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ON EXTERIOR DIFFERENTIAL IN PREMANIFOLDS OF FINITE DIMENSION

0. Introduction

In [1] there is introduced the concept of the exterior differential of differential forms and the canonical correspondence between differential k-forms and k-linear mappings of local vector fields of premanifolds (see [3] and [4]) is examined. The present paper contains the proof of the Stokes' formula for chains in a premanifold of finite dimension and the de Rham mapping for de Rham cohomology.

1. Preliminaries

Let M be a real or complex premanifold. For any vector field X tangent to M, i.e. a function X with the domain $D_X \in \text{top}M$ and such that $X(p) \in T_p(M)$ for $p \in D_X$. For $\alpha \in M$ we have the function $\partial_X\alpha$ defined on $D_\alpha \cap D_X$ by the formula $(\partial_X\alpha)(p) = X(p)(\alpha)$ for $p \in D_\alpha \cap D_X$. By definition also we have $D_{\partial_X\alpha} = D_\alpha \cap D_X$. The set of all vector fields X tangent to M such that for every $\alpha \in M$ we have $\partial_X\alpha \in M$ will be denoted by $V_{loc}(M)$. The set $V_{loc}(M)$ together with the addition and the multiplication by functions of M defined by the formulae: $(X + Y)(p) = X(p) + Y(p)$ for $p \in D_X \cap D_Y$ and $(\alpha \cdot X) = \alpha(p)X(p)$ for $p \in D_X \cap D_\alpha$, $X, Y \in V_{loc}(M)$, $\alpha \in M$ is a semigroup with local zeros (see [5]). A system $(e_1,\dots,e_m)$, where $e_1,\dots,e_m$ belong to $V_{loc}(M)$, $D_{e_1} = \dots = D_{e_m}$ and $(e_1(p),\dots,e_m(p))$ is a base for T_pM is said to be a local vector base in M. A premanifold for which any point of M has a neighbourhood with local vector base in M is said to be of a finite dimension.

A function ω such that $D_\omega \in \text{top}M$ and $\omega(p)$ is an element of $\Lambda^k(T_pM)^*$ for $p \in D_\omega$ is said to be a differential k-form in M. A k-form ω is said to be smooth in M iff for any $X_1,\dots,X_k \in V_{loc}(M)$ the function $\bar\omega$ with the domain $D_{\bar\omega(X_1,\dots,X_k)}$ equal to $D_\omega \cap D_{X_1} \cap \dots \cap D_{X_k}$ defined by the formula

$$\bar\omega(X_1,\dots,X_k)(p) = \langle X_1(p)\wedge\dots\wedge X_k(p) \mid \omega(p)\rangle \quad \text{for } p \in D_{\bar\omega(X_1,\dots,X_k)} \tag{1.0}$$

belongs to M. The set of all smooth k-forms we denote by $A^k(M)$. We adopt also $A^0(M) = M$.

The set of all $\eta : (V_{loc}(M))^k \longrightarrow M$ such that for any $X_1,\dots,X_k$, $X \in V_{loc}(M)$ and $\alpha \in M$ we have

$$\eta(X_1 + \alpha X, X_2,\dots,X_k) = \eta(X_1,\dots,X_k) + \omega\eta(X,X_2,\dots,X_k), \tag{1.1}$$

$$\eta(X_{i_1},\dots,X_{i_k}) = \operatorname{sgn} i\ \eta(X_1,\dots,X_k) \tag{1.2}$$

for any permutation $i = (i_1,\dots,i_k)$,

will be denoted by $\bar{A}^k(M)$. We adopt also $\bar{A}^0(M) = M$. In [1] we find the proof that if M is m-dimensional, then we have a one-one correspondence $\omega \mapsto \bar{\omega}$ between $A^k(M)$ and $\bar{A}^k(M)$ and the so called natural domain of $\bar{\omega}$ defined by (1.0) is equal to the domain D_ω of ω. This fact allowed to define the concept of the exterior differential $d\omega$ of $\omega \in A^k(M)$ in the following way:

$$\overline{d\omega}(X_0,\dots,X_k) = \sum_{i=0}^{k} (-1)^i \partial_{X_i} \bar{\omega}(X_0,\dots,X_{i-1},X_{i+1},\dots,X_k) +$$

$$+ \sum_{i<j} (-1)^{i+j} \bar{\omega}([X_i,X_j], X_0,\dots,X_{i-1},X_{i+1},\dots,X_{j-1},X_{j+1},\dots,X_k) \tag{1.3}$$

for $X_0,\dots,X_k \in V_{loc}(M)$.

1.1. Proposition. In a premanifold M of finite dimension we have

(i) $\langle v | d\alpha(p)\rangle = v(\alpha)$ when v is in T_pM, $p \in D_\alpha$, $\alpha \in M$,

(ii) $d(\eta + \omega) = d\eta + d\omega$ for $\eta, \omega \in A^k(M)$,

(iii) $d(\eta \wedge \omega) = d\eta \wedge \omega + (-1)^k \eta \wedge d\omega$ for $\eta \in A^k(M)$, $\omega \in A^l(M)$,

(iv) $d \circ d = 0$.

For the proof see [1] and [5].

The local property of the operation d plays an essential part in the work with the exterior differential.

1.2. Lemma. If $\omega \in A^k(M)$, $D_\omega \supset U \in \operatorname{top} M$, $\omega|U = 0$, then $(d\omega)|U = 0$.

Proof. For any $X_1,\dots,X_k \in V_{loc}(M)$ we have $\bar{\omega}(X_1,\dots,X_k)(p) = $ $= \langle X_1(p) \wedge \dots \wedge X_k(p) | \omega(p)\rangle = \langle X_1(p) \wedge \dots \wedge X_k(p) | (\omega|U)(p)\rangle =$ $= \overline{\omega|U}(X_1,\dots,X_k)(p)$ for $p \in U \cap D_{X_1} \cap \dots \cap D_{X_k}$. We have than

$$\overline{\omega|U}(X_1,\dots,X_k) = \omega(X_1,\dots,X_k)|U. \tag{1.4}$$

Applying (1.3) and (1.4) we get $\overline{(d\omega)}|U(X_o,\dots,X_k) = \overline{d(\omega|U)}(X_o,\dots,X_k) = 0$. Q.E.D.

As a consequence of 1.2 we get

1.3. Lemma. If $\omega \in A^k(M)$ and $D_\omega \supset U \in \mathrm{top}M$, $(d\omega)|U = d(\omega|U)$.

2. Pull back of differential forms. Let $f: M \longrightarrow N$ where M and N are premanifolds of finite dimensions. We have the tangent linear mapping $f_{*p}: T_pM \longrightarrow T_{f(p)}N$ defined by the formula $f_{*p}v(p) = v(\beta \circ f)$ for $\beta \in N(f(p))$, v in T_pM, where $N(q) = \{\beta;\ \beta \in \underline{N}\ \&\ q \in D_\beta\}$ for $q \in \underline{N}$. For any $\omega \in A^k(M)$ we define the f-pull-back $f^*\omega$ of ω by the formula

$$\langle v_1 \wedge \dots \wedge v_k | (f^*\omega)(p) = \langle f_{*p}v_1 \wedge \dots \wedge f_{*p}v_k | \omega(f(p)) \rangle \tag{2.0}$$

for $v_1,\dots,v_k$ of T_pM, $p \in f^{-1}D_\omega$. From this definition it follows immediately that for any $\eta \in A^k(M)$, $\omega \in A^1(M)$ and $V \in \mathrm{top}N$

$$f^*(\eta \wedge \omega) = f^*\eta \wedge f^*\omega, \tag{2.1}$$

$$f^*(\omega|V) = (f^*\omega)|f^{-1}V. \tag{2.2}$$

Thus we have

$\overline{f^*\omega}(X_1,\dots,X_k)(p) = \langle f_{*p}X_1(p) \wedge \dots \wedge f_{*p}X_k(p) | \omega(f(p)) \rangle$ for $p \in D_{X_1} \cap \dots \cap D_{X_k} \cap f^{-1}D_\omega$. It is easy to see that for $g: N \longrightarrow P$ where P is of finite dimension we have

$$(g \circ f)^*\Theta = f^*g^*\Theta \quad \text{for} \quad \Theta \in A^k(P).$$

We adopt also $f^*\omega = \omega \circ f$ for $\omega \in A^0(M)$. We will prove the following fact of importance.

2.1. Lemma. For $f: M \longrightarrow N$, where M and N are of finite dimension and for any $\omega \in A^k(N)$ we have

$$df^*\omega = f^*d\omega.$$

Proof. Let $p \in f^{-1}D_\omega$. Then $f(p) \in D_\omega$ and there exists a local vector base $(e_1,\dots,e_n)$ in N, where $e_1,\dots,e_n$ are defined in a neighbourhood V of the point f(p). For any $\beta \in N(f(p))$ we set

$$l(\beta) = (e_1(f(p))(\beta),\dots,e_n(f(p))(\beta)).$$

We have then the function l defined on the set N(f(p)) such that the image lN(f(p)) of this set is contained in $\mathbb{R}^n$ for real premanifolds or in C^n for complex ones. It is easy to see that $lN(f(p))$ is a linear subspace of R^n or C^n, respectively. If would be $lN(f(p)) \neq R^n (ln(f'p)) \neq$ $\neq C^n$, resp.), then the set $lN(f(p))$ should be a subspace of a lower dimension than n. Then should exist numbers $c_1,\dots,c_n$ not vanishing simultanously such that $c_1 e_1(f(p))(\beta) + \dots + c_n e_n(f(p))(\beta) = 0$ for $\beta \in N(f(p))$. Hence it follows that $e_1(f(p)),\dots,e_n(f(p))$ would be linearly dependent. Therefore there exist $\beta^1,\dots,\beta^n \in N(f(p))$ for which $l(\beta^j) = (\beta^j_1,\dots,\beta^j_n)$, $j=1,\dots,n$. Hence it follows that there is a neighbourhood of $f(p)$ such that for any q in this neighbourhood we have non-singular matrix

$$\left[e_i(q)(\beta^j);\ i,\ j \leqslant n\right]. \tag{2.3}$$

We may assume that V is such a neighbourhood. Taking the matrix $\left[\gamma^i_h(q);\ h,\ i \leqslant n\right]$ being inverse to (2.3) for $q \in V$ we have $\gamma^i_h(q)e_i(q)(\beta^j) =$ $= \delta^j_h$, $h, j=1,\dots,n$. We may assume that functions γ^i_h have the set V as their domains. Thus,

$$\gamma^i_h \partial_{e_i}\beta^j = \delta^j_h.$$

We have $\partial_{e_i}\beta^j \in N$, $h, j=1,\dots,n$. Therefore $\gamma^i_h \in N$. Setting

$$g_h(q) = \gamma^i_h(q)e_i(q) \quad \text{for} \quad q \in V,\ h=1,\dots,n$$

we get a vector base $(g_1,\dots,g_n)$ in M such that

$$\partial_{g_h}\beta^j = \delta^j_h,\ h,\ j=1,\dots,n.$$

Let us set for $q \in V$

$$\omega_o(q) = \sum_{i_1<\dots<i_k} \langle g_{i_1}(q)\wedge\dots\wedge g_{i_k}(q) | \omega(q)\rangle d\beta^{i_1}(q)\wedge\dots\wedge d\beta^{i_k}(q) \tag{2.4}$$

We have $\langle g_i(q) | d\beta^j(q)\rangle = g_i(q)(\beta^j) = (\partial_{g_i}\beta^j)(q) = \delta^j_i$ for $q \in V$. For any $q \in V$ we have then the base $(d\beta^1(q),\dots,d\beta^n(q))$ for $(T_qN)^*$. By (2.4), for $q \in V$ and $h_1 < \dots < h_k$ we have

$$\langle g_{h_1}(q)\wedge\dots\wedge g_{h_k}(q) | \omega_o(q)\rangle =$$

$$= \sum_{i_1<\dots<i_k} \langle g_{i_1}(q)\wedge\dots\wedge g_{i_k}(q) | \omega(q)\rangle\langle g_{h_1}(q)\wedge\dots\wedge g_{h_k}(q) | d\beta^{i_1}(q)\wedge\dots\wedge d\beta^{i_k}(q)\rangle =$$

$$= \sum_{i_1<\ldots<i_k} \langle g_{i_1}(q)\wedge\ldots\wedge g_{i_k}(q)|\omega(q)\rangle \det\left[\langle g_{h_r}(q)|\beta^{i_s}(q)\rangle;\ r,\ s\leq k\right] =$$

$$= \sum_{i_1<\ldots<i_k} \langle g_{i_1}(q)\wedge\ldots\wedge g_{i_k}(q)|\omega(q)\rangle \det\left[\delta_{h_r}^{i_s};\ r,\ s\leq k\right] =$$

$= \langle g_{h_1}(q)\wedge\ldots\wedge g_{h_k}(q)|\omega(q)\rangle$. Hence it follows that $\omega_o(q) = \omega(q)$

for $q\in V$. Setting for $q\in V$

$$\omega_{i_1\ldots i_k}(q) = \langle g_{i_1}(q)\wedge\ldots\wedge g_{i_k}(q)|\omega(q)\rangle$$

we may rewrite (2.4) as

$$\omega(q) = \sum_{i_1<\ldots<i_k} \omega_{i_1\ldots i_k}(q)\, d\beta^{i_1}(q)\wedge\ldots\wedge d\beta^{i_k}(q) \quad \text{for} \quad q\in V,$$

or equivalently in the form

$$\omega|V = \sum_{i_1<\ldots<i_k} \omega_{i_1\ldots i_k}\, d\beta^{i_1}\wedge\ldots\wedge d\beta^{i_k}. \tag{2.5}$$

From (2.5) by (2.1) it follows that

$$f^*(\omega|V) = \sum_{i_1<\ldots<i_k} f^*\omega_{i_1\ldots i_k}\, f^*d\beta^{i_1}\wedge\ldots\wedge f^*d\beta^{i_k}.$$

For any $\beta\in N$ and $q\in f^{-1}D_\beta$, by (2.0) and (i) we have $\langle v|(f^*d\beta)(q)\rangle =$ $= \langle f_{*q}v|(d\beta)(f(q))\rangle = d\beta(f(q))(f_{*q}v) = f_{*q}v(\beta) = v(f^*\beta) = (df^*\beta)(q)(v) =$ $= \langle v|(df^*\beta)(q)\rangle$ for v in T_qM. Thus

$$f^*d\beta = df^*\beta \quad \text{for} \quad \beta\in M. \tag{2.6}$$

This yields

$$f^*(\omega|V) = \sum_{i_1<\ldots<i_k} f^*\omega_{i_1\ldots i_k}\, df^*\beta^{i_1}\wedge\ldots\wedge df^*\beta^{i_k}.$$

According to (ii)-(iv), (2.6) and (2.1) we get

$$df^*(\omega|V) = \sum_{i_1<\ldots<i_k} df^*\omega_{i_1\ldots i_k}\, df^*\beta^{i_1}\wedge\ldots\wedge df^*\beta^{i_k} =$$

$$= f^* \sum_{i_1<\ldots<i_k} d\omega_{i_1\ldots i_k}\wedge d\beta^{i_1}\wedge\ldots\wedge d\beta^{i_k} = f^*d(\omega|V).$$

From 1.3 and (2.2) it follows that

$$(df^*\omega)\, f^{-1}V = d((f^*\omega)\, f^{-1}V) = df^*(\omega|V) = f^*d(\omega|V) =$$

$$= f^*((d\omega)|V) = (f^*d\omega)|f^{-1}V.$$

This ends the proof of 2.1.

3. Chains in a premanifold. For any natural number n the set R^n of all real-valued functions of C^∞ class on open sets in $\mathbb{R}^n$ is a premanifold. The set Δ_n of all points $(x^1,\ldots,x^n)$ such that $0 \leq x^i \leq 1$, $i=1,\ldots,n$ and $x^1 + \ldots + x^n \leq 1$ gives a premanifold Δ_n of the shape $R^n_{\Delta_n}$. Setting $x = (x_1,\ldots,x^n)$,

$$t_n^0(x) = 1 - \sum_{i=1}^{n} x^i \quad\text{and}\quad t_n^i(x) = x^i,\ i = 1,\ldots,n,$$

we have the baricentric coordinates of the point x.

Any smooth mapping $s:\Delta_n \longrightarrow M$ will be called a singular n-simplex in M. The set of all simplices in M will be denoted by $S_n(M)$. Every function $c: S_n(M) \longrightarrow \mathbb{R}$ such that the set of all $s \in S_n(M)$ for which $c(s) \neq 0$ is finite will be called a singular n-chain in M or, shortly, n-chain in M. The set of all n-chains in M will be denoted by $C_n(M)$.

For any $s \in S_n(M)$ and any $a \in \mathbb{R}$ we set $(as)(u) = a$ if $u = s$, and $(as)(u) = 0$ if $s \neq u \in S_n(M)$. In such a way we have defined the chain $as \in C_n(M)$. It is evident that any $c \in C_n(M)$ may be written in the form $c = c(s_1)s_1 + \ldots + c(s_k)s_k$ where $s_1,\ldots,s_k$ are all of $s \in S_n(M)$ for which $c(s) \neq 0$. It is convenient to write

$$c = \sum_s c(s)s.$$

For any smooth $f: M \longrightarrow N$ and $c \in C_n(M)$ setting $f_*c = \sum_s c(s)\circ s$ we get the mapping $f_*: C_n(M) \longrightarrow C_n(N)$. The set $C_n(M)$ may be treated in a natural way as a linear space. Now we take the standard inclusions $\Delta_{n,i}: \Delta_n \longrightarrow \Delta_{n+1}$ setting for any $x \in \Delta_n$

$$t^j_{n+1}(\Delta_{n,i}(x)) = \begin{cases} t^j_n(x) & \text{if } j < i, \\ 0 & \text{if } j = i, \\ t^{j-1}(x) & \text{if } j > i, \end{cases}$$

$i = 0,\dots,n$ and $j = 0,\dots,n+1$. This definition yields

$$\Delta_{n+1,h} \circ \Delta_{n,i} = \Delta_{n+1,i+1} \circ \Delta_{n,h} \quad \text{if } h \leqslant i. \tag{3.1}$$

For any $s \in S_n(M)$ we define (n-1)-chain ∂s by the formula

$$\partial s = \sum_{i=0}^{n} (-1)^i s \circ \Delta_{n-1,i} \tag{3.2}$$

called the border of s. For any $c \in C_n(M)$ we define its border by the formula

$$\partial c = \sum_{s} c(s)\partial s. \tag{3.3}$$

Equalities (3.1) - (3.3) yield $\partial\partial c = 0$.

4. Stokes' formula. At every point $x \in \mathbf{R}^k$ we have a base $\partial'^k_{1x},\dots,\partial'^k_{kx}$ for $T_x R^k$ of vectors defined by the formulae $\partial'^k_{ix}(\alpha) = \partial_i\alpha(x)$ for $\alpha \in R^k(x)$, ∂_i is the partial derivation with respect to i-th variable. Taking the mapping

$$id_{\underline{\Delta}_k} : \Delta_k \longrightarrow R^k$$

we remark that there exists a unique system of vectors $\partial^k_{1x},\dots,\partial^k_{kx}$ being a vector base for $T_x\underline{\Delta}_k$ such that

$$id_{\underline{\Delta}_k * x}\partial^k_{ix} = \partial'^k_{ix}, \quad i=1,\dots,k.$$

For any $s \in S_k(M)$ and $\omega \in A^k(M)$ the integral

$$\int_{\underline{\Delta}_k} \langle \partial^k_{1x} \wedge \dots \wedge \partial^k_{kx} \,|\, s^*\omega(x) \rangle dx$$

will be denoted by $\int_s \omega$. For any $c \in C_k(M)$ we set

$$\int_c \omega = \sum_c c(s) \int_s \omega .$$

The number $\int_c \omega$ just defined will be called the integral of the form ω along the chain c.

Theorem (Stokes' formula). If M is a premanifold of finite dimension, then for any $\omega \in A^k(M)$ and any $c \in C_{k+1}(M)$

$$\int_c d\omega = \int_{\partial c} \omega .$$

Proof. By linearity of the mapping $(c \mapsto \int_c \omega): C_k(M) \longrightarrow R$ it suffices to prove that

$$\int_s d\omega = \sum_{h=0}^{k+1} (-1)^h \int_{s\circ\Delta_{k,h}} \omega$$

or, what by 2.1 is equivalent to

$$\int_{i_{k+1}} \Theta = \sum_{h=0}^{k+1} (-1)^h \int_{\Delta_{k,h}} \Theta , \tag{3.4}$$

where $\Theta = s^*\omega$ and $i_{k+1}(z) = z$ for $z \in \Delta_{k+1}$. The right-hand side of (3.4) may be written as $\int_{\partial i_{k+1}} \Theta$. Formula (3.4) takes the form

$$\int_{i_{k+1}} d\Theta = \int_{\partial i_{k+1}} \Theta .$$

This formula is nothing but the Stokes' formula known in advanced calculus.

Linearity of ∂ and the equality $\partial \circ \partial = 0$ allows us to define the k-th singular homology group $H_k(M)$ of the premanifold M. Stokes' formula allows us to consider the de Rham mapping

$$((h, w) \mapsto \langle h, w\rangle): H_k(M) \times H^k(M) \longrightarrow R$$

for a premanifold of finite dimension, where for any homology class h in $H_k(M)$ and any cohomology class w in $H^k(M)$ we set

$$\langle h, w\rangle = \int_c \omega, \quad c \in h, \quad \omega \in w.$$

REFERENCES

[1] H. Matuszczyk: Exterior differential forms in premanifolds (to appear).

[2] H. Matuszczyk, W. Waliszewski: Stokes' formula for chains in a differential space of finite dimension, Colloq. Math. 53(1987), pp. 211-218.

[3] W. Waliszewski: Analytical premanifolds, Zeszyty Naukowe Politechniki Śląskiej, Seria: Matematyka-Fizyka 48 (1986), pp. 217-226.

[4] -, Complex premanifolds and foliations, Seminar on Deformations, Lublin-Łódź 1985/87, D. Reidel Publishing Company, Dordrecht.

[5] -, Abelian inverse semigroups and some skew-differential operations in semigroups (to appear).