x) \leq -\lambda,$$

using the condition b) we have $\sum_{i,j=1}^{n} a_{ij}(x)w_{x_ix_j} \leq 0$, setting $\lambda = \beta$ we get

$$F(w) = \sum_{i,j=1}^{n} a_{ij}(x)w_{x_ix_j} + \sum_{k=1}^{n} b_k(x)w_{x_k} - w_t \leq -\beta$$

But obviously from (12) we cannot conclude that there exists $\lambda > 0$ for which $\sum_{k=1}^{n} b_k(x)w_{x_k} < -\lambda w$. This requires another form of assumption c) in order to obtain the equivalence of the both assumptions (comp. Example 3).

Remark 4. When we consider the space $R^1 \ni x$, the assumption $S^1_o = S_\infty$ is obviously superfluous. If $S_\infty \subset S^1_o$, it is posible to extend the functions s on the whole S^1_o maintaining the class of regularity. In this case the Theorems 1 and 2 of [6] are simple corollaries of the above theorem.

5. Strong parabolicity of f, and Theorem 2

Now we carry into effect the modification of the condition c) of F, that we have talked about before. We will establish a property of f, which will be called the strong parabolicity.

Definition 8. We will call strongly parabolic every function f for which there exist functions $a^i: D_p \longrightarrow R^{n^2}$, $i=1,\dots,m$ such that:

1^o there exist i such that $a^i \not\equiv 0$ in D_p and for $i=1,\dots,m$,

$a^i_{jk}(t,x) = a^i_{kj}(t,x)$ in D_p,

2^o $\sum_{j,k=1}^{n} a^i_{jk}(t,x)\varkappa_j\varkappa_k \geqslant 0$ in D_p, $i=1,2,\dots,m$, for every $\varkappa = (\varkappa_1,\dots\varkappa_n)$,

3^o for every pair of symmetric matrices $r = [r_{ij}]$, $\tilde{r} = [\tilde{r}_{ij}]$ for which

$\tilde{r} \geqslant r \iff \sum_{i,j=1}^{n} (\tilde{r}_{ij} - r_{ij})\varkappa_i\varkappa_j \geqslant 0$, the following inequalities, for $i=1,\dots,m$,

$$f^i(t,x,u,q,\tilde{r},z) - f^i(t,x,u,q,r,z) \geqslant \sum_{j,k=1}^{n} a^i_{jk}(t,x)(\tilde{r}_{jk} - r_{jk}) \qquad (13)$$

for every $(t,x) \in D_p$, u,q arbitrary and $z \in \overline{C}(S_t)$, hold.

Remark 5. The Definition 8 was introduced in paper [5] (the condition H) with an additional assumption: there exists k such that $a^i_{kk}(t,x) \geqslant a^i(t) > 0$ in D_p, what will be superfluous now.

Remark 6. The strong parabolicity of f with respect to the function u, would be dificult to define, because a^i could then become dependent on u. This assumption would be too weak for our needs.

Assumption F_1. We keep in virtue all the assumptions of F, except the condition c) instead of which now we introduce:

c_1) there exists $\lambda_1 > 0$ such that

$$\mathcal{L}^i_k(t,x,\overset{k}{W}(x), \overset{k}{W}{}^i_x(x), \overset{k}{W}(\cdot)) + (-1)^k\lambda_1 \overset{k}{W}{}^i(x) +$$

$$+ (-1)^k \sum_{j,l=1}^{n} a^i_{jl}(t,x) \overset{k}{W}{}^i_{x_j x_l} > 0 \quad \text{for} \quad \text{every} \quad (t,x \in (D_p)_T.$$

Remark 7. The stronger property of of f allows us to employ weaker condition c_1) instead of c) (taking into account the inequalities

$$(-1)^k \sum_{j,l=1}^{n} a^i_{jl}(t,x) \overset{k}{W}{}^i_{x_j x_l} \geq 0).$$

We notice also that Assumption G is superfluous if f is strong parabolic.

Now we can formulate:

Theorem 2. We assume that the domain D has a property resulting from Assumption C, and that Assumptions D, B_1, B_2 holf. Moreover let f be strong parabolic and the Assumption F_1 hold in $(D_p)_{T_0}$ and H on $(\Sigma)_{T_0}$ for certain $T_0 \geq 0$. If there exists the solution s of (6) as regular as it was required in Definition 3, and satisfying the boundary conditions in agreement with Definition 7, then the Σ-regular solution of the system (1) satisfying the boundary conditions (3) has the following property

$$\lim_{t\to\infty} \| u(t,\cdot) - s(\cdot)\|_t = 0$$

We omit the proof quite analoguous to the proof of theorem 1.

6. Example 3

We will show that the criterion given by theorem 2 is essentially stronger than that given by theorem 1. It is easy to see that from c_1) it results $F(\overset{1}{W}) \leq -\beta$ (comp. (11)) for $\lambda_1 = \beta$. But now the inverse inclusion: $F(\overset{1}{W}) \leq -\beta \Longrightarrow C_1$), is true. This follows immediately, since for $\overset{1}{W}$ we have

$$-F(\overset{1}{W}) - \lambda \overset{1}{W} \geq \beta - \lambda \overset{1}{W} \geq \beta - \lambda K > 0 \quad \text{if only} \quad 0 < \lambda < \frac{\beta}{K}$$

and for $\overset{2}{W} = -\overset{1}{W}$, we verify C_1). Therefore the theorems 4 and 5 of the paper [4] become corolleries of the theorem 2.

7. The case of the limit zero, and Theorem 3

It is worthy to stress that the fundamental theorems concerning asymptotic stability proved by M. Krzyżański in [1], [2] and their generalization in [3], [5], do not result from theorems contained in [6]. Now using the strong parabolic property, we will obtain this generalization.

Assumption F_2. For $T_o \geq 0$ there exist four functions: $\mathcal{L}_k$ k=1,2 (see Def. 5) and $\overset{1}{W}, \overset{2}{W}$ as regular as it was supposed in Assumption F. Now we assume that:

1^o For every continuous function $\varphi:[0,\infty) \to R_+$ the functions $\mathcal{L}_k$, $\overset{k}{W}$, k=1,2, satisfy for i=1,...,m and arbitrary $\xi(0,\xi_o)$ the inequalities:

$$\operatorname{sgn} \overset{k}{W}\Big[f^i(t,x,\varphi(t)\overset{k}{W}(x)+\xi,\ \varphi(t)\overset{k}{W}{}^i_x(x),0,\varphi(t)\overset{k}{W}(\cdot)+\xi) -$$

$$- f^i(t,x,\xi,0,0,\xi) \leq -\varphi(t)\mathcal{L}^i_k(t,x,\overset{k}{W}(x),\overset{k}{W}{}^i_x(x),\overset{k}{W}(\cdot)) \quad \text{in } (D_p)_{T_o}.$$

2^o The functions $\overset{1}{W}, \overset{2}{W}$, satisfy the conditions a) b) d), which were formulated in Assumption F, and the condition c_1) from Assumption F_1

Assumption E_2: There exists $p_\xi: [-\delta,\infty) \to R$ such that: $\forall \xi \in (0,\xi_o)$
1^o $p_\xi(t) > 0$, 2^o $\lim\limits_{t\to\infty} p_\xi(t) = 0$, 3^o p_ξ is continuous and non-increasing,
4^o $p_\xi(t) \geq |f^i(t,x,\xi,0,0,\xi)|$ in D_p, for i = 1,...,m (comp. [6]).

Assumption H_1: Let $h^i(t,x) \geq h_o \in (0,1)$ on Σ^i, for i=1,...,m.
We have:

$$\forall \varepsilon > 0\ \exists\ T(\varepsilon) \geq 0, \text{ such that } |\varphi^i_1(t,x)| < \varepsilon \text{ for every } (t,x) \in$$
$$\in (\Sigma \setminus \Sigma^i)_{T(\varepsilon)},\ |\varphi^i_2(t,x)| < \varepsilon \text{ for every } t(x) \in (\Sigma^i)_{T(\varepsilon)}.$$

Let us notice that now we do not need neither the sets S_∞ and Σ_∞ nor the Assumption C.

Theorem 3. Assume that f is strongly parabolic in $(D_p)_{T_o}$, for $T_o \geq 0$. If B_1, B_2, H_1 and E_2 hold, and moreover F_2 (for the same T_o), then the Σ-regular solution u of system (1) satisfying the boundary conditions (3) has the following property $\lim\limits_{t\to\infty} \|u(t,\cdot)\|_t = 0$

We omit the proof which is analogous to the proof of theorem 2 in [6].

In the similar way we can formulate two theorems on the convergence of the solution to $+\infty$ or $-\infty$ under the assumption of strong parabolicity of f, using again the conditions c_1).

REFERENCES

[1] M. Krzyżański: "Sur l'allure asymptotique des solutions d'équation du type parabolique" Bull. Acad. Polon. Sci. S1, III 4(1956) 243-247.

[2] M. Krzyżański: "Sur l'allure asymptotique des solutions des problèmes der Fourier relatifs à une équation linéaire parabolique" Atti Accad. Naz. Lincei Rend 28(1960) 37-43.

[3] I. Łojczyk-Królikiewicz: L'allure asymptotigue des solutions des problèmes de Fourier relatifs aux équations linéaires normales du type parabolique dans l'espace E^{m+1}" Ann. Polon. Math. 14(1963) 1-12.

[4] I. Łojczyk-Królikiewicz: "Propriétés limites des solutions des problèmes de Fourier relatifs à l'équation presque linéaire du type parabolique" Bull. de l'acad. Polon des Sc. Vol VIII Nr 9 (1960) 597--603.

[5] I. Łojczyk-Królikiewicz: "Sur la stabilité asymptotique de la solution d'un système linéare aux dérivées partielles du type parabolique, II" Ann Polon. Math. XXI (1968) 15-20.

[6] I. Łojczyk-Królikiewicz: "The asymptotic behaviour of solutions to systems of parabolic differential-functional equations", in press, in the present monograph.

[7] J. Szarski: "Strong maximum principle for non-linear parabolic differential-functional inequalities in arbitrary domains" Ann. Polon. Math. XXXI (1975) 197-203.

[8] J. Szarski: "Differential inequalities" PWN, Warszawa "Monografie Matematyczne tom 43 (1965).