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ON THE MAP BETWEEN THE SETS OF POINTS AT INFINITY IN THE CASE OF 1-DIMENSIONAL FOLIATION OF THE TOURS

1. In [W] we proved the non-existence of totally geodesic (moreover, close to totally geodesic) foliations of compact Riemannian manifolds of negative sectional curvature. The similar problem for a complete manifold M of bounded curvature and finite volume is still open. It seems that the solution of this problem could be obtained by the following consideration: Lift a foliation F of M to the universal covering X of M. If F is totally geodesic (or, if F is reasonably close to totally geodesic) then the lifted foliation $\tilde{F}$ consists of leaves which are Hadamard manifolds, so one can try to define in a natural way a map

$$L(\infty) \longrightarrow X(\infty) \tag{1}$$

for all the leaves L of $\tilde{F}$. The configuration of the sets $L(\infty)$ in $X(\infty)$ is invariant under the group $\Gamma = \pi_1(M)$ acting on $X(\infty)$. This action is ergodic, so one could expect that invariant configurations like that do not exist. Therefore, a question we should start with is whether the map (1) could be defined. Obviously, it is so when the leaves are totally geodesic. In this note, we consider a very simple case when $M = T^2$ is a flat tours and $\dim F = 1$.

2. Recall (see [BGS]) that if X is an Hadamard manifold, then two geodesics $\gamma_1, \gamma_2 : R \longrightarrow X$ are asymptotic when the distance of $\gamma_1(t)$ and $\gamma_2(t)$ remains bounded when $t \longrightarrow +\infty$. $X(\infty)$ is defined as the set of all classes of asymptotic geodesics. When equipped with so called cone topology $X(\infty)$ is homeomorphic to S^{n-1}, $n = \dim X$. Therefore, if L is a Hadamard submanifold of X, then the map (1) could be defined when for any L-geodesic $c: R \longrightarrow L$ the X-geodesics γ_t $(t > 0)$ satisfying

$$\gamma_t(0) = c(0) \quad \text{and} \quad \gamma_t(s) = c(t),$$

where $s = d_X(c(0), c(t))$, converge to a geodesic γ on X.

In fact, if $\tilde{c}$ is a L-geodesic asymptotic to c, then the L-distance $d_L(c(t), \tilde{c}(t))$ remains bounded when $t \longrightarrow +\infty$, so the X-distance of $c(t)$ and $\tilde{c}(t)$ remains bounded as well since $d_X \leq d_L$ on $L \times L$.

Assume that $\tilde{\gamma}_t$ is a family of X-geodesics satisfying

$$\tilde{\gamma}_t(0) = \tilde{c}(0) \quad \text{and} \quad \tilde{\gamma}_t(s) = \tilde{c}(t)$$

with $s = d_X(\tilde{c}(0), \tilde{c}(t))$. Let $\tilde{\gamma}$ be the limit of $\tilde{\gamma}_t$ when $t \longrightarrow +\infty$. The X-geodesics γ and $\tilde{\gamma}$ are asymptotic: For any $u > 0$ and t big enough we have

$$d_X(\gamma(u), \tilde{\gamma}(u)) \leqslant 1 + d_X(\gamma_t(u), \tilde{\gamma}_t(u)) \leqslant$$

$$1 + \mathrm{Max}\{d_X(c(0), c(0)),\ d_X(c(t), c(t))\} \tag{2}$$

since the distance function on X is convex. The right hand side of (2) is bounded.

So, in this case the map (1) can be defined by

$$L(\infty) \ni [c] \longrightarrow \gamma \in X(\infty) \tag{3}$$

where c and γ have the same meaning as above and $[c]$ (resp., $[\gamma]$) denotes the asymptoty class of c (resp., of γ).

3. Let F be a 1-dimensional C^2-foliation of the flat torus T^2. Following Kneser classification ([K], compare [G]) we have two cases:

(i) T^2 splits into the countable union of annuli A_i bounded by closed leaves L_i and $L_{i'}$ and filled in either by closed leaves or by lines having L_i and $L_{i'}$ as limit sets (Figure 1). The number of Reeb components (Figure 1c) is finite.

(ii) All the leaves of F are dense.

Denote by $\tilde{F}$ the lift of F to the universal covering $X = R^2$ of T^2.

LEMMA. If L is the lift of a closed leaf of F, then the map (3) is well defined.

PROOF. Let $c: R \longrightarrow X$ be an arc-length parametrization of L. Then c satisfies

$$c(t + t_0) = c(t) + k \quad (t \in R) \tag{4}$$

for some $t_0 > 0$ and $k \in Z^2$. According to the previous arguments it is sufficient to show that the limit

$$\lim_{t\to\infty} \frac{c(t)}{\|c(t)\|} \tag{5}$$

exists. Since (4) it is enough to show that the sequence

$$c(nt_0)/\|c(nt_0)\|$$

converges when $n \longrightarrow +\infty$. This is an elementary exercise to prove that the limit equals to $k/\|k\|$.

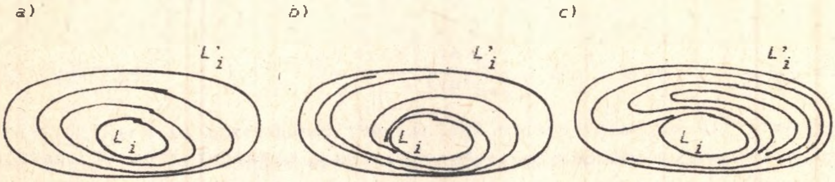

Figure 1

From our Lemma it follows that in the case (i) the map (3) is well defined for any leaf L of $\tilde{F}$. In fact, the limit (5) is the same for all closed leaves of F because it depends only on the homotopy class of $\pi \circ c$, $\pi : X \longrightarrow T^2$ being the canonical projection. Therefore, if L_1 and L_2 are lifts of closed leaves of F bounding an annulus A_1 and L is the lift of a leaf of $F|A_1$, then L stays in the region bounded by L_1 and L_2, and since $\pi(L)$ approximates $\pi(L_1)$ in infinity, the straight lines passing through the origin and a point p of L converge to P_1's when $p \longrightarrow \pm\infty$. Here, P_1 and P_2 are straight lines approximated in infinity by L_1 and L_2, respectively.

Moreover, this shows that in the case (i) there are two points z_1 and z_2 of $X(\infty)$ such that the image of $L(\infty)$ under the map (3) is contained in $Z = \{z_1, z_2\}$ for any leaf L. The image consists of a single point when L lies in a Reeb component and of two points otherwise (Figure 2).

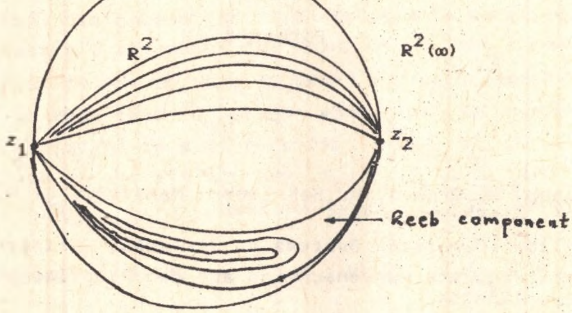

Figure 2

REMARK. Clearly, the configuration of Figure 2 cannot be invariant under ergodic actions on S^1, so the foliation $\tilde{F}$ cannot be projected onto a closed surface of genus $g > 1$. However, this observation is void: There are no foliations of such surfaces.

In the case (ii), assume that $\gamma : R \longrightarrow X$, $\|\gamma'\| = 1$, is a lift of a leaf of F and that the straight lines P_t through the origin and $\gamma(t)$ do not converge when $t \longrightarrow \infty$. Then there are two sequences (t_n) and (s_n) and two unit vectors v and w such that $t_n \longrightarrow \infty$, $s_n \longrightarrow \infty$,

$$\frac{\gamma_n(t)}{\|\gamma_n(t)\|} \longrightarrow v \quad \text{and} \quad \frac{\gamma_n(s)}{\|\gamma_n(s)\|} \longrightarrow w$$

when $n \longrightarrow \infty$. Consequently, there exists a straight line P passing through the origin and intersecting the curve γ infinitely many times: there exists a sequence (u_n) such that $u_n \longrightarrow \infty$ and $\gamma(u_n) \in P$ for any n. Without loosing generality we may assume that P is given by $x_2 = 0$, where (x_1, x_2) are Euclidean coordinates on X. The continuity argument shows easily that there are real numbers s and t for which $\gamma(s)$ and $\gamma(t)$ lie on a straight line parallel to P and $\|\gamma(s) - \gamma(t)\| = 1$ (Figure 3). It follows that $\pi(\gamma(s)) = \pi(\gamma(t))$ contradicting the assumption on F.

Therefore, we have the following.

PROPOSITION. If $\tilde{F}$ is the lift to $X = R^2$ of a foliation F of T^2, then the map (3) is well defined for any leaf L of $\tilde{F}$.

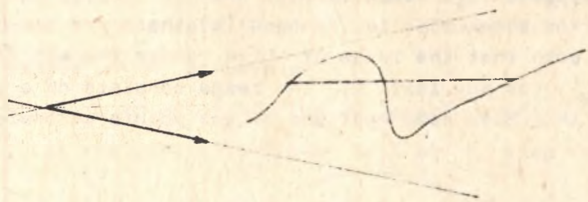

Figure 3

REFERENCES

[BGS] W. Ballmann, M. Gromov, V. Schroeder: Manifolds of Non-positive Curvature, Birkhauser 1985.

[G] C. Godbillon: Dynamical Systems on Surfaces, Springer Verlag 1983.

[K] H. Kneser: Regulare Kurvenscharen auf den Ringflaechen, Math. Ann., 91 (1923), 125-154.

[W] P.G. Walczak: Dynamics of the geodesic flow of a foliation, Ergodic Th. & Dyn. Sys., 8 (1988), 637-650.