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# ON THE EXISTENCE OF A FUNDAMENTAL FUNCTION IN FINSLER SPACES

Summary. Geometric and analytic conditions on the metric tensor $g_{ij}(x,y)$ assuring the existance of a fundamental function of an n-dimensional Finsler space (M,g) are discussed and investigated.

1. P. Finsler investigated variational problems of the form $\delta\int\mathcal{F}(x,\dot{x})dt$, and this led him in his thesis [2] written at C. Carathéodory to a geometry with a generalized metric. The name of Finsler geometry was then given by J.H. Taylor [7] in 1927. As a matter of fact the possibility of such a geometry had already been mentioned in the famous habilitation lecture of B. Riemann [6] in which he laid the fondations of Riemannian geometry. However this has completely been neglected and forgotten through the more than six decades which had elapsed from 1854.

The fundamental function $\mathcal{F}(x,\dot{x})$ of this geometry is supposed to be on its domain except for $\dot{x} = 0$ a) of class $C^3$, b) positive, and c) positively homoheneous of order 1 in $\dot{x}$. The surface $\mathcal{F}(x,\dot{x}) = 1$ of the tangent space $T_x$ of M is called indicatrix and denoted by $J(x,\ )$.

Finsler geometry was considered at the beginning as the theory of a point space (manifold) M. In this cocept however, a metrical linear connection was not realizable. Let indeed P and Q be two arbitrary points of M and C a curve connecting them. Then a linear connection $\mathcal{H}$ induces a linear map $\varphi: T_P \to T_Q$ between the tangent spaces $T_P$ and $T_Q$. $\mathcal{H}$ is called metrical if $\varphi(J(P)) = \varphi(J(Q))$ for any P, Q and C. However J(P) and J(Q) may have quite different shapes and thus, except in special cases, they cannot be transformed into each other by any linear transformation $\varphi$. Hence for a Finsler geometry $(M,\mathcal{F})$ in the above sense there exists in general no metrical linear connection. In order to assure the existence of a metrical linear connection Cartan [1] considered the 2n-1 dimensional space of the line elements, i.e. the space of points and directions as the underlying manifold of the Finsler geometry. This can be done also replacing M by the tangent space TM.

The metric tensor $g_{ij}$ of an $(M,\mathcal{F})$ is deduced from the fundamental function $\mathcal{F}(x,y)$ and has the form

$$g_{ij}(x,y) := \frac{1}{2}\frac{\partial^2 \mathcal{F}^2}{\partial y^i \partial y^j}(x,y), \qquad i,j = 1,2,\dots,n. \tag{1}$$

However it turned out that starting directly with a metric tensor $g_{ij}(x,y)$ in place of the fundamental function $\mathcal{F}(x,y)$ nearly the same edifice can be constructed [3], [4], [5]. A Finsler space over $M$ built on the metric tensor $g_{ij}$ is denoted by $(M,g)$. Here one requires from $g_{ij}$ the following properties: a) $g_{ij}$ is homogeneous of order 0 in $y$, b) it is a (0,2) tensor of class $C^2$, c) it is symmetric in $i$ and $j$, and d) it is nondegenerate.

It is clear that $(M,g)$ is somewhat more general than $(M,\mathcal{F})$. In the following we want to discuss and investigate the geometric and analytic relations between the two spaces.

2. For a fixed line element $(x_o,y_o)$ $g_{ij}(x_o,y_o)y^iy^j = 1$ is a quadric in $T_{x_o}$, and

$$\Psi(y,c): \qquad g_{ij}(x_o,c)y^iy^j - 1 = 0 \tag{2}$$

is a family of quadrics depending on the parameters $c \in R^n$ $(c \neq 0)$.

$$\Phi(c): \qquad y^i = y^i(c) := \frac{c^i}{(g_{rb}(x_o,c)c^rc^b)^{1/2}} \tag{3}$$

is a surface $\Phi \subset T_{x_o}$ in a parametrized form. We remark that the right hand side of (3) is homogeneous of order $o$ in $c^i$. Thus $y^i(c)$ depends essentially on $n-1$ parameters. We want to show that.

<u>Theorem</u>. $g_{ij}$ is deduced from an $\mathcal{F}$ by (1) iff $\Phi(c)$ is a second order envelope of $\Psi(y,c)$.

This means a geometric characterization of the relation between $(M,g)$ and $(M,\mathcal{F})$.

$\Phi(c)$ is an envelope of $\Psi(y,c)$ iff I) the point $Y(c)$ determined by (3) lies on the surface (2), and II) in these points they tangent each others.

We find a necessary and sufficient condition for this.

I) is obviously true, for

$$\Psi(y(c),c) \equiv 0, \tag{4}$$

Differentiation of (4) gives

$$\frac{\partial \Psi}{\partial y^m}(y(c),c)\,\frac{\partial y^m}{\partial c^k}(c) + \frac{\partial \Psi}{\partial c^k}(y(c),c) = 0, \tag{5}$$

where $\frac{\partial \Psi}{\partial y^m}(y(c),c) = 2g_{im}(x_o,c)y^i(c)$ are the components of the normal vector $\mathcal{N}_\Psi$ of $\Psi(y,c)$ at $Y(c)$. This normal vector never vanishes, for $\det|g_{ij}(x_o,c)| \neq 0$. $\frac{\partial y^m}{\partial k}(c)$ are components of non vanishing tangents $\mathfrak{t}_\Phi$ of the parameter lines of $\Phi(c)$. Hence in the case when $\Phi$ is an envelope of the family $\Psi$, the first term of (5) means the inner product of the perpendicular vectors $\mathcal{N}_\Psi$ and $\mathfrak{t}_\Phi$ in the tangent space $T_{x_o}$ equipped with a euclidean metric. Thus, in this case

$$\frac{\partial \Psi}{\partial c^k}(y(c),c) = 0. \tag{6}$$

Also conversely, if the second term of (5) vanishes, then the two nonvanishing vectors $\mathcal{N}_\Psi$ and $\mathfrak{t}_\Phi$ are perpendicular in $T_{x_o}$, and hence, in view of (4) $\Phi$ is an envelope of $\Psi(y,c)$. Thus $\frac{\partial \Psi}{\partial c^h}(y(c),c) = 0 \Longleftrightarrow \Phi$ is an envelope of $\Psi(y,c)$. In view of (2) and (3)

$$\frac{\partial \Psi}{\partial c^k}(y,c)\Big|_{y=y(c)} = \frac{\partial g_{ij}}{\partial c^k}(x_o,c)\,\frac{c^i}{(g_{rs}(x_o,c)c^rc^s)^{1/2}}\,\frac{c^j}{(g_{ab}(x_o,c)c^ac^b)^{1/2}} = 0.$$

Multiplying with $g_{rs}(x_o,c)c^rc^s$ and denoting the second variable of $g_{ij}$ again by $y$ we obtain.

Proposition 1. $\Phi$ is an envelope of $\Psi(y,c)$ iff

$$\frac{\partial g_{ij}}{\partial y^k}(x,y)y^iy^j = 0. \tag{7}$$

$2\frac{\partial g_{ij}}{\partial y^k}$ is Cartan's torsion tensor denoted by $C_{ijk}$.
Thus the condition of Proposition 1 can be denoted also by $C_{ijh}y^iy^j = 0$.

3. We call $\Phi$ a second order envelope of $\Psi$ if it is an envelope, i.e. it satisfies I) and II), and still III) $\Psi(y,c)$ and $\Phi$ osculate at $Y(c)$ in the second order.

Proposition 2. $\Phi$ is a second order envelope of $\Psi$ iff

$$\frac{\partial g_{ir}}{\partial y^s}(x,y)y^i = 0. \tag{8}$$

Let us consider the surface $\Gamma$ : $g_{ij}(x_o,y)y^iy^j = 1$ of $T_{x_o}$. (3) easily implies that any point $Y(c)$ of $\Phi$ lies on $\Gamma$. i.e. $\Phi \subset \Gamma$. But both $\Phi$ and $\Gamma$ have a single point on each ray in $T_{x_o}$ through $x_o$. Hence $\Phi$ is the same as $\Gamma$, $\Phi \equiv \Gamma$.

Second order osculation between $\Phi$ and $\Psi$ means the equality of

$$\frac{\partial^2\Gamma}{\partial y^r\partial y^s}(x_o,y) = \frac{\partial^2 g_{ij}}{\partial y^r\partial y^s}(x_o,y)y^iy^j + 2\frac{\partial g_{ir}}{\partial y^s}(x_o,y)y^i +$$

$$+ 2\frac{\partial g_{is}}{\partial y^r}(x_o,y)y^i + 2g_{rs}(x_o,y) \tag{9}$$

and

$$\frac{\partial^2\Psi}{\partial y^r\partial y^s}(x_o,y) = 2g_{rs}(x_o,c) \tag{10}$$

at $Y(c)$. Because of the homogeneity of order 0 of $g_{rs}$

$g_{rs}(x_o,y(c)) = g_{rs}(x_o,c)$.

Now, if $\Phi$ is a second order envelope of $\Psi$, then a) $\Phi$ is also an envelope of $\Psi$, and thus $\frac{\partial^2 g_{ij}}{\partial y^r\partial y^s}y^iy^j = 0$ by Proposition 1, and this term drops out from (9); and b) $\Phi$ and $\Psi$ osculate in the second order, what means the equality of the derivatives (9) and (10). Hence we obtain (8). Conversely we assume (8). This implies (7), and thus $\Phi$ is an envelope of $\Psi$. Moreover, in view of (7) and (8) the right hand side of (9) is $2g_{rs}$. Thus $\frac{\partial^2\Gamma}{\partial y^r\partial y^s} = \frac{\partial^2\Psi}{\partial y^r\partial y^s}$ at $Y(c)$, which yields the second order osculation between $\Phi$ and $\Psi$. These mean that $\Phi$ is a second order envelope of $\Psi$.

4. Making use of Propositions 1 and 2 we prove our Theorem.

A) (1) $\Longrightarrow$ $\Phi$ is a seconf order envelope of $\Psi$.

According to (1)

$$\frac{\partial g_{ij}}{\partial y^k}(x,y)y^i = \frac{1}{2}\frac{\partial^3\mathcal{F}^2}{\partial y^k\partial y^i\partial y^j}(x,y)y^i = \frac{1}{2}\left[\left(\frac{\partial}{\partial y^i}\frac{\partial^2\mathcal{F}^2}{\partial y^k\partial y^j}(x,y)\right)y^i\right]$$

But $\mathcal{F}^2$ is homogeneous of order 2, and $\frac{\partial^2\mathcal{F}^2}{\partial y^k \partial y^j}$ of order 0 in y. Hence, by Euler's theorem on the homogeneous functions, the expression in the square bracket vanishes. Thus (8), and also (7) are satisfied. These mean that $\Phi$ is a second order envelope of $\Psi$.

B) (1) $\Longleftarrow$ $\Phi$ is a second order envelope of $\Psi$.

Consider the function $\mathcal{F}^2(x,y) := g_{ij}(x,y)y^i y^j$. In this case $\frac{\partial\mathcal{F}^2}{\partial y^h}(x,y) = \frac{\partial g_{ij}}{\partial y^h}(x,y)y^i y^j + 2\, g_{kj}(x,y)y^j$. By Proposition 1 our assumption implies the relation (7), ans thus $\frac{\partial\mathcal{F}^2}{\partial y^h} = 2\, g_{hj}y^j$. After a repeated derivation $\frac{\partial^2\mathcal{F}^2}{\partial y^k \partial y^r} = 2\frac{\partial g_{kj}}{\partial y^r}y^j + 2\, g_{kr}$. By Proposition 2 our assumption implies the relation (8), and by using this we obtain (1).

The above considerations yield also a geometrical proof of

<u>Proposition 3</u>. $g_{ij}$ can be deducef from an $\mathcal{F}$ by (1) iff

$$\frac{\partial g_{ij}}{\partial y^k}\, y^i = 0$$

Conditions (7) and (8) in form of $C_{ijh}y^i y^j = 0$ and $C_{ijh}y^i = 0$ $\left(C_{ijh} \equiv \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}\right)$ often occur in the investigations of an (M,g). Propositions 1 and 2 enlighten their geometrical contents.

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