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ON COMPOSITION OPERATORS

The theory of functional equations is inseparably intertwined with the notion of operators on appropriate functions spaces. We are going to discuss a class we shall call composition operators.

To fix our ideas, we shall work in the Banach algebra A(I) of continuous real-valued functions on the clased unit interval. The algebra A(I) arises from the space C(I) by endowing it with pointwise multiplication. Everything we prove will be in this framework. At the end we shall mention results by R. Liedl and J. Schwaiger in a different setting. We start with.

<u>Definition</u>. A substitution (right composition) operator is a mapping (1)

$$B\varphi := \varphi \circ \beta \tag{2}$$

defined for every $\varphi \in A(I)$ where β is a fixed continuous self-mapping of I. It is seen that B is linear and multiplicative, that is an endomorphism of A(I), in particular, a bounded linear operator on C(I). The opposite is also true:

<u>Proposition</u>. Every non-trivial endomorphism of A(I) is a substituion operator. (3)

This has been known for a long time; for a proof see e.g. [Ta 70].

<u>Definition</u>. A superposition (left composition) operator is a mapping (4)

$$K\varphi := k \circ \varphi \tag{5}$$

of A(I) into itself, where k is a continuous mapping of IR into I. It is seen that K is in general neither linear nor multiplicative.

We note that the product of two substituions is not a substitution, while the product of two superpositions is a superposition.

We notice an obvious fact:

<u>Proposition</u>. Every substitution commutes with every superposition (6)

<u>Definition</u>. A composition operator on $A(I)$ is a mapping of $A(I)$ into itself of the form (7)

$$C\varphi := KB\varphi = BK\varphi = k\{\varphi[\beta(x)]\} \tag{8}$$

where K is a superposition and B is a substitution.

Thus substitutions and superpositions are special compositions. An example of (8) is $\alpha \to \alpha^{-1} \circ \varphi \circ \alpha$, a "conjugation operator".

Substitution operators wer discovered by C. Bourlet (Bo 97, Bo 97_2). For results on substitutions see [Ta 67], [Ta 81]. The latter reference also lists literature up to the end of 1980.

Superposition operators are usually defined in a more general form, as Nemytskij operators by $\varphi(x) \to f(x,\varphi(x))$. There exists a vast literature on these. Results up to 1987 are surveyed in [Ap 87]. A recent result is [Ma 87].

In this paper we shall investigate connections between substitutions and superpositions.

We already noted the commutation property (6).

<u>Proposition</u>. If a linear bounded operator on $A(I)$ commutes with the superposition Q (where $(Q\varphi)(x) := \varphi(x)^2$) then it is a substitution operator.

Proof. (Cf. [Ta 73]) Let u,v be arbitrary elements of $A(I)$. Introducing $\varphi := \frac{1}{2}(u+v)$, $\psi := \frac{1}{2}(u-v)$ we find $u = \varphi + \psi$, $v = \varphi - \psi$, thus $uv = \varphi^2 - \psi^2$ and

$$T(uv) = T(\varphi^2 - \psi^2) = TQ\varphi - TQ\psi = QT\varphi - QT\psi = (T\varphi)^2 - (T\psi)^2 =$$

$$= (T\varphi + T\psi)(T\varphi - T\psi) = T(\varphi + \psi)T(\varphi - \psi) = TuTv.$$

Thus T is an endomorphism and, by (3) a subsitution operator, QED.

If we demand a stronger commutation property we can drop the condition of linearity:

<u>Proposition</u>. If a self-mapping ("nonlinear operator") of $A(I)$ commutes with every superposition, then it is a substitution, thus linear, bounded and continuous. (10)

Proof. Let $\varphi \in A(I)$ be arbitrary. Introducing the superposition Φ: $\Phi\psi := \varphi \circ \psi$ we find $\varphi = \Phi(x)$ where x now stands for the identity mapping on I. Denoting the mapping in question by T, we have, writign $Tx =: \tau$, $T\varphi = T\Phi x = \Phi Tx = \Phi\tau = \varphi \circ \tau$, QED.

If we demand the commuting property only on a dense subset of $A(I)$, we need a stronger condition on the mapping.

By "a dense set of superpositions" we shall mean a set of superpositions K such that the functions K(id.) =: k form a set dense in A(I).

<u>Proposition</u>. If a uniformly continuous (not necessarily linear) self-mapping of A(I) commutes with a dense set of superpositions, then it is a substitution, and thus linear. (11)

Proof. Denote the mapping by T, let $\varphi \in A(I)$ be arbitrary and $\varepsilon > 0$. Then by assumption there exists a $k \in A(I)$ such that $\|\varphi - k\| < \frac{\varepsilon}{2}$ adn $\|T(\varphi) - T(k)\| < \frac{\varepsilon}{2}$ moreover $T(k)(x) = k(Tx) =: k(\tau(x))$ thus $T_k = k \circ \tau$.
Then

$$\|T(\varphi) - \varphi\circ\tau\| \leq \|T(\varphi) - T(k)\| + \|T(k) - k\circ\tau\| + \|k\circ\tau - \varphi\circ\tau\| =$$

$$= \|T(\varphi) - T(k)\| + \|(k - \varphi)\circ\tau\| < \varepsilon \qquad (12)$$

since for every $\psi \in C(I)$, $\|\psi\circ\tau\| \leq \|\psi\|$. The l.h.s. in (11) does not depend on ε, $T\varphi = \varphi\circ\tau$, QED.

We note that in the proofs for (10) and (11) we used the fact that id(x) $\in$ A(I). In function algebras which do not contain the identity mapping, the situation is more difficult.

The reverse of (10) is also true.

<u>Proposition</u>. If a self-mapping of A(I) commutes with all substitutions, then it is a superposition. (13)

Proof. Let S be the self-mapping of A(I), and $\Phi \in A(I)$. We can introduce the substitution operator Φ by $\Phi\psi = \psi\circ\varphi$ where ψ ranges over A(I). Then we can represent φ in the form $\varphi(x) = \Phi x$. Since S commutes with Φ we have, by putting $Sx =: \sigma(x)$

$$(S\varphi)(x) = S\Phi x = \Phi Sx = (\Phi\sigma)(x) = (\sigma\circ\varphi)(x) \qquad (13)$$

which was to be proven.

Let us conclude by pointing out some unsolved problems. We wanted to characterize the basic composition operators, substitutions and superpositions by their commutation properties. We did so in the following propositions, stated informally:

in (9): "linearity and commuting with x^2 implies substitution"

in (10): "commuting with all superpositions implies substitution"

in (13): "commuting with all substitutions implies superposition"

There are obviously two things here to be improved. Linearity in (9) is alien to the spirit of our approach: everything should be formulated in terms of commutation only. On the other hand, "commuting with all..."

is much too strong. Here (11) already points to an improvement. Since the Weierstrass approximation theorem can be formulated in terms of polynomials with rational coefficients, we have from (11): "commuting with a certain countably infinite set of superpositions implies substitution".

We are now able to formulate our

Question (15)

(a) is it possible to show that a bounded operator on A(I) is a substitution (and thus linear), if it commutes with certain, finitely many, superpositions?

(b) is it possible to show that a bounded operator on A(I) is a superposition, if it commutes with certain, finitely many substitutions?

Two questions probably leading farther afield are the following:

Question. Is it possible to show linearity of a uniformly continuous selfmapping of A(I) from the condition that it commutes with certain, finitely many composition operators (cf. (7))? (16)

Question. In what way can the condition of additivity (one of the constituent parts of linearity) be replaced by the condition of multiplicativity? (17)

As a comment on Question (17) let us analyze Proposition (9). It says the following:

If a self-mapping T of A(I) is additive $(T(\varphi_1 + \varphi_2) = T\varphi_1 + T\varphi_2)$ and commutes with the superposition $\Lambda\,(\Lambda\varphi := \lambda\varphi)$ and $Q(Q(\varphi) = \varphi^2)$ then A is a substitution. Here we singled out one constituent of linearity, namely additivity, while the other constituent, homogeneity, was written as commutation with the superposition Λ.

We can prove a result in a sens "dual" to (9), thereby contributing towards an answer to Question (17)

Proposition. If a uniformly continuous self-mapping T on A(I) is multiplicative: $(T(\varphi_1\varphi_2) = T\varphi_1 T\varphi_2)$ and commutes with the superpositions $\Lambda(\Lambda\varphi := \lambda\varphi)$ and $E\ ((E\varphi)(x) = \varphi(x) + 1)$ then it is a substituion, thus linear.

Proof. We have the conditions

$$\text{(a)}\quad T(\varphi_1\varphi_2) = T(\varphi_1)T(\varphi_2) \tag{19}$$

$$\text{(b)}\quad \forall_{\lambda\in\mathbb{R}}\lambda \quad T(\lambda\varphi) = \lambda T(\varphi)$$

$$\text{(c)}\quad T[\varphi(x) + 1] = (T\varphi)(x) + 1$$

We first show by induction that T is a substitution when acting on a polynomial. Denoting by 1 the constant function with value 1, we have by (19) (a) $T(1) = T(1^2) = (T1)^2$ thus either $T(1) = 0$ or $T(1) = 1$; the former is not possible, since then by (19) (a) $T(\varphi) = T(1 \cdot \varphi) = T(1)T(\varphi) = 0$ and T were the trivial zero operator; but then (19) (c) would result in $0 = 1$. Thus

$$T(1) = 1 \tag{20}$$

In the sequel let us again define $Tx = \tau(x)$.

Let now P_n be a polynomial of degree n; every such polynomial is of the form

$$P_n(x) = x^k\left[a + x^m P_{n-m-k}(x)\right] \quad \text{where } a \neq 0.$$

From here

$$T\left[P_n(x)\right] = \tau(x)^k T\left[a + x^m P_{n-m-k}(x)\right] = \tau(x)^k a T\left[1 + \frac{x^m}{a} P_{n-m-k}(x)\right] =$$

$$= \tau(x)^k a\left[1 + \frac{1}{a} T(x^m P_{n-m-k}(x)\right] = \tau(x)^k\left[a + \tau(x)^m T\left[P_{n-m-k}(x)\right]\right]$$

Applying the same procedure to the polynomial P_{n-m-k} (unless it is a constant) we find in finitely many steps $(TP)(x) = P\left[\tau(x)\right]$ thus

$$TP = P \circ \tau \tag{21}$$

for every polynomial in $A(I)$. Now the proof is analogous to the proof of (11).

Since the set of all polynomials is dense in $A(I)$, we have for every $\varphi \in A(I)$ and every $\varepsilon > 0$ a polynomial P such the $\|\varphi - P\| < \frac{\varepsilon}{2}$ and $\|T(\varphi) - T(P)\| < \frac{\varepsilon}{2}$.

Then

$$\|T(\varphi) - \varphi \circ \tau\| \leq \|T(\varphi) - T(P)\| + \|T(P) - p \circ \tau\| + \|p \circ \tau - \varphi \circ \tau\| \tag{22}$$

Since the middle term is zero, we have $\|T(\varphi) - \varphi \circ \tau\| < \varepsilon$ for Proposition (18) is proven.

The idea of characterizing substitutions by means of commutation properties originated with R. Liedl and J. Schwaiger, who have unpublished results on this in the context of C^∞ - mappings of n-dimensional manifolds ([LS 87]).

REFERENCES

[Ap 87] Appell J.: The superposition operator in function spaces - a survey. Report No 141, Institut für Mathematik, Universität Augsburg 1987.

[Bo 97] Bourlet C.: Sur certaines équations analogues aux équations différentielles. Comptes Rendus Acad. Sci. Paris 124 (1897), 1431-1433.

[Bo 97_2] Bourlet C.: Sur les transmutations. Bull. Soc. Math. France 25 (1897), 132-140.

[LS 87] Liedl R. and Schwaiger J.: Manuscript (1987).

[Ma 87] Matkowski J.: On Nemytzkij operator. Proceedings of the II. International Conference on Functional Equations and Inequalities, June 22-27, 1987, Szczawnica (Poland).

[Ta 67] Targoński Gy.: Zur Klassifizierung der linearen Operatoren auf Funktionenalgebren. Math. Z. 97 (1967), 238-250.

[Ta 70] Targoński Gy.: Linear endomorphisms of function algebras and related functional equations. Indiana Univ. Mathematics J. 20 (1970), 579-589.

[Ta 73] Targoński Gy.: Charakterisierung von Operatoren durch Funktionalgleichungen. Demonstratio Mathematica 6 (1973), 861-870.

[Ta 81] Targoński Gy.: Topics in Iteration Theory. Vandenboeck & Ruprecht, Göttingen, Zürich 1981.