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ON SUBSEMIGROUPS AND SUBGROUPS OF THE GROUP L_4^1 WITH A FAITHFUL PARAMETRIZATION

Let R denote the set of real numbers and $R_o := R\setminus\{0\}$. In the set $G = R_o \times R^3$ we consider the following operation "."

$$\langle\beta_1,\beta_2,\beta_3,\beta_4\rangle\cdot\langle\alpha_1,\alpha_2,\alpha_3,\alpha_4\rangle = \langle\beta_1\alpha_1, \beta_1\alpha_2 + \beta_2\alpha_1^2, \tag{1}$$

$$\beta_1\alpha_3 + 3\beta_2\alpha_1\alpha_2 + \beta_3\alpha_1^3, \alpha_4\beta_1 + \beta_4\alpha_1^4 + 4\beta_2\alpha_1\alpha_3 + 6\beta_3\alpha_1^2\alpha_2 + 3\beta_2\alpha_2^2\rangle .$$

The set G with the operation defined above is a group (cf. [1] p. 24, [5]). The neutral element is $\langle 1,0,0,0\rangle$ and the inverse element for $\langle\beta_1,\beta_2,\beta_3,\beta_4\rangle$ is

$$\left\langle \frac{1}{\beta_1}, -\frac{\beta_2}{\beta_1^3}, \frac{3\beta_2^2}{\beta_1^5} - \frac{\beta_3}{\beta_1^4}, \frac{10\beta_2\beta_3}{\beta_1^6} - \frac{15\beta_2^3}{\beta_1^7} - \frac{\beta_4}{\beta_1^5}\right\rangle.$$

This group is denoted by L_4^1 (cf. [5]).

In this paper we shall find certain subsemigroups and subgroups of the group L_4^1 with a faithful parametrization. Certain such subsemigroups and subgroups are determined in [6].

Subsemigroups and subgroups with a faithful parametrization of a certain two-parameter group are considered also in [2].

1. J. Dhombres defined a set with a faithful parametrization [4]

Definition 1. ([4] p. 267) Let G, F be two non-empty sets and H - a subset of the cartesian product $G\times F$. We say that H has a faithful parametrization if there exist a set E and a mapping $g: E \xrightarrow{\text{onto}} H$ $(g(u) = \langle\alpha(u),\beta(u)\rangle,\ u\in E)$ such that either

(i) $\beta(E) = F$ and $(\beta(u) = \beta(u') \Longrightarrow \alpha(u) = \alpha(u'))$

or

(ii) $\alpha(E) = G$ and $(\alpha(u) = \alpha(u') \Longrightarrow \beta(u) = \beta(u'))$.

We generalize this definition.

<u>Definition 2</u>. Let $G_1,\dots,G_n$ be non empty sets and $H \subset G_1 \times \dots \times G_n$. We say that H has a faithful parametrization if there exist a set E and a mapping $g: E \xrightarrow{\text{onto}} H$ $(g(u) = \langle\alpha_1(u),\dots,\alpha_n(u)\rangle)$ such that

$$\bigvee_{i \in \{1,\dots,n\}} \Big[\alpha_i(E) = G_i \ \text{ and } \ (\alpha_i(u) = \alpha_i(u') \Longrightarrow \alpha_j(u) = \alpha_j(u'))$$

$$\text{for } j \in \{1,\dots,n\}\Big]. \tag{2}$$

We prove

<u>Theorem 1</u>. A set $H \subset G_1 \times \dots \times G_n$ has a faithful parametrization if and only if there exist $i \in \{1,\dots,n\}$ and mappings $f_k: G_i \to G_k$ for $k \in \{1,\dots,i-1,i+1,\dots,n\}$ such that

$$H = \{\langle f_1(\alpha),\dots,f_{i-1}(\alpha),\alpha,f_{i+1}(\alpha),\dots,f_n(\alpha)\rangle : \alpha \in G_i\}. \tag{3}$$

<u>Proof</u>. Let $H \subset G_1 \times \dots \times G_n$ and H be of form (3). Let $E = G_i$ and $g(u) := \langle f_1(u),\dots,f_{i-1}(u),u,f_{i+1}(u),\dots,f_n(u)\rangle$ for $u \in E$. Then $g: E \xrightarrow{\text{onto}} H$ and (2) holds.

Conversely, let H has a faithful parametrization. There exists a set E and a function $g: E \xrightarrow{\text{onto}} H$ such that (2) is fulfilled. Therefore H is of form

$$H = \{\langle\alpha_1(u),\alpha_2(u),\dots,\alpha_n(u)\rangle : u \in E\}, \tag{4}$$

and $\alpha_i(E) = G_i$.

Choose $w \in G_i$. We obtain from (2) that $A_k := \alpha_k(\alpha_i^{-1}(\{w\}))$ is an one-element set for every $k \in \{1,\dots,n\}$. Therefore

$$f_k(w) := \}\alpha_k(\alpha_i^{-1}(\{w\}))\{ \ ^{x)}$$

is the mapping from G_i into G_k.

Let $u \in E$ and $\alpha_i(u) = w$. Then $u \in \alpha_i^{-1}(\{w\})$ and it follows from (5) that

$$\langle\alpha_1(u),\alpha_2(u),\dots,\alpha_n(u)\rangle = \langle f_1(w),f_2(w),\dots,f_{i-1}(w),w,f_{i+1}(w),\dots,f_n(w)\rangle$$

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x) $\}\{a\}\{$ denotes element of an one-element set $\{a\}$.

Therefore

$$H \subset \left\{ \langle f_1(w), \dots, f_{i-1}(w), w, f_{i+1}(w), \dots, f_n(w) \rangle : w \in G_i \right\}.$$

For any $w \in G_i$ there exists $u \in E$ such that $\alpha_i(u) = w$, and we obtain from (5) and (4)

$$\left\{ \langle f_1(w), \dots, w, \dots, f_n(w) \rangle : w \in G_i \right\} \subset H.$$

This completes the proof.

In virtue of Theorem 1, the problem of determining of all subsemigroups or subgroups of the group L_4^1 with a faithful parametrization reduces to the description of subsemigroups or subgroups of this group of the following four forms:

$$G_1 := \left\{ \langle \alpha, \varphi_2(\alpha), \varphi_3(\alpha), \varphi_4(\alpha) \rangle : \alpha \in R_o \right\}, \text{ where } \varphi_i : R_o \longrightarrow R \text{ for } i = 2,3,4. \tag{6}$$

$$G_2 := \left\{ \langle \varphi_1(\alpha), \alpha, \varphi_3(\alpha), \varphi_4(\alpha) \rangle : \alpha \in R \right\}, \text{ where } \varphi_1 : R \longrightarrow R_o \text{ and} \tag{7}$$

$\varphi_i : R \longrightarrow R$ for $i = 3, 4$.

$$G_3 := \left\{ \langle \varphi_1(\alpha), \varphi_2(\alpha), \alpha, \varphi_4(\alpha) \rangle : \alpha \in R \right\}, \text{ where } \varphi_1 : R \longrightarrow R_o \text{ and} \tag{8}$$

$\varphi_i : R \longrightarrow R$ for $i = 2, 4$.

$$G_4 := \left\{ \langle \varphi_1(\alpha), \varphi_2(\alpha), \varphi_3(\alpha) \alpha \rangle : \alpha \in R \right\}, \text{ where } \varphi_1 : R \longrightarrow R_o \text{ and} \tag{9}$$

$\varphi_i : R \longrightarrow R$ for $i = 2, 3$.

In this paper we shall consider four above cases. We shall find all subsemigroups and subgroups of form (6). In the case (7) we assume that φ_1 is continuous and we find all subsemigroups and subgroups of this form under this assumption. In the cases (8) and (9) we give only some examples and some partial results connected with this problem.

2. Subsemigroups and subgroups of the group L_4^1 of form (6)

It follows directly from the definition of the operation in L_4^1 that a subset G_1 is a subsemigroup of the group L_4^1 if and only if functions $\varphi_2, \varphi_3, \varphi_4$ satisfy the following system of functional equations:

$$\varphi_2(\beta\alpha) = \beta\varphi_2(\alpha) + \alpha^2\varphi_2(\beta), \tag{10}$$

$$\varphi_3(\beta\alpha) = \beta\varphi_3(\alpha) + 3\alpha\varphi_2(\beta)\varphi_2(\alpha) + \alpha^3\varphi_3(\beta), \tag{11}$$

$$\varphi_4(\beta\alpha) = \beta\varphi_4(\alpha) + \alpha^4\varphi_4(\beta) + 4\alpha\varphi_2(\beta)\varphi_3(\alpha) + 6\alpha^2\varphi_2(\alpha)\varphi_3(\beta) + + 3\varphi_2(\beta)\varphi_2^2(\alpha)^{x)} \tag{12}$$

A subsemigroup G_1 is a subgroup of the group L_4^1 if and only if functions φ_2, φ_3, φ_4 satisfy the following system of equations:

$$\varphi_2\left(\frac{1}{\beta}\right) = -\frac{\varphi_2(\beta)}{\beta^3}, \tag{13}$$

$$\varphi_3\left(\frac{1}{\beta}\right) = \frac{3\varphi_2^2(\beta) - \beta\varphi_3(\beta)}{\beta^5} \tag{14}$$

$$\varphi_4\left(\frac{1}{\beta}\right) = \frac{10\varphi_2(\beta)\varphi_3(\beta)}{\beta^6} - \frac{15\varphi_2^3(\beta)}{\beta^7} - \frac{\varphi_4(\beta)}{\beta^5}. \tag{15}$$

This follows from the definition of the inverse element for $\langle\beta_1, \beta_2, \beta_3, \beta_4\rangle$.

Now we give the solution of system (10)-(12) in the class of mappings $\varphi_i : R_0 \longrightarrow R$ for i = 2,3,4.

The general solution of the equation (10) is the set of following functions:

$$\varphi_2(\alpha) = p(\alpha^2 - \alpha), \tag{16}$$

where p is an arbitrary real constant (cf. [5] Théorème 1).

Suppose that $(\varphi_2, \varphi_3, \varphi_4)$ satisfies the system (10)-(12). We get from (11) and (16)

$$\varphi_3(\beta\alpha) = \beta\varphi_3(\alpha) + \alpha^3\varphi_3(\beta) + 3p^2\alpha(\alpha^2 - \alpha)(\beta^2 - \beta). \tag{17}$$

If φ_3 satisfies (17), then it is of form

$$\varphi_3(\alpha) = l(\alpha^3 - \alpha) + 3p^2(\alpha - \alpha^2), \tag{18}$$

where $l \in R$ (cf. [6] Théorème 2).

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x) In this paper $\varphi^k(\alpha)$ denotes $[\varphi(\alpha)]^k$.

Now, by (16), (18) and (12)

$$\varphi_4(\beta\alpha) = \beta\varphi_4(\alpha) + \alpha^4\varphi_4(\beta) + 4\alpha pl(\beta^2-\beta)(\alpha^3-\alpha) +$$
$$+ 12\alpha p^3(\beta^2-\beta)(\alpha-\alpha^2) + 6\alpha^2 p(\alpha^2-\alpha)\left[l(\beta^3-\beta) + 3p^2(\beta-\beta^2)\right] +$$
$$+ 3p^3(\beta^2-\beta)(\alpha^2-\alpha)^2. \tag{19}$$

Let us denote

$$F(\alpha,\beta):= 4\alpha pl(\beta^2-\beta)(\alpha^3-\alpha) + 12\alpha p^3(\beta^2-\beta)(\alpha-\alpha^2) + 6p\alpha^2(\alpha^2-\alpha)\cdot$$
$$\cdot\left[l(\beta^3-\beta) + 3p^2(\beta-\beta^2)\right] + 3p^3(\beta^2-\beta)(\alpha^2-\alpha)^2.$$

Then (19) implies

$$\varphi_4(\alpha\beta) = \alpha\varphi_4(\beta) + \beta^4\varphi_4(\alpha) + F(\beta,\alpha). \tag{20}$$

It follows from the above equality that

$$\varphi_4(\beta\alpha) = \beta\varphi_4(\alpha) + \alpha^4\varphi_4(\beta) + F(\alpha,\beta), \tag{21}$$

and in virtue of (20) and (21), we have

$$\varphi_4(\alpha)(\beta^4-\beta) = \varphi_4(\beta)(\alpha^4-\alpha) + F(\alpha,\beta) - F(\beta,\alpha) \tag{22}$$

Setting in (22) $\beta = -1$ we obtain

$$\varphi_4(\alpha) = \frac{\varphi_4(-1)}{2}(\alpha^4-\alpha) + \frac{F(\alpha,-1) - F(-1,\alpha)}{2}.$$

Put $p_1 = \frac{\varphi_4(-1)}{2}$. Then, in view of the form of F, we have

$$\varphi_4(\alpha) = p_1(\alpha^4-\alpha) + (\alpha^2-1)\left[2pl\alpha(2\alpha-3) - 15p^3\alpha^2\right]. \tag{23}$$

It is easy to verify that every three functions $\varphi_2, \varphi_3, \varphi_4$ of form (16), (18) and (23) respectively, satisfy the system (10)-(12).

We have proved

<u>Theorem 2</u>. The general solution of the system (10)-(12) is the set of triplets of functions $\langle\varphi_2,\varphi_3,\varphi_4\rangle$ $\varphi_i: R_0 \to R$, where

$$\varphi_2(\alpha) = p(\alpha^2-\alpha),$$

$$\varphi_3(\alpha) = l(\alpha^3-\alpha) + 3p^2(\alpha-\alpha^2),$$

$$\varphi_4(\alpha) = p_1(\alpha^4-\alpha) + (\alpha^2-1)\left[2pl\alpha(2\alpha-3) - 15p^3\alpha^2\right],$$

and p, p_1, l are arbitrary real constants.

Therefore we have that a subset G_1 of the set G of form (6) is a subsemigroup of the group $(G,\cdot)$ if and only if φ_2, φ_3, φ_4 are of the form (16), (18) and (23) respectively.

Every triplet $\langle\varphi_2,\varphi_3,\varphi_4\rangle$ which is a solution of the system (10)-(12) satisfies (13)-(15) too. Therefore we have

Theorem 3. Any subsemigroup of form (6) of the group L_4^1 is its subgroup.

3. Subsemigroups and subgroups of the group L_4^1 of form (7)

Using (1) and (7) one can easily prove that a subset G_2 of form (7) is a subsemigroup of L_4^1 if and only if the triplet $\langle\varphi_1,\varphi_3,\varphi_4\rangle$ is the solution of the following system of functional equations:

$$\varphi_1(\alpha\varphi_1(\beta) + \beta\varphi_1^2(\alpha)) = \varphi_1(\alpha)\varphi_1(\beta), \tag{24}$$

$$\varphi_3(\alpha\varphi_1(\beta) + \beta\varphi_1^2(\alpha)) = \varphi_1(\beta)\varphi_3(\alpha) + 3\alpha\beta\varphi_1(\alpha) + \varphi_3(\beta)\varphi_1^3(\alpha), \tag{25}$$

$$\varphi_4(\alpha\varphi_1(\beta) + \beta\varphi_1^2(\alpha)) = \varphi_4(\alpha)\varphi_1(\beta) + \varphi_4(\beta)\varphi_1^4(\alpha) + 4\beta\varphi_1(\alpha)\varphi_3(\alpha) + 6\alpha\varphi_3(\beta)\varphi_1^2(\alpha) + 3\beta\alpha^2. \tag{26}$$

It follows from the form of the inverse element to $\langle\beta_1,\beta_2,\beta_3,\beta_4\rangle$ that a subsemigroup G_2 of form (7) is a subgroup of L_4^1 if and only if functions φ_1, φ_3, φ_4 satisfy the following system of functional equations:

$$\varphi_1\left(-\frac{\beta}{\varphi_1^3(\beta)}\right) = \frac{1}{\varphi_1(\beta)}, \tag{27}$$

$$\varphi_3\left(-\frac{\beta}{\varphi_1^3(\beta)}\right) = \frac{3\beta^2 - \varphi_3(\beta)\cdot\varphi_1(\beta)}{\varphi_1^5(\beta)}, \tag{28}$$

$$\varphi_4\left(-\frac{\beta}{\varphi_1^3(\beta)}\right) = \frac{10\beta\varphi_3(\beta)}{\varphi_1^6(\beta)} - \frac{15\beta^3}{\varphi_1^7(\beta)} - \frac{\varphi_4(\beta)}{\varphi_1^5(\beta)}. \tag{29}$$

Now we prove

Theorem 4. The general solution of the system (24)-(26) in the class of triplets $<\varphi_1,\varphi_3,\varphi_4>$ of mappings such that $\varphi_1: R \to R_o$, φ_1 is continuous, φ_3, $\varphi_4 : R \to R$ is the set of following sequences $<\varphi_1,\varphi_3,\varphi_4>$:

$$\varphi_1(\alpha) = 1, \tag{30}$$

$$\varphi_3(\alpha) = c\alpha + \frac{3}{2}\alpha^2, \tag{31}$$

$$\varphi_4(\alpha) = g(\alpha) + 3\alpha(\alpha^2-1) + 5c(\alpha^2-\alpha), \tag{32}$$

where g is an additive function and c -a real constant.

Proof. Let $<\varphi_1,\varphi_2,\varphi_3>$ satisfies the given system. It was proved in [7] that if φ_1 is continuous and satisfies (24), then $\varphi_1(\alpha) \equiv 1$. By (25), (26) and the above result we obtain

$$\varphi_3(\alpha+\beta) = \varphi_3(\alpha) + \varphi_3(\beta) + 3\alpha\beta, \tag{33}$$

$$\varphi_4(\alpha+\beta) = \varphi_4(\alpha) + \varphi_4(\beta) + 4\beta\varphi_3(\alpha) + 6\alpha\varphi_3(\beta) + 3\beta\alpha^2. \tag{34a}$$

Let $h(\alpha) := \varphi_3(\alpha) - \frac{3}{2}\alpha^2$

If φ_3 satisfies (33) then h is additive.

Therefore if φ_3 is a solution of (33) then there exists an additive function $h: R \to R$ such that

$$\varphi_3(\alpha) = h(\alpha) + \frac{3}{2}\alpha^2.$$

In view of the form of φ_3 and (34a) we have

$$\varphi_4(\alpha+\beta) = \varphi_4(\alpha) + \varphi_4(\beta) + 4\beta h(\alpha) + 6\alpha h(\beta) + 9(\alpha+\beta)\alpha\beta. \tag{34b}$$

Since the left hand side of the above equality is symmetric, we obtain

$$\bigwedge_{\alpha,\beta} \alpha h(\beta) = \beta h(\alpha).$$

Therefore for $\beta = 1$ and $c := h(1)$ we have that $h(\alpha) = c\alpha$. We obtain from (34b)

$$\varphi_4(\alpha+\beta) = \varphi_4(\alpha) + \varphi_4(\beta) + 10c\alpha\beta + 9\alpha\beta(\alpha+\beta). \tag{35}$$

Now we define

$$g(\alpha) := \varphi_4(\alpha) - 3(\alpha^3-\alpha) - 5c(\alpha^2-\alpha). \tag{36}$$

We prove that g is an additive function;

$$g(\alpha+\beta) = \varphi_4(\alpha+\beta) - 3(\alpha+\beta)((\alpha+\beta)^2 - 1) - 5c((\alpha+\beta)^2 - \alpha - \beta) =$$

$$= \varphi_4(\alpha) + \varphi_4(\beta) + 10c\alpha\beta + 9\alpha\beta(\alpha+\beta) - 3(\alpha^3-\alpha) - 3(\beta^3-\beta) -$$

$$- 9\alpha\beta(\alpha+\beta) - 5c(\alpha^2-\alpha) - 5c(\beta^2-\beta) - 10c\alpha\beta = g(\alpha) + g(\beta).$$

Then by the additivity of g and (36), φ_4 is of form (32). Hence, if $\langle\varphi_1, \varphi_3, \varphi_4\rangle$ satisfies the system (24)-(26) then $\varphi_1, \varphi_3, \varphi_4$ are of form (30), (31), (32) respectively, where c is a real constant and g is a certain additive function and $g: R \to R$.

It is easy to verify that every triplet of form

$$\langle 1,\ c\alpha + \frac{3}{2}\alpha^2,\ g(\alpha) + 3(\alpha^3-\alpha) + 5c(\alpha^2-\alpha)\rangle,$$

where c is a real constant and g - additive function, is a solution of (24)-(26).
The proof of Theorem 4 is completed.

It follows from the above theorem that a subset G_2 of the set G of form (7), under the assumption of the continuity of φ_1 is a subsemigroup of L_4^1 if and only if $\varphi_1, \varphi_3, \varphi_4$ are of form (30), (31), (32) respectively, where g is an arbitrary additive function, $g: R \to R$ and c is a real constant.

The proof of the following theorem is obvious

Theorem 5. Every subsemigroup of L_4^1 of form (7) with the assumption of the continuity of φ_1 is a subgroup of this group.

4. Subsemigroups of the group L_4^1 of form (8)

It is easy to prove that a subset G_3 of form (8) is a subsemigroup of L_4^1 if and only if functions $\varphi_1, \varphi_2, \varphi_4$ are solutions of the following system of functional equations:

$$\varphi_1(\alpha\varphi_1(\beta) + 3\varphi_2(\beta)\varphi_2(\alpha)\varphi_1(\alpha) + \beta\varphi_1^3(\alpha)) = \varphi_1(\beta)\varphi_1(\alpha), \tag{37}$$

$$\varphi_2(\alpha\varphi_1(\beta) + 3\varphi_2(\beta)\varphi_2(\alpha)\varphi_1(\alpha) + \beta\varphi_1^3(\alpha)) = \varphi_1(\beta)\varphi_2(\alpha) + \varphi_2(\beta)\varphi_1^2(\alpha), \tag{38}$$

$$\varphi_4(\alpha\varphi_1(\beta) + 3\varphi_2(\beta)\varphi_2(\alpha)\varphi_1(\alpha) + \beta\varphi_1^3(\alpha)) = \varphi_4(\alpha)\varphi_1(\beta) + \varphi_4(\beta)\varphi_1^4(\alpha) +$$
$$+ 4\alpha\varphi_2(\beta)\varphi_1(\alpha) + 6\beta\varphi_1(\alpha)\varphi_2(\alpha) + 3\varphi_2(\beta)\varphi_2^2(\alpha), \tag{39}$$

where $\varphi_1: R \to R_0$ and $\varphi_i: R \to R$ for $i = 2,4$.

A subsemigroup G_3 of form (8) of the group L_4^1 is its subgroup if and only if $\varphi_1, \varphi_2, \varphi_4$ satisfy the following system

$$\frac{1}{\varphi_1(\beta)} = \varphi_1\left(\frac{3\varphi_2^2(\beta)}{\varphi_1^3(\beta)} - \frac{\beta}{\varphi_1^4(\beta)}\right), \tag{40}$$

$$-\frac{\varphi_2(\beta)}{\varphi_1^3(\beta)} = \varphi_2\left(\frac{3\varphi_2^2(\beta)}{\varphi_1^5(\beta)} - \frac{\beta}{\varphi_1^4(\beta)}\right), \tag{41}$$

$$\frac{10\beta\varphi_2(\beta)}{\varphi_1^6(\beta)} - \frac{15\varphi_2^2(\beta)}{\varphi_1^7(\beta)} - \frac{\varphi_4(\beta)}{\varphi_1^5(\beta)} = \varphi_4\left(\frac{3\varphi_2^2(\beta)}{\varphi_1^5(\beta)} - \frac{\beta}{\varphi_1^4(\beta)}\right). \tag{42}$$

We prove

Theorem 6. Let $\varphi_2 \equiv 0$ and φ_1 has the Darboux property. Then the general solution of the system (37)-(39) is a set of functions of form

$$\varphi_1(\alpha) \equiv 1, \tag{43}$$

φ_4 - an arbitrary additive function. (44)

Proof. Let us consider the triplet $\langle 1, 0, h \rangle$, where h is additive. Then $\langle 1, 0, h \rangle$ satisfies the system (37)-(39). Conversely; let φ_1 be continuous, $\varphi_2 \equiv 0$, and $\langle \varphi_1, \varphi_2, \varphi_4 \rangle$ satisfy (37)-(39). We obtain from (37) and the assumption for φ_2 that φ_1 is a solution of the following equation:

$$\varphi_1(\alpha\varphi_1(\beta) + \beta\varphi_1^3(\alpha)) = \varphi_1(\alpha)\varphi_1(\beta).$$

In view of the theorem from [7], we obtain that $\varphi_1 \equiv 1$. Setting in (39) $\varphi_1(\alpha) \equiv 1$ and $\varphi_2(\alpha) = 0$ we have

$$\varphi_4(\alpha+\beta) = \varphi_4(\alpha) + \varphi_4(\beta),$$

which completes the proof.

We can verify that all subsemigroups of the group L_4^1 given in Theorem 6 are subgroups of this group.

5. Subsemigroups of the group L_4^1 of form (9)

A subset G_4 of form (9) is a subsemigroup of the group L_4^1 if and only if the triplet $\langle\varphi_1,\varphi_2,\varphi_3\rangle$ is a solution of the following system of functional equations:

$$\varphi_1(\alpha\varphi_1(\beta)+\beta\varphi_1^4(\alpha)+4\varphi_2(\beta)\varphi_3(\alpha)\varphi_1(\alpha)+6\varphi_3(\beta)\varphi_1^2(\alpha)\varphi_2(\alpha)+$$
$$+3\varphi_2^2(\alpha)\varphi_2(\beta)) = \varphi_1(\alpha)\varphi_1(\beta). \tag{45}$$

$$\varphi_2(\alpha\varphi_1(\beta)+\beta\varphi_1^4(\alpha)+4\varphi_2(\beta)\varphi_3(\alpha)\varphi_1(\alpha)+6\varphi_3(\beta)\varphi_1^2(\alpha)\varphi_2(\alpha)+$$
$$+3\varphi_2^2(\alpha)\varphi_2(\beta)) = \varphi_1(\beta)\varphi_2(\alpha)+\varphi_2(\beta)\varphi_1^2(\alpha), \tag{46}$$

$$\varphi_3(\alpha\varphi_1(\beta)+\beta\varphi_1^4(\alpha)+4\varphi_2(\beta)\varphi_3(\alpha)\varphi_1(\alpha)+6\varphi_3(\beta)\varphi_1^2(\alpha)\varphi_2(\alpha)+$$
$$+3\varphi_2^2(\alpha)\varphi_2(\beta)) = \varphi_1(\beta)\varphi_3(\alpha)+3\varphi_2(\beta)\varphi_1(\alpha)\varphi_2(\alpha)+\varphi_3(\beta)\varphi_1^3(\alpha), \tag{47}$$

where $\varphi_1:R\to R_o$, $\varphi_i:R\to R$ for $i = 2,3$.
This follows from the definition of the operation "$\cdot$" in L_4^1.

Now, the form of the inverse element for $\langle\alpha_1,\alpha_2,\alpha_3,\alpha_4\rangle$ yields that a subsemigroup of L_4^1 of form (9) is its subgroup if and only if $\langle\varphi_1,\varphi_2,\varphi_3\rangle$ is a solution of the system

$$\varphi_1\left(\frac{10\varphi_2(\alpha)\varphi_3(\alpha)}{\varphi_1^6(\alpha)}-\frac{15\varphi_2^3(\alpha)}{\varphi_1^7(\alpha)}-\frac{\alpha}{\varphi_1^5(\alpha)}\right)=\frac{1}{\varphi_1(\alpha)}, \tag{48}$$

$$\varphi_2\left(\frac{10\varphi_2(\alpha)\varphi_3(\alpha)}{\varphi_1^6(\alpha)}-\frac{15\varphi_2^3(\alpha)}{\varphi_1^7(\alpha)}-\frac{\alpha}{\varphi_1^5(\alpha)}\right)=-\frac{\varphi_2(\alpha)}{\varphi_1^3(\alpha)}, \tag{49}$$

$$\varphi_3\left(\frac{10\varphi_2(\alpha)\varphi_3(\alpha)}{\varphi_1^6(\alpha)}-\frac{15\varphi_2^3(\alpha)}{\varphi_1^7(\alpha)}-\frac{\alpha}{\varphi_1^5(\alpha)}\right)=\frac{3\varphi_2^2(\alpha)}{\varphi_1^5(\alpha)}-\frac{\varphi_3(\alpha)}{\varphi_1^4(\alpha)}. \tag{50}$$

Analogously to Theorem 6 we can prove

Theorem 7. If $\varphi_2 \equiv 0$ and φ_1 is continuous then the general solution of the system (45)-(47) as well as of the system (45)-(50) is the set of triplets $\langle 1,0,h\rangle$ where $h:R\to R$ is any additive function.

Therefore, the set $\{\langle 1,0,h(\alpha),\alpha\rangle : \alpha\in R\}$ with h - additive is a subgroup of L_4^1.

The problem of finding all subsemigroups and all subgroups of L_4^1 in the cases (8) and (9) is open.

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