ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1990

Seria: MATEMATYKA-FIZYKA z. 64 Nr kol. 1070

DEDICATED TO PROFESSOR MIECZYSŁAW KUCHARZEWSKI
WITH BEST WISHES ON HIS 70TH BIRTHDAY

Józef TABOR

RESTRICTED STABILITY OF THE CAUCHY AND THE PEXIDER EQUATIONS

D.H. Hyers [7] introduced the notion of stability of a functional equation. Lately this notion has been generalized [9], [8] and combined with the notion of "almost everywhere" [6]. The purpose of this paper is to show that the Cauchy and the Pexider equations are "ideally stable". Here "ideally stable" means that the respective inequality and equality hold everywhere except of elements of some "small" sets.

1. Introduction

Let $(X,+)$ be a group (not necessarily commutative). By a proper linearly invariant ideal or in short p.l.i. ideal [4] we mean a nonempty family $I \subset 2^X$ satisfying the following conditions:

1. $A, B \in I \Rightarrow A \cup B \in I$,
2. $A \in I,\ B \subset A \Rightarrow B \in I$,
3. $x \in X,\ A \in I \Rightarrow x - A \in I$,
4. $X \notin I$.

If we replace 1 in the above definition by

1. $A_i \in I,\ i = 0,1,2,\ldots \Rightarrow \bigcup_{i=0}^{\infty} A_i \in I$,

then I will be called a p.l.i. σ-ideal.

For $A \subset X$, $n = 0,1,2,\ldots$ we put

$$A_n := \{x \in X:\ 2^n x \in A\}, \qquad A^* := \bigcup_{n=0}^{\infty} A_n.$$

In the sequel we shall assume the following condition

(c) $A \in I \Rightarrow A^* \in I$.

Following notations of [4], we put for given p.l.i. ideal I in X:

$$\Omega(I) := \{M \subset X \times X : \underset{U(M) \in I}{\exists} \ \underset{x \in X \setminus U(M)}{\forall} \ M_x := \{y \in X : (x,y) \in M \in I\}.$$

$\Omega(I)$ is a p.l.i. ideal in $(X \times X, +)$.

Thoroughout the paper $U(M)$ and M_x will denote the sets distinguished in the definition of $\Omega(I)$.

Let Y be a sequentially complete linear topological space over the filed Q of rationals. Following [9], Hyers' definition of stability[1)] may be formulated as follows:

Definition 1. The Cauchy equation

$$f: X \to Y, \ f(x+y) = f(x) + f(y) \quad \text{for} \quad x,y \in X \tag{1}$$

is said to be stable if there exists a constant $K \in Q$, $K \neq 0$ such that for every Q-convex set $V \subset Y$, symmetric with respect to zero, containing zero and bounded and every $g: X \to Y$ satisfying the condition

$$g(x+y) - g(x) - g(y) \in V \quad \text{for} \quad x,y \in X$$

there exists a solution f of equation (1) such that

$$f(x) - g(x) \in K \operatorname{seqcl} V \quad \text{for} \quad x \in X,$$

where seqcl V denotes the sequential closure of V.

We propose the following definition of the ideal stability.

Definition 2. The Cauchy equation is said to be ideally stable or in short I-stable if there exists a constant $K \in Q$, $K \neq 0$ such that for every $V \subset Y$ possesing in Definition 1 specified properties, for every $M \in \Omega(I)$ and every $g: X \to Y$ satisfying the condition

$$g(x+y) - g(x) - g(y) \in V \quad \text{for} \quad (x,y) \in X^2 \setminus M$$

there exist a set $W \in I$ and a solution f of equation (1) such that

$$f(x) - g(x) \in K \operatorname{seqcl} V \quad \text{for} \quad x \in X \setminus W.$$

Observe that if $I = \{\emptyset\}$, where $\emptyset$ denotes the empty set, then I-stability coincides with stability.

---

1) There are also other definitions of the stability in use (cf. [2], [3], [8]).

The main purpose of the paper is to prove that the Cauchy and the Pexider equation

$$f,g,h: X \to Y, \quad f(x+y) = g(x) + h(y) \quad \text{for} \quad x,y \in X$$

are I-stable. Cauchy's equation is I-stable not only in X but also in some class of subsemigroups of X.

2. I-stability of the Cauchy equation

In our considerations we shall apply some properties of ideals. One can prove easily that if I is a p.l.i. ideal in X then

$$x \in X,\ A \subset I \Rightarrow -x + A,\ x + A,\ A + x,\ A - x \in I.$$

For further considerations we need the following general hypotheses:

(H1) $(X,+)$ is a group, I is a p.l.i. ideal in X satisfying condition (c).

(H2) S is a subsemigroup of X, $S - S = X$, $S \notin I$.

(H3) $M \in \Omega(I)$.

(H4) $\bigcup_{k=0}^{\infty} \left[M_{2^k x}\right]^* \in I \quad \text{for} \quad x \in S \setminus \left[U(M)\right]^*.$

(H5) Y is a sequentially complete linear topological space over the field Q. $V \subset Y$ is a Q-convex set, symmetric with respect to zero bounded and containing zero.

(H6) $f: S \to Y,\ f(x+y) - f(x) - f(y) \in V \quad \text{for} \quad (x,y) \in S^2 \setminus M.$

(H7) $\forall_{x,y \in S}\ \forall_{n \in N_0}\ \exists_{m \in N_0,\ m \geq n} \quad f(2^m(x+y)) = f(2^m x + 2^m y).$

We start with some useful lemmas.

Lemma 1. If (H1) holds and $A \subset X$ then

$$x \in X \setminus A^* \Rightarrow 2^n x \in X \setminus A^* \quad \text{for} \quad n \in N_0 := N \cup \{0\}.$$

Proof. If $2^k x \in A^*$ for some $k \in N_0$ then there exists an $n \in N_0$ such that $2^n \cdot 2^k x = 2^{n+k} x \in A$, which means that $x \in A^*$. □

Lemma 2. If (H1) and (H3) hold then

$$x\in X\setminus[U(M)]^{*}\Rightarrow M_x\cup M_{2x}\cup(-x+M_x)\in I.$$

Proof. Let $x\in X\setminus[U(M)]^{*}$. But $U(M)\subset[U(M)]^{*}$, hence $x\in X\setminus U(M)$. Therefore, in view of the definition of $\Omega(I)$, $M_x\in I$ and further $-x+M_x\in I$. By Lemma 1 $2x\in X\setminus U(M)$, so, again by the definition of $\Omega(I)$, $M_{2x}\in I$.

Lemma 3. If (H1), (H2), (H3), (H4) hold then

(i) $$\tilde{M}:=\{(x,y)\in S^2:\ x\in[U(M)]^{*}\ \text{or}\ y\in[U(M)]^{*}\ \text{or}\ x+y\in[U(M)]^{*}$$
$$\text{or}\ (x\notin[U(M)]^{*}\ \text{and}\ y\in\bigcup_{k=0}^{\infty}[M_{2^kx}]^{*})\}\in\Omega(I).$$

(ii) $(x,y)\in S^2\setminus\tilde{M}\Rightarrow(2^nx,2^ny)\in S^2\setminus M.$

Proof. (i). Consider any $x\in S\setminus[U(M)]^{*}$. We have

$$\tilde{M}_x=([U(M)]^{*}\cup\{y\in X:\ x+y\in[U(M)]^{*}\cup\{y\in X:\ y\in\bigcup_{k=0}^{\infty}[M_{2^kx}]^{*})\cap S$$
$$=([U(M)]^{*}\cup(-x+[U(M)]^{*}\cup\bigcup_{k=0}^{\infty}[M_{2^kx}]^{*})\cap S.$$

By (H4) $\bigcup_{k=0}^{\infty}[M_{2^kx}]^{*}\in I$, hence $\tilde{M}_x\in I$, which proves that $\tilde{M}\in\Omega(I)$.

(ii). Let $(x,y)\in S^2\setminus\tilde{M}$. Then $x\in S\setminus[U(M)]^{*}$ and $y\in S\setminus\bigcup_{k=0}^{\infty}[M_{2^kx}]^{*}$. Suppose, for an indirect proof, that $(2^nx,2^ny)\in M$ for some $n\in N_0$. By Lemma 1 $2^nx\in S\setminus[U(M)]^{*}$ and hence $2^ny\in M_{2^nx}$.

Thus $y\in[M_{2^nx}]^{*}\subset\bigcup_{k=0}^{\infty}[M_{2^kx}]^{*}$, which leads to a contradiction. □

Lemma 4. If (H1), (H2), (H3), (H5), (H6) hold then

$$f(2x)-2f(x)\in 3V\quad\text{for}\quad x\in S\setminus[U(M)]^{*}. \tag{2}$$

Proof. Consider an arbitrary $x\in S\setminus[U(M)]^{*}$. On account of Lemma 2 $M_x\cup M_{2x}\cup(-x+M_x)\in I$, so $S\setminus(M_x\cup M_{2x}\cup(-x+M_x))\neq\emptyset$. Choose arbitrarily a $y \in S \setminus (M_x \cup M_{2x} \cup (-x + M_x))$. Then $(x,y) \in S^2 \setminus M$, $(2x,y) \in S^2 \setminus M$ and $y \in S \setminus (-x + M_x)$ i.e. $(x,x+y) \in S^2 \setminus M$. Now applying (H6) we get

$$f(2x + y) - f(2x) - f(y) \in V,$$

$$f(2x + y) - f(x) - f(x+y) = f[x + (x+y)] - f(x) - f(x+y) \in V,$$

$$f(x+y) - f(x) - f(y) \in V.$$

These three conditions yield

$$f(2x) - 2f(x) = -f(2x+y) + f(2x) + f(y) + f(2x+y) - f(x) -$$

$$- f(x+y) + f(x+y) - f(x) - f(y) = -V + V + V = 3V.$$

Lemma 5. If (H1), (H2), (H3), (H5), (H6) hold then

$$2^{-n}f(2^m x) - f(2^{m-n}x) \in 3(1 - 2^{-n})V \tag{3}$$

for

$$x \in S \setminus [U(M)]^*, \quad n,m \in N_0, \quad n \leq m.$$

Proof. We shall use induction with respect to n. For $n = 0$ condition (3) evidently holds. Consider now an $x \in S \setminus [U(M)]^*$, $n,m \in N_0$, $n + 1 \leq m$ and suppose that

$$2^{-n}f(2^m x) - f(2^{m-n}x) \in 3(1 - 2^{-n})V.$$

Then we have

$$2^{-n-1}f(2^m x) - 2^{-1}f(2^{m-n}x) \in \frac{3}{2}(1 - 2^{-n})V. \tag{4}$$

On account of Lemma 1 $2^{m-n-1}x \in S \setminus [U(M)]^*$. Applying now Lemma 4 (with x replaced in (2) by $2^{m-n-1}x$) we get

$$f(2^{m-n}x) - 2f(2^{m-n-1}x) \in 3V,$$

which yields

$$2^{-1}f(2^{m-n}x) - f(2^{m-n-1}x) \in \frac{3}{2} V. \tag{5}$$

Relations (4) and (5) together give

$$2^{-n-1}f(2^m x) - f(2^{m-n-1}x) \in \frac{3}{2}(1 - 2^{-n})V + \frac{3}{2} V = 3(1 - 2^{-n-1})V$$

which completes the inductive proof of (3). □

The following theorem is the main result of the paper. It generalizes the result of R. Ger [6] p. 265, Theorem. However the method of our proof is different from R. Ger's one.

Theorem 1. If (H1), (H2), (H3), (H4), (H5), (H6), (H7) hold then there exists an additive mapping $\alpha: X \to Y$ and a set $T \in I$ such that

$$\alpha(x) - f(x) \in 3 \text{ seqcl } V \quad \text{for } x \in S \setminus T, \tag{6}$$

where seqcl V denotes sequential closure of V.
Moreover, if Y is a T-1 space then α is unique.

Proof. Consider an $x \in S \setminus [U(M)]^*$. Following D.H. Hyers [7] we are going to show that $\frac{f(2^n x)}{2^n}$ is a Cauchy sequence. Let V_0 be a neighbourhood of zero. Replacing in Lemma 5 n by m-n we get for $m,n \in N_0$, $n \leq m$:

$$\frac{f(2^m x)}{2^m} - \frac{f(2^n x)}{2^n} = \frac{2^{n-m} f(2^m x) - f(2^n x)}{2^n} \in \frac{3(1 - 2^{n-m})}{2^n} V.$$

Since V is bounded, for sufficiently large $n,m \in N_0$, $m \geq n$

$$\frac{3(1 - 2^{n-m})}{2^n} V \subset V_0.$$

Thus $\frac{f(2^n x)}{2^n}$ is a Cauchy sequence and hence there exists its limit (generally not unique).

We set

$$h(x) = \begin{cases} \lim\limits_{n \to \infty} \frac{f(2^n x)}{2^n} & \text{for} \quad x \in S \setminus [U(M)]^* \\ g(x) & \text{for} \quad x \in [U(M)]^*, \end{cases}$$

where $\lim\limits_{n\to\infty} \frac{f(2^n x)}{2^n}$ denotes any one limit of the sequence $\frac{f(2^n x)}{2^n}$ and $g: [U(M)]^* \to Y$ is an arbitrary function.

Consider the set $\tilde{M}$ defined as in Lemma 3. By this lemma $\tilde{M} \in \Omega(I)$. We are going to show that

$$h(x+y) - h(x) - h(y) \in \text{cl}\{0\} \quad \text{for} \quad (x,y) \in S^2 \setminus \tilde{M}. \tag{7}$$

Take for the proof a pair $(x,y) \in S^2 \setminus \tilde{M}$. In virtue of (H7) there exists a sequence of positive integers $\{n_k\}$, $n_k \to \infty$ as $k \to \infty$, such that

$$f(2^{n_k}(x+y)) = f(2^{n_k}x + 2^{n_k}y) \quad \text{for all} \quad n_k.$$

By Lemma 3 (ii) $(2^{n_k}x,\ 2^{n_k}y) \in S^2 \setminus M$ and hence, in view of (H6)

$$f(2^{n_k}x + 2^{n_k}y) - f(2^{n_k}x) - f(2^{n_k}y) \in V.$$

Thus

$$f(2^{n_k}(x+y)) - f(2^{n_k}x) - f(2^{n_k}y) \in V$$

and so

$$\frac{f(2^{n_k}(x+y))}{2^{n_k}} - \frac{f(2^{n_k}x)}{2^{n_k}} - \frac{f(2^{n_k}y)}{2^{n_k}} \in \frac{1}{2^{n_k}} V. \tag{8}$$

It results from the definition of $\tilde{M}$ that $x,\ y,\ x+y \in S \setminus [U(M)]^*$. Therefore, by the first part of the proof, the sequences of the left hand side of (7) are convergent. Letting $n_k \to \infty$ in (8) and making use of boundedness of V we get directly (7).

Following the method used in [9] we write Y as the direct sum of $cl\{0\}$ (which is a subspace of V) and some subspace Y_1:

$$Y = Y_1 + cl\{0\}.$$

Let π_1, π_0 be the projections of Y on Y_1 and $cl\{0\}$, respectively. Evidently π_1 and π_0 are additive and

$$y = \pi_1(y) + \pi_0(y) \quad \text{for} \quad y \in Y. \tag{9}$$

We set

$$A(x) := \pi_1[h(x)] \quad \text{for} \quad x \in S.$$

From (7) and additivity of π_1, we get now for $(x,y) \in S^2 \setminus \tilde{M}$
$A(x+y) - A(x) - A(y) = \pi_1[h(x+y) - h(x) - h(y)] = \pi(0) = 0.$
Thus A is almost $\Omega(I)$-additive and hence in consequence of Theorem 1 of [5] there exist an additive mapping $\alpha : X \to Y$ and a set $W \in I$ such that

$$\alpha(x) = A(x) \quad \text{for} \quad x \in S \setminus W. \tag{10}$$

Put

$$T: = W^* \cup [U(M)]^*.$$

Setting in (3) m = n, we obtain

$$\frac{f(2^n x)}{2^n} - f(x) \in 3(1 - 2^{-n})V \quad \text{for} \quad x \in S \setminus T. \tag{11}$$

Making use of (9), we get

$$h(x) - A(x) = h(x) - \pi_1[h(x)] = \pi_0[h(x)] \in \text{cl}\{0\},$$

and hence, as the translation is a homeomorphism,

$$A(x) \in h(x) + \text{cl}\{0\} = \text{cl}\{h(x)\}.$$

Since $\frac{f(2^n x)}{2^n} \xrightarrow[n\to\infty]{} h(x)$ for $x \in S \setminus T$ and $A(x) \in \text{cl}\{h(x)\}$, we have

$$\frac{f(2^n x)}{2^n} \xrightarrow[n\to\infty]{} A(x) \quad \text{for} \quad x \in S \setminus T.$$

This condition and (11) yield

$$A(x) - f(x) \in \text{seqcl } 3V = 3 \text{ seqcl } V \quad \text{for} \quad x \in S \setminus T,$$

which, in view of (10), gives (6) and hence completes the proof of the first part of the theorem.

An easy proof of uniqueness of α is given in 6. □

Under some additional assumption on the exceptional set M, the proof of Theorem 1 can be shorten and better estimation than in (6) can be found.

Theorem 2. Let all assumptions of Theorem 1 be fulfilled and let additionally

(H8) $M_D: = \{x \in S: (x,x) \in M\} \in I.$

Then there exist an additive mapping $\alpha: Y \to Y$ and a set $\Gamma \in I$ such that

$$\alpha(x) - f(x) \in \text{seqcl } V \quad \text{for} \quad x \in S \setminus T.$$

Before we shall give an idea of the proof of this theorem, we have to do some modifications in Lemmas 4 and 5.

Lemma 4'. Under assumptions of Theorem 2

$$f(2x) - 2f(x) \in V \quad \text{for} \quad x \in S \setminus [U(M) \cup M_D]^*.$$

Proof. Let $x \in S \setminus [U(M) \cup M_D]^*$. Then $x \in S \setminus M_D$, and hence $(x,x) \in S^2 \setminus M$. Applying (H6) we get

$$f(2x) - 2f(x) \in V. \square$$

Lemma 5'. Under assumptions of Theorem 2

$$2^{-n}f(2^m x) - f(2^{m-n}x) \in (1 - 2^{-n})V$$

for $x \in S \setminus [U(M) \cup M_D]^*$, $n,m \in N_0$, $n \leq m$.

Proofs of Lemma 5' and that of Theorem 2 run analogusly as the proofs of Lemma 5 and Theorem 1, respectively. We should only replace, in respective places: 3 by 1, $U(M)$ by $U(M) \cup M_D$, Lemmas 4 and 5 by Lemmas 4' and 5', respectively. $\square$

As it can be seen in the previous theorems, the estimation of the expression $\alpha(x) - f(x)$ depends strongly on the shape of the exceptional set M. In the case when the exceptional set M has the shape of a cross, $\alpha(x) - f(x)$ can be estimated for all x, instead of "almost all" x, as it occurs in the preceding theorems. It leads to the following corollary which is a generalisation of Proposition of R. Ger [6], p. 273.

Corollary. Let (H1), (H5) hold and let the following conditions (H9), (H10) be fulfilled.

(H9) $f: X \to Y$, $f(x+y) - f(x) - f(y) \in V$ for $x,y \in X \setminus W$, where $W \in I$,

(H10) $$\forall_{x,y \in X} \ \forall_{n \in N_0} \ \exists_{m \in N_0,\, m \geq n} \quad f(2^m(x+y)) = f(2^m x + 2^m y).$$

Then there exists an additive mapping $\alpha: X \to Y$ such that

$$\alpha(x) - f(x) \in 3 \text{ seqcl } V \quad \text{for} \quad x \in X.$$

Moreover, if Y is a T-1 space then α is unique.

Proof. Put $S := X$, $M := W \times W$. Obviously $M \in \Omega(I)$ and $M_D = W \in I$, i.e. (H3) and (H8) hold. Furthermore, in our case, $W \subset U(M)$ and so $M_x = W$ for $x \in X \setminus U(M)$, which proves that

$$\bigcup_{k=0}^{\infty} [M_{2^k x}]^* \in I \quad \text{for} \quad x \in X \setminus [U(M)]^*.$$

Thus the assumptions of Theorem 2 are fulfilled. Therefore there exist an additive mapping $\alpha: X \to Y$ and a set $T \in I$ such that

$$\alpha(x) - f(x) \in \text{saqcl } V \quad \text{for} \quad x \in X \setminus T. \tag{12}$$

Without loss of generality we may assume that $W \subset T$. Since $T \in I$, we have (cf. [4] Lemma 3)

$$(X \setminus T) + (X \setminus T) = X.$$

Let us write $x \in X$ in the form $x = x_1 + x_2$ with $x_1, x_2 \in X \setminus T$. Then, as $W \subset T$; $x_1, x_2 \in X \setminus W$. Making use of (H7), additivity of α and (12) we get

$$\alpha(x) - f(x) = \alpha(x_1+x_2) - f(x_1+x_2) = \alpha(x_1) + \alpha(x_2) - f(x_1) - f(x_2) - V$$
$$= [\alpha(x_1) - f(x_1)] + [\alpha(x_2) - f(x_2)] + V$$
$$\subset \text{seqcl } V + \text{seqcl } V + V \subset 3 \text{ seqcl } V. \;\square$$

3. Remarks and comments

Remark 1. If I is a p.l.i. σ-ideal in X such that

$$A \in I \Rightarrow A_n \in I,$$

then (H4) holds.

In fact, if $A \in I$ than $A^* = \bigcup_{n=0}^{\infty} A_n \in I$. Further, if $x \in X \setminus [U(M)]^*$ then, according to Lemma 1, $2^k x \in X \setminus [U(M)]^*$ and hence $M_{2^k x} \in I$ and so does $[M_{2^k x}]^*$. Since I is a σ-ideal, $\bigcup_{k=0}^{\infty} [M_{2^k x}]^* \in I$ i.e. (H4) holds.

But on the other hand, it may happen that (H4) holds, however, I is not a σ-ideal. To see this, put

$(X,+) = (R,+)$, I - the family of bounded sets,

W - a bounded set, $M = W \times W$.

This example also shows that a p.l.i. ideal satisfying condition (c) need not be a σ- ideal.

Remark 2. From the stability point of view hypothesis (H7) may seem a little inconvenient. One may prefer to make assumptions rather on X and Y than on f. Hypothesis (H7) results directly from the following "weak commutativity condition"

(H7′) $$\underset{x,y\in S}{\forall}\ \underset{n\in N_0}{\forall}\ \underset{m\in N_0,\ m\geq n}{\exists}\quad 2^m(x+y) = 2^m x + 2^m y.$$

The converse implication is not true, even if we assume hypothesis (H6) additionally. To prove it put, similarly as in [9]:

$$(X,+) = (S,+) = (GL(2,R),\cdot),\ (Y,+) = (R,+),$$

$$f(x) = c \text{ (constant)},\ x_0 = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\quad y_0 = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}.$$

Then (H6) and (H7) hold but (H7) is not fulfilled for x_0, y_0. Condition (H7) also results from the following one.

(H7″) $$\underset{k\in N_0}{\exists}\ \underset{x,y\in S}{\forall}\quad f(2^k(x+y)) = f(2^k x + 2^k y).$$

In fact, by induction we derive from (H7″)

$$f(2^{kn}(x+y)) = f(2^{kn}x + 2^{kn}y) \quad \text{for } x,y\in X,\ n\in N_0,$$

which simply implies (H7).

Similarly, if we cancel f in (H7″) we get a condition, which implies (H7′).

There exist noncommutative groups satisfying (H7) e.g. multiplicative group of quaternions $\{1,-1,i,-i,j,-j,k,-k\}$. Observe also that if each group of the family $\{G_l\}_{l\in L}$ satisfies (H7) then $\bigoplus_{l\in L} G_l$ satisfies it, too.

Remark 3. If we replace in Theorem 1 (H7) by (H7′) then, in view of Remarks 1 and 2, this theorem proves that the Cauchy equation on the semigroup S is I-stable with respect to every p.l.i. σ-ideal satisfying (H1) and (H2).

Remark 4. It is assumed in the Corollary, contrary to Theorems 1 and 2, that the function f is defined on X. For $f: S\to Y$ the Corollary is not true, even if we replace 3 by any constant $k\in Q$, $k\neq 0$. Set

$$(X,+) = (Y,+) = (R,+),\quad S = \langle 0,\infty),$$

$$f(x) = \begin{cases} c\neq 0 & \text{for } x = 0\\ x & \text{for } x > 0.\end{cases}$$

Then

$$f(x+y) - f(x) - f(y) = 0 \quad \text{for} \quad x,y>0,$$

but $f(0) = c \neq 0$.

4. I-stability of the Pexider equation

R. Ger proved in [4] that if $(X,+)$ and $(Y,+)$ are abelian groups, I is a p.l.i. ideal in $(X,+)$, $M \in \Omega(I)$, $M^{-1} : \{(y,x)\ X^2: (x,y) \in M\} \in \Omega(I)$, $F,G,H : X \to Y$ are functions such that $F(x+y) = G(x) + H(y)$ for $(x,y) \in X \times X \setminus M$, then
there exists a unique homomorphism $f: X \to Y$ constants $a,b \in Y$ and a set $W \in I$ such that

$$F(x) = f(x) + a + b \qquad \text{for} \quad x \in X \setminus W,$$

$$G(x) = f(x) + a \qquad \text{for} \quad x \in X \setminus W,$$

$$H(x) = f(x) + b \qquad \text{for} \quad x \in X \setminus W.$$

Following some ideas of the proof of this theorem we shall obtain now analogous results for the Pexider equation

$$F(x+y) = G(x) + H(y)$$

as in section 3 for the Cauchy equation.

Consider the following hypotheses:

(H1′) $(X,+)$ is a group, I is a p.l.i. σ- ideal in X satisfying condition (c).

(H3′) $M \in \Omega(I)$, $M^{-1} := \{(y,x) \in X^2: (x,y) \in M\} \in \Omega(I)$.

(H7′) $\underset{x,y \in X}{\forall} \quad \underset{n \in N_0}{\forall} \quad \underset{m \in N_0,\ m \geq n}{\exists} \qquad 2^m(x+y) = 2^m x + 2^m y.$

(H11) $f,g,h: X \to Y$, $f(x+y) - g(x) - h(y) \in V$ for $(x,y) \in X^2 \setminus M$.

<u>Theorem 3</u>. If (H_1'), (H_3'), (H5), (H7′), (H11) hold then there exist a solution $F, G, H: X \to Y$ of Pexider equation and a set $W \in I$ such that

$$F(x) - f(x) \in 9 \text{ seqcl } V \quad \text{for} \quad x \in X \setminus W,$$

$$G(x) - g(x) \in 10 \text{ seqcl } V \quad \text{for} \quad x \in X \setminus W,$$

$$H(x) - h(x) \in 10 \text{ seqcl } V \quad \text{for} \quad x \in X \setminus W.$$

<u>Proof</u>. We set

$$M' := M \cup M^{-1}.$$

By (H3') $M' \in \Omega(I)$. Take now $U(M')$ and fix an $x_0 \in X \in U(M')$.
Put

$$\bar{M} := \{(x,y) \in X^2 : x \in M'_{x_0} \text{ or } y \in M'_{x_0} \text{ or } (x,y) \in M\}$$
$$= (M'_{x_0} \times X) \cup (X \times M'_{x_0}) \cup M.$$

It is a routine matter to check that $\bar{M} \in \Omega(I)$.
In view of (H11) we have

$$f(x+y) - g(x) - h(y) \in V \quad \text{for} \quad (x,y) \in X^2 \setminus M. \tag{13}$$

If $x \in X \setminus M'_{x_0}$ then $(x, x_0) \in X^2 \setminus M$. Therefore, making use of (H11) we get

$$f(x+x_0) - g(x) - h(x_0) \in V \quad \text{for} \quad x \in X \setminus M'_{x_0}. \tag{14}$$

Similarly, if $y \in X \setminus M'_{x_0}$ then $(x_0, y) \in X^2 \setminus M$. Hence, again by (H11)

$$f(x_0+y) - g(x_0) - h(y) \in V \quad \text{for} \quad y \in X \setminus M'_{x_0}. \tag{15}$$

(13), (14) and (15) imply directly

$$f(x+y) - f(x+x_0) - f(x_0+y) + g(x_0) + h(x_0) \in 3V \quad \text{for} \quad (x,y) \in X^2 \setminus \bar{M}. \tag{16}$$

Consider now the set

$$\bar{M}(x_0) := \{(x,y) \in X^2 : (x_0+x,\ y+x_0) \in \bar{M}\} = (-x_0, 0) + \bar{M} + (0, -x_0).$$

Since $\Omega(I)$ is a p.l.i. ideal, $\bar{M}(x_0) \in \Omega(I)$.
Replacing in (16) x by x_0+x and y by $y+x_0$, we obtain

$$f(x_0+x+y+x_0) - f(x_0+x+x_0) - f(x_0+y+x_0) + g(x_0) + h(x_0) \in 3V \quad \text{for} \quad (x,y) \in X^2 \setminus \bar{M}(x_0). \tag{17}$$

Put

$$\phi(x) := f(x_0+x+x_0) - g(x_0) - h(x_0). \tag{18}$$

(17) and (18) imply directly the following condition

$$\varphi(x+y) - \varphi(x) - \varphi(y) \in 3V \quad \text{for} \quad (x,y) \in X^2 \setminus \bar{M}(x_0).$$

Since I is a σ-ideal satisfying condition (c) condition (H4) (with $S = X$) holds (see Remark 1).

Thus, on account of Theorem 1, there exist an additive function $\alpha : X \to Y$ and a set $T \in I$ such that

$$\alpha(x) - \varphi(x) \in 3 \text{ seqcl } 3V = 9 \text{ seqcl } V \quad \text{for} \quad x \in X \setminus T. \tag{19}$$

Replacing in this condition x by $-x_0+x-x_0$, applying additivity of α and (18), we get

$$\alpha(x) - f(x) - \alpha(x_0) - \alpha(x_0) + g(x_0) + h(x_0) \in 9 \text{ seqcl } V \tag{20}$$
$$\text{for} \quad x \in X \setminus (x_0 + T + x_0).$$

By the similar argument, substituting in (19) x by $x+x_0$, we obtain

$$\alpha(x) - f(x+x_0) - \alpha(x_0) + g(x_0) + h(x_0) \in 9 \text{ seqcl } V \tag{21}$$
$$\text{for} \quad x \in X \setminus (x_0 + T).$$

Conditions (14) and (21) yield

$$\alpha(x) - g(x) - \alpha(x_0) + g(x_0) \in 10 \text{ seqcl } V \tag{22}$$
$$\text{for} \quad x \in X \setminus \left[M'_{x_0} \cup (x_0 + T)\right].$$

Inserting in (20) x_0+x in the place of x we get

$$\alpha(x) - f(x_0+x) - \alpha(x_0) + g(x_0) + h(x_0) \in 9 \text{ seqcl } V \tag{23}$$
$$\text{for} \quad x \in X \setminus (T + x_0).$$

Interchanging in (15) y on x we obtain

$$f(x_0+x) - g(x_0) - h(x) \in V \quad \text{for} \quad x \in X \setminus M'_{x_0},$$

which, together with (23), leads to

$$\alpha(x) - h(x) + h(x_0) \in 10 \text{ seqcl } V \tag{24}$$
$$\text{for} \quad x \in X \setminus \left[M'_{x_0} \cup (T + x_0)\right].$$

Put:

$$W: = (x_o + T + x_o) \cup (T + x_o) \cup (x_o + T) \cup M'_{x_o},$$

$$p: = -\alpha(x_o) + g(x_o),$$

$$q: = -\alpha(x_o) + h(x_o),$$

$$F(x): = \alpha(x) + p + q,$$

$$G(x): = \alpha(x) + p,$$

$$H(x): = \alpha(x) + q.$$

Clearly, $W \in I$ and the triplet (F,G,H) satisfies the Pexider equation (cf. [1], p. 142).

Applying the above denotations, we get now from (20), (22) and (24)

$$F(x) - f(x) \in 9 \text{ seqcl } V \quad \text{for} \quad x \in X \setminus W,$$

$$G(x) - g(x) \in 10 \text{ seqcl } V \quad \text{for} \quad x \in X \setminus W,$$

$$H(x) - h(x) \in 10 \text{ seqcl } V \quad \text{for} \quad x \in X \setminus W,$$

which completes the proof. □

Now we shall prove an analogue of Theorem 2 for the Pexider equation.

Theorem 4. Let all the assumptions of Theorem 3 be fulfilled and let, additionally, there exists an $x_o \in X \in U(M')$ such that

$$\{x \in X: (x_o+x, x+x_o) \in M\} \in I.$$

Then there exist a solution $F,G,H: X \to Y$ of the Pexider equation and a set $W \in I$ such that

$$F(x) - f(x) \in 3 \text{ seqcl } V \quad \text{for} \quad x \in X \setminus W,$$

$$G(x) - g(x) \in 4 \text{ seqcl } V \quad \text{for} \quad x \in X \setminus W,$$

$$H(x) - h(x) \in 4 \text{ seqcl } V \quad \text{for} \quad x \in X \setminus W.$$

Proof. The proof runs similarly to the proof of Theorem 3. Only on essential change is needed. Using the denotations of the previous proof, we get from our additional assumption

$$[\bar{M}(x_o)]_D = \{x \in X: (x_o+x, x+x_o) \in \bar{M}\}$$

$$= \{x \in X: x_o+x \in M'_{x_o} \quad \text{or} \quad x+x_o \in M'_{x_o} \quad \text{or} \quad (x_o+x, x+x_o) \in M\} =$$

$$= (-x_o + M'_{x_o}) \cup (M'_{x_o} - x_o) \cup \{x \in X: (x_o+x, x+x_o) \in M\} \in I.$$

Hence, $\bar{M}(x_0)$ satisfies condition (H8). Therefore, instead of Theorem 1, we may apply, to the function φ, Theorem 2. Then, instead of (19), we obtain

$$\alpha(x) - \varphi(x) \in 3 \text{ seqcl } V \quad \text{for} \quad x \in X \setminus T. \tag{19'}$$

The remaining part of the proof runs analogusly as before; we should only keep in mind consequences of (19'). ▭

Theorem 5. If Y is a T-1 space then the functions F, G, H occuring in Theorems 3 and 4 are determined uniquely up to additive constants.

Proof. The general solution of the Pexider equation has the form (cf. [1], p. 142)

$$F(x) = \alpha(x) + p + q,$$
$$G(x) = \alpha(x) + p,$$
$$H(x) = \alpha(x) + q,$$

where α is an additive function and p, q are constants.
Consider any two solutions of the Pexider equation

$$F_1(x) = \alpha_1(x) + p_1 + q_1; \qquad F_2(x) = \alpha_2(x) + p_2 + q_2.$$
$$G_1(x) = \alpha_1(x) + p_1; \qquad G_2(x) = \alpha_2(x) + p_2.$$
$$H_1(x) = \alpha_1(x) + q_1; \qquad H_2(x) = \alpha_2(x) + q_2.$$

Suppose that these solutions satisfy Theorem 3. Then there exist $W_1, W_2 \in I$ such that

$$F_1(x) - f(x) \in 9 \text{ seqcl } V \quad \text{for} \quad x \in X \setminus W_1,$$
$$F_2(x) - f(x) \in 9 \text{ seqcl } V \quad \text{for} \quad x \in X \setminus W_2.$$

Thus we have

$$F_2(x) - F_1(x) \in 18 \text{ seqcl } V \quad \text{for} \quad x \in X \setminus (W_1 \cup W_2)$$

i.e.

$$\alpha_2(x) - \alpha_1(x) + p_2 + q_2 - p_1 - q_1 \in 18 \text{ seqcl } V \quad \text{for} \quad x \in X \setminus W,$$

where $W := W_1 \cup W_2$.

Since V is bounded, the last condistion means that $\alpha_2(x) - \alpha_1(x)$ is bounded on $X \setminus W$. Hence, there exists a bounded set $Z \subset Y$ such that

$$\alpha_2(x) - \alpha_1(x) \in Z \quad \text{for} \quad x \in X \setminus W. \tag{25}$$

Take now an $x \in X \setminus W^*$. Then, by Lemma 1, $2^n x \in X \setminus W^*$ for $n \in N_0$. According to (25) we have now

$$2^n[\alpha_2(x) - \alpha_1(x)] = \alpha_2(2^n x) - \alpha_1(2^n x) \in Z,$$

i.e.
$$\alpha_2(x) - \alpha_1(x) \in \frac{1}{2^n} Z.$$

Letting $n \to \infty$, making use of the boundedness of Z and of the fact that Y is a T-1 space, we get

$$\alpha_2(x) - \alpha_1(x) \in \operatorname{seqcl} \{0\} \subset \operatorname{cl}\{0\} = \{0\}.$$

Thus

$$\alpha_2(x) = \alpha_1(x) \quad \text{for} \quad x \in X \setminus W^*.$$

But, in view of Lemma 3 of [4], $X \setminus W^*$ generates X, so

$$\alpha_2(x) = \alpha_1(x) \quad \text{for all} \quad x \in X,$$

whence, we obtain

$$F_2(x) - F_1(x) = c_1 \quad \text{for all} \quad x \in X,$$
$$G_2(x) - G_1(x) = c_2 \quad \text{for all} \quad x \in X,$$
$$H_2(x) - H_1(x) = c_3 \quad \text{for all} \quad x \in X,$$

where c_1, c_2, c_3 are respective constant. □

Remark 6. The assumptions of Theorems 3 and 4 can be slightly weakened. Namely, the crucial step is the application of Theorem 1 to the function Φ defined by (18). Therefore, instead of assuming that I is a σ-ideal, it is sufficient to assume that hypothesis (H4) with M replaced by $M(x_0)$ holds. Similarly, instead of (H7'), we may assume that the function Φ satisfies (H7). Then we would get respective conditions on f, x_0, M, $g(x_0)$, $h(x_0)$. However, it is clear that such assumptions are inconvenient to formulate.

Remark 7. Analogues of Theorems 4 and 5 for a semigroup S are not true. Put

$$(X,+) = (Y,+) = (R,+), \quad S = (1,\infty),$$

$$F(x) = \begin{cases} \frac{1}{x-1} & \text{for} \quad x \in (1,2) \\ x & \text{for} \quad x \in \langle 2,\infty), \end{cases}$$

$$G(x) = H(x) = x \quad \text{for} \quad x \in (1,\infty).$$

Let $I \subset 2^X$ be a family of countable sets. Then

$$F(x+y) - G(x) - H(y) = 0 \quad \text{for all} \quad x,y \in S,$$

but none additive function α exists such that

$$F(x) = \alpha(x) + c \quad \text{for} \quad x \in S \setminus W,$$

where $W \in I$.

5. Open problems

The following problems remain open.

1) To characterize groups (or semigroups) satisfying the following so called "weak commutativity condition"

$$\underset{x,y \in X}{\forall} \quad \underset{n \in N_0}{\forall} \quad \underset{m \in N_0,\ m \geq n}{\exists} \quad 2^m(x+y) = 2^m x + 2^m y.$$

2) To determine whether the estimation of $\alpha(x) - f(x)$ in Theorem 1 can be improved. We mean here if 3 can be replaced by some $K \in Q$, $0 < K < 3$.

3) To give an example of a group X and a non-trivial p.l.i. ideal I in X (if there exists any) such that the Cauchy equation is stable but not I-stable.

REFERENCES

[1] Aczél J.: Lectures on functional equations and their applications, New York and London 1966.

[2] Baker J.A., Lawrence J., Zorzitto F.: The stability of the equation $f(x+y) = f(x)f(y)$, Proc. Amer. Math. Soc. 74 (1979), 242-246.

[3] Brydak D.: On the stability of the linear functional equation $\Phi[f(x)] = g(x)\Phi(x) + F(x)$, Proc. Amer. Math. Soc. 26 (1970), 455-460.

[4] Ger R.: On some functional equation with the restricted domain, Fund. Math. 89 (1975), 131-149.

[5] Ger R.: Almost additive functions on semigroups and a functional equation, Publ. Math. 26 (1979), 219-228.

[6] Ger R.: Almost aproximately additive mappings, General inequalities 3, Birkhauser Verlag, Basel-Boston-Stuttgart 1983, 263-276.

[7] Hyers D.H.: On the stability of the linear functional equation, Proc. Math. Acad. Sci. USA 27 (1951), 222-224.

[8] Moszner Z.: Sur la stabilité de l'équation d'homomorphisme, Aeq. Math. 29 (1985), 290-306.

[9] Rätz J.: On approximately additive mappings, General Inequalities 2, Birkhauser Verlag, Basel-Boston-Stuttgart 1980, 233-251.

[10] Schaefer H.H.: Topological vector spaces, New York and London 1966.