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THE ASYMPTOTIC BEHAVIOUR OF SOLUTIONS TO SYSTEMS OF PARABOLIC DIFFERENTIAL-FUNCTIONAL EQUATIONS

1. Introduction

In this paper we will investigate the solutions of the system

$$u_t^i = f^i(t,\ x,\ u,\ u_x^i,\ u_{xx}^i,\ u\ \ (t,\cdot)) \tag{1}$$

$i = 1,\dots,m$, with linear boundary conditions.
We will establish certain sufficient propositions under which the solution has the limit zero, or ∞, or $-\infty$ as $t \to \infty$.

The studies of these problems were begun by M. Krzyżański ([1], [2]) and were generalized later in [3], [4].

The results of the present paper base on the theorems on differential-functional inequalities given in Remark 5 below.

Definitions and notations

Let D be an open set in the R^{n+1} space of the variables $(t,\ x) = (t,x_1,\dots,x_n)$, and assume that interval $(0,\infty)$ constitutes the projection of D onto the t-axis, and S_0^1 is the projection of D on $R^n \ni x = (x_1,\dots,x_n)$. S_0^1 may be bounded or not.

For an arbitrary set $E \subset R^{n+1}$ and $T \geq 0$ let us denot by E_T the set:

$$E_T = E \cap \{(t,x)\ :\ t > T, x \in R^n\}$$

Let D_p be a subset of these points $(\tilde{t},\tilde{x}) \in \overline{D}$, for which there exists a half-neighbourhood:

$$\left\{(t,x)\ :\ (t < \tilde{t}) \wedge \sum_{j=1}^{n} (x_j - \tilde{x}_j)^2 + (t - \tilde{t})^2 < r^2\right\}$$

containing in the domain D. It is obvious that $D \subset D_p$.

$S_{\tilde{t}}(\tilde{t} > 0)$, denotes the projection of the set $D_p \cap \{(t,x)\ :\ t = \tilde{t}\}$ onto R^n. $S_{\tilde{t}}$ is an open set for any $\tilde{t} > 0$.

Let $\sum$ be such subset of the boundary ∂D, that:

1^0 $\sum = \partial D \cap \{(t,x) : t > 0\}$, $2^0 \sum \cap D_p = \emptyset$. Denote $S_0 = \partial D \cap \{(t,x) : t = 0\}$

Assumption A. S_0 and $S_{\tilde{t}}$ for any $\tilde{t} > 0$, are bounded sets.

Let $g^i : \sum^i \to R_+$, $h^i : \sum^i \to R_+$ $i = 1,\dots,m$, where $\sum^i \subset \sum$ ($\sum^i$ may be empty for certain i).

In every point $(t,x) \in \sum^i$, there exists the direction $l^i(t,x)$ orthogonal to t-axis, and there exists an open interval of the half-line of the direction l^i emerging from the point (t,x) which is also the point of this interval, contained in D_p.

For every $t > 0$, we denote by $\bar{C}(S_t)$ the space of the continuous and bounded functions $Z(\cdot) = (Z^1(\cdot),\dots,Z^m(\cdot))$: $S_t \ni x \to Z(x) \in R^m$, which are put in order in the following manner:

$$Z(\cdot) \leq \tilde{Z}(\cdot),\ (Z < \tilde{Z}) \Longleftrightarrow Z^j(x) \leq \tilde{Z}^j(x),\ (Z^j(x) < \tilde{Z}^j(x)),$$

for every $x \in S_t$, $i = 1,\dots,m$.

In this space we introduce the norm:

$$\|Z\|_t = \max_{1 \leq i \leq m} \sup_{x \in S_t} |Z^i(x)| \tag{2}$$

Let f be the function of argument (t,x,u,q,r,s) where $(t,x) \in D_p$, $u = (u^1,\dots,u^m) \in R^m$, $q = (q_1,\dots,q_n) \in R^n$, $r = (r_{11}, r_{12},\dots,r_{nn}) \in R^{n^2}$ and $r_{ij} = r_{ji}$, $i,j = 1,\dots,n$, $s \in \bar{C}(S_t)$, its values belong to R^m. Denote $f = (f^1,\dots,f^m)$.

Definition 1. A function $u : \bar{D} \to R^m$, is said to be $\sum$-regular if u^i are continuous functions for $i = 1,\dots,m$ in $\bar{D}$, u^i_x, u^i_{xx}, u^i_t are continuous in D_p, and in every point $(t,x) \in \sum^i$, $i = 1,\dots,m$, there exists the derivative $\frac{du^i}{dl^i}$ in the direction $l^i(t,x)$.

Definition 2. We say that u is the $\sum$-regular solution of the system (1) if for every $(t,x) \in D_p$ and $u(t,\cdot) \in \bar{C}(S_t)$, u is a function $\sum$-regular, and if it constitutes the solution of the system (1).

The function $u(t,\cdot)$ we define as follow: $u(t,\cdot) : S_t \to R^m$, $u(t,\cdot)(x) = u(t,x)$.

<u>Definition 3</u>. We say that f is parabolic with respect to the $\sum$-regular function u, if for every pair of arguments $r, \tilde{r} \in R^{n^2}$, the inequality

$$f^i(t,x,u(t,x), u_x^i(t,x),\tilde{r},u(t,\cdot)) - f^i(t,x,u(t,x),u_x^i(t,x),r,u(t,\cdot)) \geq 0 \tag{4}$$

for $(t,x) \in D_p$, $i = 1,\dots,m$ and $\tilde{r} \geq r$, holds.

$\tilde{r} \geq r$ means, that for every $\alpha = (\alpha_1,\dots,\alpha_n) \in R^n$, $\sum_{j,k=1}^{n} (\tilde{r}_{jk} - r_{jk})\alpha_j\alpha_k \geq 0$, where $r_{ij} = r_{ji}$ and $\tilde{r}_{ij} = \tilde{r}_{ji}$ (comp. [5]).

<u>Definition 4</u>. We say that $\sum$-regular function u, satisfies strong boundary inequalities if there exists $T \geq 0$ such that for $i = 1,\dots,m$ we have (comp. [5])

1^0 $u^i(T,x) < 0$ for every $x \in \bar{S}_T$

2^0 $M^i(u)(t,x) = h^i(t,x)u^i(t,x) - g^i(t,x)\frac{d^i}{dl}u^i(t,x) < 0$

for every $(t,x) \in \sum_t^i$, and

3^0 $u^i(t,x) < 0$ for every $(t,x) \in (\sum \setminus \sum^i)_T$.

Using the theorems about the differential-functional inequalities given in Remark 5 bolew, we are going to formulate the Lemma 1 under the Assumption B_1 and Lemma 2 under the Assumption B_2.

We consider the system (1) with the following boundary conditions:

$$\left.\begin{aligned} &u^i(t,x) = \varphi_1^i(t,x) \quad \text{for } (t,x) \in \textstyle\sum \setminus \sum^i \\ &M^i(u)(t,x) = \varphi_2^i(t,x) \quad \text{for } (t,x) \in \textstyle\sum^i \\ &u^i(0,x) = \varphi_0^i(t,x) \quad \text{for } (x) \in S_0 \end{aligned}\right\} \tag{5}$$

<u>Lemma 1</u>. Let $T \geq 0$, and the $\sum$-regular function V satisfies for $(t,x) \in (D_p)_T$, $i = 1,\dots,m$, the inequalities:

$$V_t^i(t,x) > f^i(t,x,V(t,x),V_x^i(t,x),V_{xx}^i(t,x),V(t,\cdot)) \tag{6}$$

where $V(t,\cdot) \in \bar{C}(S_t)$. We assume also that u is the $\sum$-regular solution of the system (1) in D_p, u satisfies the boundary condition (5), and f is parabolic with respect to u. If the difference $u - V$

satisfies the strong boundary inequalities according to definition 4 then $u \leq V$ in $(D_p)_T$.

Lemma 2. Let $T \geq 0$, and the $\sum$-regular function v satisfies for $(t,x) \in (D_p)_T$, $i = 1,\dots,m$, the inequalities:

$$v_t^i(t,x) < f^i(t,x),v,v_x^i(t,x),v_{xx}^i(t,x),v(t,\cdot)) \tag{7}$$

where $v(t,\cdot) \in \bar{C}(S_t)$. If u satisfies the Assumption of Lemma 1 and the difference $v - u$ satisfies the strong boundary inequalities then $v \leq u$ in $(D_p)_T$.

Corollary 1. If all the Assumptions of the Lemmas 1 and 2 hold for $T \geq 0$ and besides for $i = 1,\dots,m$:

1^o $\forall (t,x) \in \bar{D}_T$ $V^i(t,x) > 0$, $v^i(t,x) < 0$

2^o $\forall \varepsilon > 0$, $\exists$ $T_o \geq T$, $\forall (t,x) \in (D_p)_{To}$

$V^i(t,x) < \varepsilon$ and $v^i(t,x) > -\varepsilon$

then $\lim_{t\to\infty} \|u(t,\cdot)\|_t = 0$ the norm $\|\cdot\|_t$ is defined analogously as in (2).

Corollary 2. If we create two families of functions V_ε and v_ε such that:

1^o $\forall \varepsilon > 0$ $\exists$ T_o such that in $(D_p)_{T_o}$ all the assumptions of Lemmas 1 and 2 hold,

2^o $\forall \varepsilon > 0$, $V_\varepsilon(t,x) > 0$, $v_\varepsilon(t,x) < 0$ in $(D_p)_{T_o}$

3^o $\forall \varepsilon > 0$, for T_o from 1^o, $\exists$ $T_1 \geq T_o$ such that $V_\varepsilon(t,x) < \varepsilon$ and $v_\varepsilon(t,x) > -\varepsilon$ in $(D_p)_{T_1}$

then $\lim_{t\to\infty} \|u(t,\cdot)\|_t = 0$

Proof. It results from Lemmas 1, 2 that: $\forall \varepsilon > 0$

$V_\varepsilon(t,x) > u(t,x) > v_\varepsilon(t,x)$ in $(D_p)_{T_o}$

Therefore $-\varepsilon < u(t,x) < \varepsilon$ in $(D_p)_{T_1}$. Hence $\|u(t,\cdot)\|_t < \varepsilon$ for $t > T_1$, what complete the proof.

Remark 1. The assertions of the Lemmas are valid independently of the initial condition φ_o on S_o, if we are only able to construct the function v and V just such that $v(T,x) < u(T,x) < V(T,x)$ on S_T.

We will establish conditions for the function f, which will enable the effective construction of the function v and V.

2. Homogeneous boundary conditions

Assumption C. There exists $p: [-\delta, \infty) \to R$ where $\delta > 0$ is fixed but arbitrary, satisfying:

1^o $p(t) > 0$

2^o $\lim_{t\to\infty} p(t) = 0$

3^o p is continuous, non-increasing

4^o $p(t) \geqslant |f^i(t,x, 0,0,0,0)|$ in D_p, $i = 1,\dots,m$.

We can construct the function p effectively if we introduce:

Assumption D. We denote $\zeta^i(t,x) = f^i(t,x,0,0,0,0)$ for $(t,x) \in D_p$ and assume fhat $\zeta = (\zeta^1,\dots,\zeta^m)$ is a continuous function in $\bar{D}$ and $\lim_{t\to\infty} \|\zeta(t,\cdot)\|_t = 0$.

We put: $p(t) = \sup_{\tau \geqslant t} \|\zeta(\tau,\cdot)\|_\tau$ for $t \geqslant 0$ and $p(t) = p(0)$ for $-\delta < t < 0$

If there exists an index i such fhat $\zeta^i(t,x) \not\equiv 0$ for every $T \geqslant 0$ in D_T then function p constructed above satisfies all the conditions 1^o-4^o. If for every index i there exists $T_i \geqslant 0$ such that $\zeta^i(t,x) \equiv 0$ in D_{T_i}, then $p(t)$ may be constructed arbitrarily, according to the Assumption C.

As the Assumption C is weaker than D, we will use it in further consideration.

With the help of the function p we can construct the following function

$$J_\lambda(t) = \int_{-\delta}^{t} p(S)\exp\lambda(S - t)dS$$

which, for every $\lambda > 0$, has properties important for us (these properties were proved in [4])

1^o $J_\lambda(t) > 0$ for $t \geqslant 0$

2^o $J_\lambda(t)$ is continuous and has the continuous derivative for $t \geqslant 0$

3^o $\lim_{t\to\infty} J_\lambda(t) = 0.$

Now we can construct the function $V = (V^1,\dots,V^m)$ in the following way:

$v^i(t,x) = J(t)w^i(x)$, $i = 1,\dots,m$, but we shall still need another assumptions for the adequate choice of functions w^i.

Definition 5. Let L be the function of argument (t,x,u,q,s), where $(t,x) \in (D_p)_{T_0}$, $u \in R^m$, $q \in R^n$, $s \in \bar{C}(S_t)$ its values belong to R^m.

Assumption E. For $T_0 \geqslant 0$, there exist four functions: L_k, $k = 1,2$ (see Def. 5) and $\overset{1}{w} : S_0^1 \longrightarrow R_+^m$ and $\overset{2}{w} : S_0^1 \longrightarrow R_-^m$ continuous in $\bar{S}_0^1$, of class C^2 in S^1, which have the derivatives $\frac{d}{dl^i(t,x)} \overset{k}{w}{}^i(x)$, $k = 1,2$ if $(t,x) \in \sum^i$, $i = 1,\dots,m$. We assume that:

1^o For every continuous function $\varphi : [0,\infty) \longrightarrow R_+$, the functions L_k, $\overset{k}{w}$, $k = 1,2$, satisfy the inequalities, for $i = 1,\dots,m$,

$$\operatorname{sgn} \overset{k}{w}\Big[f^i(t,x,\varphi(t)\overset{k}{w}(x),\varphi(t)\overset{k}{w}{}^i_x(x),0,\varphi(t)\overset{k}{w}(\cdot)) -$$
$$-f^i(t,x,0,0,0,0)\Big] \leqslant -\varphi(t)L^i_k(t,x,\overset{k}{w}(x),\overset{k}{w}{}^i_x(x),\overset{k}{w}(\cdot))$$

in the domain $(D_p)_{T_0}$.

2^o The functions $\overset{1}{w}$, $\overset{2}{w}$ satisfy the following conditions, for $i = 1,\dots,m$, $k = 1, 2$:

a) $1 \leqslant (-1)^{k+1} \overset{k}{w}{}^i(x) \leqslant K$ for $x \in S_0^1$,

b) $(-1)^{k+1} \sum_{j,l=1}^{m} \overset{k}{w}_{x_j x_l}(x)\, \alpha_j \alpha_l \leqslant 0$ for $x \in S_0^1$ and every $\alpha = (\alpha_1,\dots,\alpha_n)$

c) there exists $\lambda > 0$, such that $L^i_k(t,x,\overset{k}{w}(x),\overset{k}{w}{}^i_x(x), \overset{k}{w}(\cdot)) +$
$+ (-1)^k \lambda \overset{k}{w}{}^i(x) > 0$ in $(D_p)_{T_0}$,

d) $(-1)^{k+1} M^i(\overset{k}{w})(t,x) > 0$ on $(\sum^i)_{T_0}$.

Assumption F. Let us denote:

$H_0 = \{\Phi : \bar{D} \rightarrow R^m,\ \Phi^i(t,x) = \varphi(t)\, w^i(x), i = 1,\dots,m\}$ where $\varphi \in C^1([0,\infty), R_+)$ and $w^i(x) = \overset{k}{w}{}^i(x)$, $k = 1, 2$ are as regular as it was assumed in Assumption E, and satisfy the conditions a), b). We assume that f is parabolic with respect to every function $\Phi \in H_0$.

We prove now

Theorem 1. Let us suppose that f satisfies Assumptions F, B_1, B_2 and the Assumtions C and E hold, for $T_o = 0$, in D_p. Let u be the $\sum$-regular solution of the system (1) in D_p, satisfying the boundary condition (5) with:

$$\varphi_1^i \equiv 0 \quad \text{on} \sum \setminus \sum{}^i, \quad \varphi_2^i \equiv 0 \quad \text{on} \quad \sum{}^i \quad \text{and} \quad \varphi_0^i \equiv 0 \quad \text{on} \quad S_o.$$

We assume moreover that f is parabolic with respect to the u. Then $\lim_{t\to\infty} \|u(t,\ )\|_t = 0$.

Proof: We take $\lambda > 0$, for which the condition c) of Assumption $_2E$ is satisfied. Put $J(t) = J_\lambda(t)$ and $V^i(t,x) = J(t)\overset{1}{w}{}^i(x), v^i(t,x) = J(t)\overset{2}{w}{}^i(x)$, $i = 1,\dots,m$ in D_p. Notice fhat $V = (V^1,\dots,V^m)$ and $v = (v^1,\dots,v^m)$ are the $\sum$-regular function in D_p and $V(t,\cdot)$, $v(t,\cdot)$ belong to $\bar{C}(S_t)$ for $t > 0$.

We will prove that the functions V and v, constructed above satisfy the propositions of Lemmas 1, 2. First we verify inequalities (6). We apply succesively the Assumption C, inequality (8) for $k = 1$, conditions a), c) and b) of the Assumption E, and finally Assumption F, and by this way we abtain

$$V_t^i(t,x) = \overset{1}{w}{}^i(x)J'(t) \geqslant p(t) - \lambda\overset{1}{w}{}^i(x)J(t) \geqslant -\lambda\overset{1}{w}{}^i(x)J(t) +$$

$$f^i(t,x,0,0,0,0) \geqslant J(t)\Big[-\lambda\overset{1}{w}{}^i(x) + L_1^i(t,x,\overset{1}{w}(x),\overset{1}{w}{}_x^i(x),\overset{1}{w}(\cdot)\Big] +$$

$$+ f^i(t,x,V(t,x),\ V_x^i(t,x),0,V(t,\cdot)) >$$

$$> f^i(t,x,V(t,x),\ V_x^i(t,x),\ V_{xx}^i(t,x),\ V(t,\cdot)) \quad \text{in } D_p.$$

Analogously we shaw that v satisfy the inequality (7) in D_p.

Now we will deal with the boundary conditions. On $\sum\setminus\sum^i$ for $i = 1,\dots,m$, and on S_o we have $v^i(t,x) < 0 \equiv u(t,x) < V^i(t,x)$. It results from the condition d) of the Assumption E that we get

$$M^i(u^i - V^i)(t,x) = -h^i(t,x)V^i(t,x) + g^i \frac{d}{dl^i} V^i(t,x) =$$

$$= J(t)(g^i(t,x) \frac{d}{dl^i} \overset{1}{w}{}^i(x) - h^i\overset{1}{w}{}^i(x)) < 0$$

and similarly $M^i(v^i - u^i)(t,x) < 0$ on $\sum^i$, $i = 1,\dots,m$.

Furthermore applying the condition 3° of the function J (and the inequality a) of Assumption E) we conclude that all the propositions of the corollary 1 hold in the domain D, that completes the proof.

Under linear boundary conditions the theoreme 1 in [4] is a particular case of the above theorem.

3. Some examples

Example 1. We consider the case when the system (1) is reduced to the one differential equation of the parabolic type:

$$u_t(t,x) = \sum_{i,j=1}^{n} a_{ij}(t,x)u_{x_i x_j}(t,x) + \sum_{k=1}^{n} b_k(t,x)u_k(t,x) +$$

$$+ c(t,x)u(t,x) - f_1(t,x) \tag{9}$$

Now the inequality (8) has the following form:

$$\operatorname{sgn} \overset{k}{w}\left\{\varphi(t)\left[\sum_{j=1}^{n} b_j(t,x)\overset{k}{w}_{x_j}(x) + c(t,x)\overset{k}{w}(x)\right]\right\} \leqslant$$

$$\leqslant -\varphi(t)L_k(t,x,\overset{k}{w}(x),\overset{k}{w}_x(x)),\ k = 1,\ 2,\ \text{for every } (t,x) \in D_p.$$

We note that in the case of $\operatorname{sgn} w = 1$ we can put

$$L_1(t,x,\overset{1}{w}(x),\overset{1}{w}(x)) = -\sum_{j=1}^{n} b_j(t,x)\overset{1}{w}_{x_j}(x) - c(t,x)\overset{1}{w}(x)$$

and if $\operatorname{sgn} \overset{2}{w} = -1$ then

$$L_2(t,x,\overset{2}{w}(x),\overset{2}{w}_x(x)) = \sum_{j=1}^{n} b_j(t,x)\overset{2}{w}_{x_j}(x) + c(t,x)\overset{2}{w}(x).$$

Hence the inequalities c) can be written in the form

$$\left.\begin{aligned} &\sum_{k=1}^{n} -b_k(t,x)\overset{1}{w}_{x_k}(x) - (c(t,x) + \lambda)\overset{1}{w}(x) > 0 \\ &\sum_{k=1}^{n} b_k(t,x)\overset{2}{w}_{x_k}(x) + (c(t,x) + \lambda)\overset{2}{w}(x) > 0 \end{aligned}\right\} \tag{10}$$

The coefficients b_k and c are known, therefore taking an arbitrary parameter $\lambda > 0$ we can find the solutions of the inequalities (10). We can give an example of such solutions:

Obviously we have $\overset{2}{w} = -\overset{1}{w}$. Assume for $k = 1,\dots,m$, $b_k(t,x) \leqslant \leqslant -B_o < 0$, $c(t,x) \leqslant c_o$, $c_o \geqslant 0$ and let the following condition $B_o - Kc_o > 0$ hold true for $K > 1$. Suppose S_o^1 is bounded, denote $R = \operatorname{diam} S_o^1$ and for suitable fixed i_o we set $w(x) = K - \exp(R - x_{i_o})$, assuming that the origin of coordinates $\in S_o^1$.

Then we have $1 \leqslant \overset{1}{w}(x_{i_o}) \leqslant K$, where $K > \max(\beta + 1, \beta \frac{g_o + h}{h_o})$, $\beta = \exp 2R$, $0 < g^1(t,x) \leqslant g_o$ and $0 < h_o \leqslant h^1(t,x) \leqslant h$ on $\sum^1$. If we set $0 < \lambda < \frac{B_o - Kc_o}{K}$ then the first one of the inequalities (10) obtains the form:

$$-\sum_{k=1}^{n} b_k(t,x)\overset{1}{w}_{x_k}(x) - (c(t,x) + \lambda)\overset{1}{w}(x) \geqslant B_o - K(c_o + \lambda) > 0.$$

We see that $\overset{2}{w} = -\overset{1}{w}$ satisfies the second of the inequalities (10), for the same number λ.

Yet we shall examine the conditions d) of the Assumption E. We have

$$g^1(t,x)\frac{d}{dl^1}\overset{1}{w}(x) - h^1(t,x)\overset{1}{w}(x) \leqslant (g_o + h)\exp(R - x_{i_o}) - h_o K < 0$$

for the above fixed K. So is also for $\overset{2}{w} = -\overset{1}{w}$, $M^1(\overset{2}{w})(t,x) < 0$.

If we suppose that f is continous in D and $\lim_{t\to\infty} f_i(t,x) = 0$ uniformly with respect to x, then we can construct the function p in the same way as we have done it on the page 165 (under the Assumption D) and then the Assumption C is held. Thus all the Assumptions of the Theorem 1 are satisfied and we obtain $\lim_{t\to\infty} u(t,x) = 0$ as the result.

Analogously if $b_k(t,x) \geqslant B_o > 0$, taking $w(x) = K - \exp(R + x_{i_o})$, we can prove the same property of the solution of the equation (9).

We have supposed, that K satisfies two inequalities: $\max(\beta + 1, \beta\frac{g_o + h}{h_o}) < K < \frac{B_o}{c_o + \lambda}$. For simplicity of our consideration let us establish $\beta + 1 < K < \frac{B_o}{c_o + \lambda}$ hence

$$(\beta + 1)(c_o + \lambda) < B_o \tag{11}$$

Setting c_0 as a constant we obtain from (11) the restriction of B_0, or for the diameter R, as $\beta = \exp 2R$. If we want the inequality (11) to hold for arbitrary R it is sufficient to put a stronger condition on the coefficient $c(t,x)$, for instance $c(t,x) \leqslant c_0 \exp(-\varkappa t)$, $\varkappa > 0$. Then for sufficiently large $t \geqslant T_0 > 0$ and convenient $\lambda > 0$, the inequality (11) holds in D_{T_0} (comp.th.2 in [4]). The inequality $c(t,x) \leq c_0 \exp(-\varkappa t)$ is the particular case of the Assumption C_1 od the theorem 2 (comp. remark 2 below).

Example 2. We will give the example of certain equation of which the solution does not converge to zero and we shall prove that the Assumption E does not hold in this case.

The function $u(t,x) = \frac{t^2}{1+t^2} \cdot \sin \frac{x}{\sqrt{a}}$ where $a > 0$ is arbitrary, satisfies the equation $a \cdot u_{xx} + u + \frac{2t}{(1+t^2)^2} \cdot \sin \frac{x}{\sqrt{a}} = u_t$ in the domain $D = (0 < x < \pi\sqrt{a}) \times (0,\infty)$, and u is equal to zero for $x_1 = 0$, $x_2 = \pi\sqrt{a}$, and for $t = 0$, but u does not converge to zero, as $t \to \infty$, in the whole interval $[0, \pi\sqrt{a}]$. For $\Phi \in H_0$, sgn $\Phi = 1$, the condition (8) has now the following form $f(t,x,\Phi(t,x), \Phi_x(t,x),0) - f(t,x,0,0,0) = \Phi(t,x) \leqslant -\varphi(t)\, L(t,x,w(x),w_x(x))$. Setting $\Phi(t,x) = \overset{1}{J}(t)w(x)$, in virtue of γ) we get $\Phi(t,x) = \overset{1}{J}(t)w(x) < -\overset{1}{J}(t)\lambda w(x) \Longrightarrow \overset{1}{w}(x)(1+\lambda) < 0$. This however contradicts the condition a).

3. Non - homogeneous boundary conditions

We shall get a generalisation of the theorem 2 of [4] only for the linear boundary conditions[1)].

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1) The proof of the theorem 2 in [4] is not quite correct. A mistake was made in the proof of the inequalities

$(*)$ $V_t^i \geqslant f^i(t,x,V,V_x^i,V_{xx}^i)$ $i = 1,\dots,m.$

We shall now carry on this fragment of reasoning in the correct way. For $\varepsilon > 0$ and $T(\varepsilon) > 0$ the following estimates hold:

$$\frac{\sigma^i(t-T)}{J(t)} \leqslant \frac{d \exp[-\varkappa(t-T)]}{x \exp(-\lambda t)} = \frac{d_1}{s} \exp(\lambda - \varkappa)t \leqslant \frac{d_1}{s}$$

for $\lambda < \varkappa$, where $d_1 = (\exp \varkappa T)d$, $s = \frac{p(0)}{\lambda}[1 - \exp(-\lambda\delta)]$.

Applying the assumption (24) from page 249 in [4] we have

$$A = -\lambda K - \sigma^i(t-T)mK - \frac{\sigma^i(t-T)}{J(t)} m\varepsilon + \mu^i(t-T)\, w_x \geqslant -K\lambda + \gamma - \frac{d_1}{s} m\varepsilon,$$

but $\frac{d_1 \varepsilon}{s} m = \frac{d_1 m \varepsilon \lambda}{p(0)[1 - \exp(-\delta\lambda)]}$.

The following definitions and assumptions will be used:

Assumption E_1. For $T_0 \geq 0$, there exist four functions: L_k, $k = 1, 2$, (see Def. 5) and $\overset{k}{w}$, $k = 1, 2$, which are as regular as it was supposed in Assumption E. We assume that:

1^0. For every $\xi \in (0, \xi_0)$, where ξ_0 arbitrary but fixed number, and for every continuous function $\varphi: [0,\infty) \to R_+$, the functions L_k, $\overset{k}{w}$, $k = 1,2$, satisfy for $i = 1,\ldots,m$, the inequalities:

$$\operatorname{sgn} \overset{k}{w}\Big[f^i(t,x,\varphi(t)\overset{k}{w}(x) + \xi,\varphi(t)\overset{k}{w}{}^i_x(x),0,\varphi(t)\overset{k}{w}(\cdot) + \xi) - f^i(t,x,\xi,0,0,\xi)\Big] \leq -\varphi(t)L^i_k(t,x,\overset{k}{w}(x),\overset{k}{w}{}^i_x(x),\overset{k}{w}(\cdot)) \tag{12}$$

in the domain $(D_p)_{T_0}$.

2^0. The functions $\overset{k}{w}$, $k = 1, 2$, satisfy the conditions a), b), c), d) of the Assumption E.

Assumption C_1. For every $\xi \in (0, \xi_0)$ exists $p_\xi : [-\delta,\infty) \to R$, for $\delta > 0$ arbitrary fixed, such that:

1^0 $p_\xi(t) > 0$

2^0 $\lim\limits_{t\to\infty} p_\xi(t) = 0$

3^0 p_ξ is continous and non - increasing

4^0 $p_\xi(t) \geq |f^i(t,x,\xi,0,0,\xi)|$ in D_p, $i = 1,\ldots,m$.

Remark 2. In the case of one differential equation of the form (9) we have $f(t,x,\xi,0,0) = c(t,x)\xi - f_1(t,x)$. Assuming that the functions c and f_1 converge uniformly to zero, the Assumption C_1 holds. The inequality (12) is satisfied by just the same functions L_1 and L_2 as those in the Example 1.

The non - homogeneous boundary conditions will be considered under suitable:

Assumption G_1. Let $h^i(t,x) \geq h_0$, where $0 < h_0 < 1$, on Σ^i $i = 1,\ldots,m$. Suppose that for every $\varepsilon > 0$ exists $T > 0$ such that: $|\varphi^i_1(t,x)| < \varepsilon$ for every $(t,x) \in (\Sigma\setminus\Sigma^i)_T$ and $|\varphi^i_2(t,x)| < \varepsilon$ for every $(t,x) \in (\Sigma^i)_T$.

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Since up to now δ has been an arbitrary number so we can set $\delta = \frac{1}{\lambda}$ and therefore $\frac{d_1 m\varepsilon}{s} = \frac{d_1 m\varepsilon\lambda}{p(0)(1-e^{-1})}$ and hence $A \geq \lambda\left[-K - \frac{d_1 m\varepsilon}{p(0)(1-e^{-1})}\right] + \gamma$. Since $\lambda < \varkappa$ can be arbitraily small, we notice that $A \geq 0$ and hence the inequality (*) holds.

Theorem 2. Let f satisfies Assumptions F, B_1, B_2, and let Assumptions C_1, E_1 hold in $(D_p)_{T_0}$. Let u be the $\sum$-regular solution of the system (1) in D_p such, that f is parabolic with respect to u. If u satisfies the boundary conditions (5) under the Assumption G_1 then

$$\lim_{t\to\infty} \|u(t,\cdot)\|_t = 0.$$

Proof. We establish for arbitrary $\varepsilon = \frac{\gamma}{2} > 0$ the number $\varepsilon_1 = h_0\varepsilon$. We choose a suitable $T_1 \geq T_0$ for this ε_1, so that the inequalities of Assumption G_1 hold on $\sum_{T_1}$.

Let λ_1 be established according to the Assumption E_1 in $(D_p)_{T_1}$. We denote $\|u(T_1,\cdot)\|_{T_1} = K_0$.

Let $p_\varepsilon(t)$ be the function defined by Assumption C_1. We construct the function $\overline{p}$, which satisfies the conditions 1^0-3^0 of C_1, such that $\overline{p}(t) \geq p_\varepsilon(t)$ for $t > 0$ and

$$\overline{p}(0) > \max\left\{p_\varepsilon(0), \frac{K_0\lambda_1}{(1-\frac{1}{e})\exp(-\lambda_1 T_1)}\right\} \tag{13}$$

We set $\overline{p}(t) = \overline{p}(0)$ for $-\frac{1}{\lambda_1} \leq t \leq 0$.

Taking $\overset{1}{J}(t) = J_{\lambda_1}(t) = \int_{-\frac{1}{\lambda_1}}^{t} \overline{p}(\tau)\exp[\lambda_1(\tau-t)]d\tau$, we can define $V_\varepsilon^i(t,x) = \overset{1}{J}(t)\overset{1}{w}{}^i(x) + \varepsilon$ and $v_\varepsilon^i(t,x) = \overset{1}{J}(t)\overset{2}{w}{}^i(x) - \varepsilon$, $i = 1,\dots,m$. Due to the conditions which satisfy $\overset{1}{w}$ and $\overset{2}{w}$ (see Assumption E_1), the function V_ε and v_ε satisfy the inequalities (6) and (7) in the domain $(D_p)_{T_1}$. The proof of this fact is similar to the adequate part of the proof of the Theorem 1. Now we should verify the boundary conditions. Let us notice that (comp [4]: the property 6^0 of the function J, and assumption (13) above) for every $x \in S_{T_1}$, $i = 1,\dots,m$ we have

$$V_\varepsilon^i(T_1,x) = \overset{1}{J}(T_1)\overset{1}{w}{}^i(x) + \varepsilon > \int_{-\frac{1}{\lambda_1}}^{0} \overline{p}(\tau)\exp[\lambda_1(\tau - T_1)]d\tau \geq$$

$$\geq \overline{p}(0)\frac{1}{\lambda_1}(1-\frac{1}{e})\exp(-\lambda_1 T_1) > K_0 \geq u^i(T_1,x)$$

and similarly

$$v_\varepsilon^i(T_1,x) < -\overset{1}{J}(T_1) \leqslant -\int_{-\frac{1}{\lambda_1}}^{0} \bar{p}(\tau)\exp\left[\lambda_1(\tau - T_1)\right]d\tau < -K_o \leqslant u^i(T_1,x).$$

It follows from the condition d) of the Assumption E_1 that

$$M^i(u^i - V_\varepsilon^i)(t,x) = \varphi_2^i(t,x) +$$

$$+ \overset{1}{J}(t)\left(g^i(t,x)\frac{d}{dl^i}\overset{1}{w}{}^i(x) - h^i(t,x)\overset{1}{w}{}^i(x)\right) - \varepsilon h^i(t,x) <$$

$$< \varphi_2^i(t,x) - h_o\varepsilon = \varphi_2^i(t,x) - \varepsilon_1 < 0 \quad \text{for every } (t,x) \in (\Sigma^i)_{T_1}$$

Analogously $M^i(u^i - v_\varepsilon^i)(t,x) > 0$ on $(\Sigma^i)_{T_1}$

We have also $V_\varepsilon^i(t,x) > \overset{1}{J}(t) + \varepsilon > h_o\varepsilon = \varepsilon_1$ and $v_\varepsilon^i(t,x) < -\varepsilon_1$ on $(\Sigma\setminus\Sigma^i)_{T_1}$. Hence $v_\varepsilon^i(t,x) < u^i(t,x) < V_\varepsilon^i(t,x)$ on $(\Sigma\setminus\Sigma^i)_{T_1}$.

Applying the Lemmas 1 and 2 we get $v_\varepsilon < u < V_\varepsilon$ in $(D_p)_{T_1}$.

Now we see that: $\forall\varepsilon = \frac{\eta}{2} > 0,\ \exists\ T_2 \geqslant T_1$ so that $0 < V_\eta(t,x) < \eta$ and $0 > v_\eta(t,x) > -\eta$ for $(t,x) \in (D_p)_{T_2}$ in virtue of Corollary 2 we have $\lim\limits_{t\to\infty}\|u(t,\cdot)\|_t = 0$ what closes the proof.

Let us notice that u has the limit zero independently of the initial conditions (comp. Remark 1).

4. The case of the improper limits

Assumption C_2: There exists $p_1 : [0,\infty) \to R$, which has the following preperties

1^o $p_1(t) \geqslant 0$

2^o $\lim\limits_{t\to\infty} p_1(t) = \infty$

3^o p_1 is continuous non - decreasing function

4^o there exists $T_3 \geqslant 0$ such that for $i = 1,\dots,m$,

$p_1(t) \leqslant f^i(t,x,0,0,0,0)$ in $(D_p)_{T_3}$.

Remark 3. We denote $\zeta^i(t,x) = f^i(t,x,0,0,0,0)$ for $(t,x) \in D_p$ and assume that $\zeta = (\zeta^1,\dots,\zeta^m)$ is a continuous function in D. Setting $|\zeta(t,\cdot)|_t = \min_{1\le i\le m} \inf_{S_t} |\zeta^i(t,x)|$, we assume $\lim_{t\to\infty} |\zeta(t,\cdot)|_t = \infty$, then we can put $p_1(t) = \inf_{\tau\ge t} |\zeta(\tau,\cdot)|_\tau$

Now we introduce a new assumption concerning the boundary condition (5).

Assumption G_2. There exists $\varkappa_1 > 0$ and $T_2 \geq 0$, such that for $i = 1,\dots,m$ we have $\varphi_1^i(t,x) \geq \frac{p_1(t)}{\varkappa_1}$ on $(\Sigma\setminus\Sigma^i)_{T_2}$, $\varphi_2^i(t,x) \geq \frac{p_1(t)}{\varkappa_1}$ on $(\Sigma^i)_{T_2}$, $u_i(T_2,x) \geq \frac{p_1(t)}{\varkappa_1}$ on S_{t_2}.

Remark 4. If $T_2 = 0$ and the last condition in Asumption G_2 holds, then $\varkappa_1 \cdot \varphi_0^i \geq f^i(0,x,0,0,0,0)$, $i = 1,\dots,m$, that means that the solution is essentially dependent on the initial condition.

Assumption E_2. For $T_0 \geq 0$ there exist two functions: L_3 (see Def 5) and $\overset{3}{w} : S_0^1 \to R_+^m$ as regular as it was assumed in E. We assume that:

1° For every continuous function $\varphi : [0,\infty) \to R_+$ the function L_3 and $\overset{3}{w}$ satisfy the inequalities:

$$f^i(t,x,\varphi(t)\overset{3}{w}(x),\ \varphi(t)\overset{3}{w}{}^i_x(x),\ 0,\ \varphi(t)\overset{3}{w}(\cdot)) -$$

$$- f^i(t,x,0,0,0,0) \geq -\ \varphi(t)L_3^i(t,x,\overset{3}{w}(x),\ \overset{3}{w}{}^i_x(x),\ \overset{3}{w}(\cdot))$$

in $(D_p)_{T_0}$, for $i = 1,\dots,m$.

2° The function $\overset{3}{w}{}^i(x)$ for $i = 1,\dots,m$, satisfy the conditions:

a_1) $0 < w_0 \leq \overset{3}{w}{}^i(x) \leq 1$ for every $x \in \overline{S_0^1}$

b_1) $\sum_{j,k=1}^{n} \overset{3}{w}{}^i_{x_jx_k}(x)\,\alpha_j\alpha_k > 0$ for every $x \in S_0^1$ and every $\alpha = (\alpha_1,\dots,\alpha_n)$

c_1) there exists $\lambda_3 > \varkappa_1$ such that the following inequalities hold, for $i = 1,\dots,m$:

$$L_3^i(t,x,\overset{3}{w}(x),\ \overset{3}{w}{}^i_x(x),\overset{3}{w}(\cdot)) - \lambda_3\overset{3}{w}{}^i(x) < 0 \text{ for every } (t,x) \in (D_0)_{T_0}.$$

d_1) $M^i(\overset{3}{w}(x))(t,x) < 1$ for every $(t,x) \in \Sigma^i_{T_0}$.

<u>Assumption F_1</u>. Let us denote:

$$H_1 = \left\{ \Phi : \overline{D} \to R^m, \ \Phi^i(t,x) = \varphi(t) w^i(x), \ i = 1,\dots,m \right\}$$

where $\varphi \in C^1([0,\infty), R_+)$ and $w^i(x) = \overset{3}{w}{}^i(x)$ as regular as it was assumed in Assumption E and satisfying the conditions a_1), b_1).
We assume that f is parabolic with respect to every function $\Phi \in H_1$.

Let us denot $J_\lambda(t) = \int_0^t p_1(\tau)\exp[\lambda(\tau - t)]d\tau$ for arbitrary $\lambda > 0$.
We have obviously

$$\lim_{t\to\infty} J_\lambda(t) = \infty . \qquad (15)$$

Now we prove

<u>Theorem 3</u>. Let T_0 be so established that the Assumptions E_2 in $(D_p)_{T_0}$ and C_2 for $T_3 = T_0$, hold. Suppose that f satisfies the Assumptions F_1 and B_2. Let u be the $\sum$-regular solution of the system (1) in D_p, such that f is parabolic with respect to u in $(D_p)_{T_0}$. Furthermore let us suppose that u satisfies boundary conditions (5) under the Assumption G_2 for $T_2 = T_0$.

Then we have $\lim_{t\to\infty} |u(t,\ )|_t = \infty$ (comp Remark 3).

<u>Proof</u>. We establish λ_3 according with the condition c_1) of E_2 and we put $\overset{3}{J} = J_{\lambda_3}$, $v^i(t,x) = \overset{3}{J}(t)\overset{3}{w}{}^i(x)$, $i = 1,\dots,m$. It is easy to prove that v satisfies the system (7).

Since $\overset{3}{J}(t) \le p_1(t)\frac{1}{\lambda_3}\left[\exp\lambda_3(\tau - t)\right]^t_{-\frac{1}{\lambda_3}} =$

$$= \frac{p_1(t)}{\lambda_3} - \frac{p_1(t)e^{-1}}{\lambda_3} \exp -(\lambda_3 t) < \frac{p_1(t)}{\lambda_3}$$

therefore for $\lambda_3 > \varkappa_1$ the function v - u satisfies the strong boundary conditions (def. 4). In virtue of Lemma 2, $v < u$ in $(D_p)_{T_0}$. From the conditions (15) and a_1) of the Assumption E_2 follows $\lim_{t\to\infty} |u(t,\cdot)|_t = \infty$

Establishing the symmetric assumptions we can give the conditions for which $u \to -\infty$ for $t \to \infty$.

<u>Example 4</u>. Let $u: \overline{D} \to R$ be the $\sum$-regular solution of the equation (9) of parabolic type. We set $L_3(t,x,\overset{3}{w}(x),\overset{3}{w}_x(x)) = -\sum_{k=1}^{n} b_k(t,x)\overset{3}{w}_{x_k} -$ $- c(t,x)\overset{3}{w}(x)$ similarly as it was made in the example 1. Let the index i_0 be established so, that $-R \le x_{i_0} \le 0$, where $0 < R < \infty$ is the diameter

of S_o^1. Let us take $\overset{3}{w}(x) = w(x) = \exp x_{i_o}$. Now we have $w_o = \exp(-R) \leqslant$ $\leqslant w(x) \leqslant 1$ and $\sum_{i,j=1}^{n} w_{x_i x_j} \alpha_i \alpha_j = w_{x_{i_o} x_{i_o}} \alpha_{i_o}^2 \geqslant 0$. Let us suppose that $0 < b_o \leqslant b_k(t,x)$ $k = 1,\dots,n$, and $c_o \leqslant c(t,x) \leqslant 0$ then $(-\lambda \overset{3}{w} + L_3(\overset{3}{w}))(t,x) =$

$$= - \sum_{k=1}^{n} b_k(t,x) w_{x_k}(x) - (c(t,x) + \lambda) w(x) \leqslant - \exp(-R)(\lambda + b_o) - c_o.$$

Setting $\lambda > \max(\varkappa_1, - c_o \exp R - b_o)$ we see that C_1) holds.

Let us assume now that on $\sum^1$ we have $0 < h(t,x) < 1$ and $0 < g(t,x) <$ $< 1 - h(t,x)$. Hence $h(t,x)w(x) - g(t,x) \frac{d}{dl} w(x) < h(t,x) + g(t,x) < 1$ on $\sum^1$. We see that all conditions of the Assumption E_2 hold. If we set $\lim_{t\to\infty} -f_1(t,x) = \infty$ uniformly with respect to $x \in S_o^1$, and furtherly $- f_1(t,x) > 0$ for $t \geqslant T_o$, then we can define $p_1(t) = \inf_{\tau \geqslant t} |-f_1(\tau,\cdot)|_{\tau}$, for $t \geqslant T_o$, $p_1(t) = p_1(T_o)$ for $0 \leqslant t < T_o$ and after that we can use $p_1(t)$ to construct $J_\lambda(t)$. In virtue of the theorem 3 we get:

Corollary 3. Let us suppose that: $\lim_{t\to\infty} - f_1(t,x) = \infty$ and the following inequalities hold in $(D_p)_{T_o}$: $0 < b_o \leqslant b_k(t,x)$ $k = 1,\dots,n$,

$c_o \leqslant c(t,x) \leqslant 0$, $\sum_{ij=1}^{n} a_{ij}(t,x)\alpha_i\alpha_j \geqslant 0$. If on the boundary $(\sum^1)_{T_o}$ there is $0 < h(t,x) < 1$ and $0 < g(t,x) < 1 - h(t,x)$, then $\sum$-regular solution of (9), which satisfies boundary conditions (5) under following assumptions:

$$u(t,x) > \frac{p_1(t)}{\varkappa_1} \quad \text{on } (\Sigma \setminus \Sigma^1)_{T_o}, \quad h(t,x)u(t,x) - g(t,x) \frac{d}{dl} u(t,x) >$$

$$> \frac{p_1(t)}{\varkappa_1} \quad \text{on } (\Sigma^1)_{T_o}, \quad u(T_o,x) > \frac{p_1(T_o)}{\varkappa_1} \quad \text{on } S_{T_o}$$

converges to $+\infty$ for $t \to \infty$, uniformly with respect to x.

Remark 5. Now we will formulate two theorems concerning differential-functional inequalities, which were basic for the two Lemmas 1 and 2 given at the beginning.

Assjmption B_k, $k = 1,2$. Let u, v be $\sum$-regular. We denote $N_k^i = \{(t,x) \in D_p, (-1)^{k+1} u^i(t,x) > (-1)^{k+1} v^i(t,x)\}$. We assume that $(-1)^{k+1} u_t^i \leqslant (-1)^{k+1} f^i(t,x,u,u_x^i,u_{xx}^i,u(t,\cdot))$

$$(-1)^{k+1}v_t^i \leqslant (-1)^{k+1}f^i(t,x,v,v_x^i,v_{xx}^i,v(t,\cdot))$$

for $(t,x) \in N_k^i$. Next we assume that there exists $M:\{(t,x,s,q,s(t,\cdot))\} \to R^m$, where $s \in \bar{C}(S_t, R^m)$, such that for every $i = 1,\dots,m$ and every pair of arguments of f^i we have

$$\operatorname{sgn}(x^i - \bar{s}^i)\left[f^i(t,x,s,q,r,s(t,\cdot)) - f^i(t,x,\bar{s},\bar{q},r,\bar{s}(t,\cdot))\right] \leqslant$$

$$\leqslant M^i(t,x,x-\bar{s},q-\bar{q},s(t,\cdot)-\bar{s}(t,\cdot)) \text{ for } (t,x) \in D_p \text{ and arbitrary } r \in R^{n^2}.$$

Next we assume that for every $z: D_p \to R^m$, bounded from above in the set N_1^i (bounded from below in the set N_2^i), in every point of the set N_k^i, in which $\max_p \left[(-1)^{k+1} z^p(t,x)\right] > 0$, we have

$$M^i(t,x,z(t,x),0,z(t,\cdot)) \leqslant \max_p \sup_{S_t} \left[(-1)^{k+1} z^p(t,x)\right] K, \text{ for a certain } K \in \mathbb{R}.$$

Theorem 4 Let u, v be $\sum$-regular for which the Assumption B_1 holds and f^i are parabolic with respect to u. If u-v satisfies boundary inequalities according to Def. 4, then $u \leqslant v$ in $(D_p)_T$.

Theorem 5. Let u, v be $\sum$-regular for which the Assumption B_2 holds and f^i are parabolic with respect to u. If v-u satisfies boundary inequalities according to Def. 4, then $u \geqslant v$ in $(D_p)_T$.

Proofs are easier then these of the Theorems 6 and 8 given in paper [7], therefore we omit them.

Notice that from the Theorem 1 it results, that if the posed boundary problem for system (1) has a solution, then this solution is unique.

Remark 6. The Theorems 1 and 2 of the paper 4 can be proved under the Assumptions B_1 and B_2 too, as they are particular cases of the above presented Theorems 1 and 2.

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