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ON JENSEN EQUATION ON A GRAPH

DEDICATED TO PROFESSOR MIECZYSŁAW KUCHARZEWSKI
ON THE OCCASION OF HIS 70-TH BIRTHDAY

Summary. In the paper we consider functional equation in the form:

$$f(x + \varphi(x)) + f(x - \varphi(x)) = 2f(x) \tag{1}$$

where φ: $(0,\infty) \longrightarrow \mathbb{R}$ is a function such that $\varphi(x) \in (0,x)$, $x>0$. Some conditons are given for any solution $f:(0,\infty) \longrightarrow \mathbb{R}$ of (1) such that $\lim_{x\to 0}(f(x)-b)/x = \lim_{x\to\infty} f(x)/x$ for a constant $b \in \mathbb{R}$, solves the Jensen equation

$$f(x+y) + f(x-y) = 2f(x) \tag{2}$$

An example is also given showing that the assertion is no more valid when one assumes only differentiability of f at 0 as it is the case in an analogous problem related to the Cauchy equation.

Perhaps the most celebrated and intensively studied functional equation is the one which takes its name after Cauchy. Since the beginning of seventies the investigation of this equation on restricted domain has involved many mathematicians and a large number of results have been published. One of the interesting aspects of this investigation is the following problem. Let X be a set and let "+": $X \times X \longrightarrow X$ be a binary operation. Suppose that $h: X \longrightarrow X$ is a function. Further, let Y be a set and "$\oplus$": $Y \times Y \longrightarrow Y$ a binary operation. The question is when a solution $f: X \longrightarrow Y$ of the equation

$$f(x+h(x)) = f(x) \oplus f(h(x)) \tag{1}$$

which holds for $x \in X$ is additive i.e. fulfils

$$f(x+y) = f(x) \circledast f(y) \tag{2}$$

for every $x,y \in X$. Equation (1) is an example of Cauchy equation on restricted domain, the restricted domain here is the graph of the function h. Note that (1) expresses additivity of f postulated for some pairs $(x,y) \in X \times Y$ only, namely for $(x,y) \in \mathrm{Gr}\, h = \{(x,h(x)) : x \in X\}$.

Of course the results concerning (1) are obtained under some specific assumptions, usually in the case where $X=Y=\mathbb{R}$, or $X = (0,\infty)$ and binary operations are usual addition. Some assumptions are made also on the function h. The reader interested in the topic should address e.g. to Dhombres [1], Forti [3] and [4], Sablik [7], Zdun [8], Matkowski [6] and Jarczyk [5] (the last author deals with the case where $h=f$ is unknown as well).

A natural extension of these results would be to answer similar questions for some other functional equations. In this paper we present an attempt to do it for Jensen functional equation which we write in the following from

$$f(x+y) + f(x-y) = 2\, f(x) \tag{3}$$

and assume to hold for $x,y \in \mathbb{R}$.

We will restrict ourselves to the case where $x \in (0,\infty)$ and ask for the conditions quaranteeing that a solution of

$$f(x + \varphi(x)) + f(x - \varphi(x)) = 2\, f(x), \tag{4}$$

where $\varphi : (0,\infty) \longrightarrow \mathbb{R}$ is a function such that $\varphi(x) \in (0,x)$, $x>0$, solves (3) is a sum of an additive function and a constant. Thus one might think that our problem should be very close to the one mentioned above. This is not quite true. In the study of (1) it is usually assumed that h is rather a regular function, e.g. such that $\mathrm{id}+h$ is a homeomorphism. Even in this case solutions of (1) are not additive in general. Therefore the authors look for the equivalence of (1) and (2) in some specific classes of functions. Many results establish the equivalence in the class $A_o = \{f : (0,\infty) \longrightarrow \mathbb{R} : \lim_{x \to 0} f(x)/x$ exists$\}$. However, the class A_o happens to be too large to assure the equivalence of (3) and (4). To show this we exhibit the following

Example. Let $\varphi: (0,\infty) \longrightarrow \mathbb{R}$ be given by $\varphi(x) = \frac{x}{2}$ and consider the set $G = \{x\in(0,\infty): \bigvee_{k,l\in\mathbb{Z}} x=2^k3^l\}$. Define a function $g_1:G\to\mathbb{R}$ by

$$g_1(2^k 3^l) = \begin{cases} 0 & \text{if } (k\geq 0 \wedge l\geq 1) \vee (k+l\leq -1), \\ \binom{k}{-l} 4^{-k} 3^{-l} & \text{if } (k+l\geq 0 \wedge l \leq 0), \\ 4^{-k} 3^{-l} & \text{if } (k+l = 0 \wedge l > 0), \\ \binom{l-1}{-k-1}(-1)^{l+k} 4^{-k} 3^{-l} & \text{if } (k+l\geq 0 \wedge k\leq -1) \end{cases} \tag{5}$$

Further, let $g: (0,\infty)\to\mathbb{R}$ be defined by $g_{|G} = g_1$ and $g_{|(0,\infty)\setminus G} = 0$. Finally let $f: (0,\infty) \to\mathbb{R}$ be given by $f(x) = x\, g(x) \in [0,\infty)$. Of course f is not solution of (3) for, as it is well known, g should be constant then. On the other hand $f \in A_o$. To show this it is sufficient to check that $\lim_{x\to 0} g_1(x)=0$.

Let us take a sequence $(x_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}}$ such that $\lim_{n\to\infty} x_n = 0$. Write $x_n = 2^{k_n}3^{l_n}$, $n \in \mathbb{N}$. Divide $(x_n)_{n\in\mathbb{N}}$ into two subsequences $(x_{p_n})_{n\in\mathbb{N}}$ and $(x_{q_n})_{n\in\mathbb{N}}$ such that $l_{p_n} > 0$, $n\in\mathbb{N}$, and $l_{q_n} \leq 0$, $n\in\mathbb{N}$. Observe that $\lim_{n\to\infty} x_{p_n} = 0$ implies $k_{p_n} + l_{p_n} < 0$ for n sufficiently large. Thus $g_1(x_{p_n}) = 0$ for large n which proves our conjecture in this case. To deal with the other case we may assume without loss of generality that $k_{q_n} + l_{q_n} \geq 0$. Convergence of $(x_{q_n})_{n\in\mathbb{N}}$ to zero implies now that $\lim_{n\to\infty} l_{q_n} = -\infty$. Thus $\lim_{n\to\infty} k_{q_n} = \infty$.

Writing $g_1(x_{q_n})$ in the form (cf. (5))

$$g_1(x_{q_n}) = \binom{k_{q_n}}{-l_{q_n}} \left(\frac{3}{4}\right)^{-l_{q_n}} \left(\frac{1}{4}\right)^{k_{q_n}+l_{q_n}}$$

we see that $\lim g_1(x_{q_n}) = 0$ (cf. Feller [2]). It remains to show that f actually stisfies (4).

This can be done by a straightforward although tedious calculation showing that $x \longrightarrow xg_1(x)$ satisfies (4) for $x \in G$ and observing that $f|_{(0,\infty)\setminus G} = 0$ satisfies (4) as well (note that $x \in G$ if and only if $\frac{3}{2} x \in G$ and $\frac{1}{2} x \in G$).

The above example makes it reasonable to look for the equivalence of (3) and (4) in a class different from A_o. Actually consider first the class

$$A_{a;b} = \left\{ f:(0,\infty) \longrightarrow \mathbb{R}: \frac{f(x)-b}{x} \text{ is bounded in a neighbourhood of } 0 \text{ and } \lim_{x\to\infty} \frac{f(x)}{x} = a \right\},$$

where $a \in [-\infty, +\infty]$ and $b \in \mathbb{R}$ are some fixed constants.

We have the following

Theorem 1. Let $\alpha \in (0,1)$. If $f:(0,\infty) \longrightarrow \mathbb{R}$ is a solution of the equation

$$2 f(x) = f(x + \alpha x) + f(x - \alpha x) \tag{6}$$

and belongs to $A_{a;b}$ for some $a \in [-\infty, +\infty]$ and $b \in \mathbb{R}$ then $a \in \mathbb{R}$ and $f(x) = ax+b$, $x \in (0,\infty)$ (and hence solves (3)).

Proof. This is a straight forward matter to check that f solves (6) if and only if $g:(0,\infty) \longrightarrow \mathbb{R}$ given by $g(x) = \frac{f(x)-b}{x}$, $x \in (0,\infty)$ solves the equation

$$g(x) = \frac{c}{2} g(cx) + \frac{d}{2} g(dx) \tag{7}$$

where $c=1+\alpha$, $d=1-\alpha$. Moreover, if $f \in A_{a;b}$ then

$$g \text{ is bounded in a neighbourhood of } 0 \text{ and } \lim_{x\to\infty} g(x) = a. \tag{8}$$

We show first that

$$\text{for every } R > 0 \text{ there exists a } B_R > 0 \text{ such that } |g(y)| \le B_R \text{ for } y \le R. \tag{9}$$

Indeed, by (8) $|g(x)| \le B$ for $x \in (0,r)$, where $B > 0$ and $r > 0$ are some constants. Now, from (7) we get

$$g(cx) = \frac{2}{c}\, g(x) - \frac{d}{c}\, g(dx)$$

for $x > 0$. Taking into account that

$$0 < d < 1 < c < 2 \tag{10}$$

we get

$$|g(cx)| \le 2|g(x)| + |g(dx)|$$

whence

$$|g(y)| \le 3B$$

for every $y \in (0, cr)$. An easy induction shows that

$$|g(y)| \le 3^n B$$

for every $y \in (0, c^n r)$. To show (9) it is enough to put $B_R = 3^N B$, where N is such that $c^N r \ge R$.
The following two relations can be easily proved by a simple calculation (cf. (10))

$$0 < -\frac{\ln d}{\ln \frac{c}{d}} < \frac{c}{2}\,, \tag{11}$$

for every $n \in \mathbb{N}$, $k \in \{0,1,\ldots\ldots,n\}$, $x > 0$ and $R > 0$,
inequalities (i) $k/n \ge (1/n)(\ln(R/x)/\ln((c/d)) - (\ln d)/\ln(c/d)$ (12)

and

$$\text{(ii)}\quad c^k d^{n-k} x \ge R$$

are equivalent.
Now, we show that $a \in \mathbb{R}$. Indeed, suppose that $a = +\infty$. Let $D > 0$ be an arbitrary number and choose $R > 0$ so that

$$g(y) \ge D \qquad \text{for} \quad y \ge R. \tag{13}$$

Fix $x \in (0,\infty)$ arbitrarily and let $\delta > 0$ be such that (cf. (11))

$$A: = -\frac{\ln d}{\ln(c/d)} + \delta < \frac{c}{2} - \delta. \tag{14}$$

Finally, let $N \in \mathbb{N}$ be so large that

$$\frac{1}{n}\left[\ln (R/x)/\ln(c/d)\right] < \delta \tag{15}$$

for every $n \geq N$. By (14) and (15) we get inequality (12) (i) for every $n \geq N$ and $k \in \{0,1,\ldots\ldots,n\}$ such that $k \geq A \cdot n$. Thus (12)(ii) holds as well.

From (7) we obtain by an easy induction

$$g(y) = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{c}{2}\right)^k\left(\frac{d}{2}\right)^{n-k} g(c^k d^{n-k} y) \tag{16}$$

for every $y \in (0,\infty)$ and $n \in \mathbb{N}$.

Using the last equality for $n \geq N$ we obtain in view what has been written above (cf. (9), (12) and (13))

$$\begin{aligned} |g(x)| &\geq \Big|\sum_{k\geq nA} \binom{n}{k}\left(\frac{c}{2}\right)^k\left(\frac{d}{2}\right)^{n-k} g(c^k d^{n-k} x)\Big| - \\ &\sum_{k<nA} \binom{n}{k}\left(\frac{c}{2}\right)^k\left(\frac{d}{2}\right)^{n-k} |g(c^k d^{n-k} x)| \geq \\ &D\sum_{k\geq nA} \binom{n}{k}\left(\frac{c}{2}\right)^k\left(\frac{d}{2}\right)^{n-k} - B_R \sum_{k<nA} \binom{n}{k}\left(\frac{c}{2}\right)^k\left(\frac{d}{2}\right)^{n-k} \end{aligned} \tag{17}$$

Using Bernoulli's law of large numbers and (14) we get by (17)

$$\begin{aligned} |g(x)| &\geq \limsup_{n\to\infty} \left[D \cdot P\left(\frac{k}{n} \geq A\right) - B_R\, P\left(\frac{k}{n} < A\right)\right] \geq \\ &\limsup_{n\to\infty} \left[D\, P\left(\frac{k}{n} \geq \frac{c}{2} - \delta\right) - B_R\, P\left(\frac{k}{n} < \frac{c}{2} - \delta\right)\right] \geq \\ &\limsup_{n\to\infty} \left[D \cdot P\left(\left|\frac{k}{n} - \frac{c}{2}\right| \leq \delta\right) - B_R \cdot P\left(\left|\frac{k}{n} - \frac{c}{2}\right| > \delta\right)\right] = D. \end{aligned}$$

Thus $|g| \ge D$ because x has been arbitrary. But D has been arbitrary, too, whence $|g| = \infty$ contrary to (8).

If $a = -\infty$, then we can repeat the above argument for $-g$ to get a contradiction.

To prove our assertion it remains to show, that $g(x) \equiv \text{const} = a$. To this aim fix $x \in (0,\infty)$ and $\varepsilon > 0$ arbitrarily. Choose > 0 so that (cf. (8))

$$|g(y) - a| < \varepsilon \qquad \text{for} \quad y \ge R. \tag{18}$$

We can choose a $\delta > 0$ such that (14) holds. Similary as above we have (12) (ii) for n large enough and $k \ge nA$ $(k \in \{0,1,\dots,n\})$. By (16) and (18) we obtain

$$|g(x) - a| \le \sum_{k<nA} \binom{n}{k}\left(\frac{c}{2}\right)^k\left(\frac{d}{2}\right)^{n-k}(B_R+2|a|+\varepsilon)+\varepsilon \sum_{k\ge nA} \binom{n}{k}\left(\frac{c}{2}\right)^k\left(\frac{d}{2}\right)^{n-k},$$

whence, similarly as above, the inequality

$$|g(x) - a| \le \varepsilon$$

follows. Hence $g(x) = a$ because ε has been arbitrary. This ends the proof.

In the case where φ is not linear we look for the equivalence of (3) and (4) in the class

$$A_{a;b,a'} = \left\{ f \in A_{a;b}\colon \lim_{x\to 0} \frac{f(x)-b}{x} = a' \right\},$$

with $a \in [-\infty, \infty]$, $a', b \in \mathbb{R}$.

We have the following

Theorem 2. Let $\varphi\colon (0,\infty)\to(0,\infty)$ be such a mapping that $\varphi(x) \in (0,x)$ for every $x > 0$ and $x \to x + \varphi(x)$ maps homeomorphically $(0,\infty)$ onto itself. If $f\colon (0,\infty) \to \mathbb{R}$ belongs to $A_{a;b,a}$ for some $a,b \in \mathbb{R}$ and f solves (4) then $f(x) = ax+b$ (and hence f solves (3)).

Proof. Define $g\colon (0,\infty) \to \mathbb{R}$ putting $g(x) = \dfrac{f(x) - b}{x} - a$ for $x > 0$. Then it is easy to check that $f \in A_{a;b,a}$ and f solves (4) iff $\lim_{x\to 0^+} g(x) = \lim g(x) = 0$ and g solves

$$g(x) = \frac{u(x)}{2x} g(u(x)) + \frac{v(x)}{2x} g(v(x)) \tag{19}$$

for $x > 0$, where $u(x) = x+\varphi(x)$ and $v(x) = x-\varphi(x)$. In a very similar way as in the proof of Theorem 1 we can prove that (9) holds for g and thus g is bounded in $(0, \infty)$ for $\lim_{x\to\infty} g(x)$ exists.

Suppose that $B = \sup\{g(x): x \in (0,\infty)\} > 0$ and fix an $\varepsilon < \frac{B}{2}$. Then there exist $r > 0$ and $R > 0$ such that $r \le R$ and

$$|g(x)| < \varepsilon \qquad \text{for} \quad x \notin [r,R]. \tag{20}$$

Put $N = \min\{n \in \mathbb{N}: u^{-n}(R) < r\}$. We will show now that

for every $n \in \{1,\dots,N\}$ there exists a constant $c_n < B$ such that

$$g(x) \le c_n \tag{21}$$

for all $x \in (u^{-n}(R), \infty)$.

Indeed, for $x > R$ we have $g(x) < \varepsilon < \frac{B}{2}$ and for $x \in (u^{-1}(R),R]$ we get by (15), (20) and the choice of ε

$$g(x) \le \frac{u(x)}{2x}\varepsilon + \frac{v(x)}{2x} B \le B\left[\frac{u(x)}{4x} + \frac{v(x)}{2x}\right] = B\left(\frac{1}{2} + \frac{v(x)}{4x}\right) < \frac{3}{4} B,$$

because $v(x) \in (0,x)$, $x > 0$. Thus, putting $c_1 = \frac{3}{4} B$ we obtain (21) for $n = 1$. Suppose (21) holds for some $n \in \{1,\dots,N-1\}$ and take any $x \in (u^{(n+1)}(R), u^{-n}(R)]$. Then $u(x) \in (u^{-n}(R),\infty)$ and we get from (19) by induction hypothesis

$$g(x) \le \frac{u(x)}{2x} c_n + \frac{v(x)}{2x} B.$$

The function $[u^{-(n+1)}(R), u^{-n}(R)] \ni x \longrightarrow \frac{u(x)}{2x} c_n + \frac{v(x)}{2x} B$ is continuous and hence (cf. induction hypothesis) bounded above by a constant $d_n < B$. Putting $c_{n+1} = \max(c_n, d_n)$ we get the assertion which ends the proof of (21).

Now, by (21), choice of ε, r, R and N we get for every $x \in (0,\infty) = (0,r) \cup (u^{-N}(R), \infty)$

$$g(x) \le \max\{c_N, \varepsilon\} \le \max\{c_N, \frac{B}{2}\} < B$$

which contradicts the definition of B. Thus $g(x) \le 0$ for every $x > 0$. Since $-g$ is also a solution of (19) with the same asymptotic properties as g, we get also $-g \le 0$. Thus $g=0$ which means that $f(x) = ax+b$, $x > 0$ and ends the proof.

From the above theorem we get

Corollary. Let φ be as in Theorem 2 and fix $a,b,a' \in \mathbb{R}$, $a \neq a'$. Then (4) has at most one solution $f:(0,\infty) \to \mathbb{R}$ belonging to the class $A_{a;b,a'}$.

Proof. Let $f_1, f_2 \in A_{a;b,a'}$ be two solutions of (4). Then $f_1 - f_2 \in A_{0;0,0}$ is also a solution of (4) and hence $f_1 - f_2 = 0$ by Theorem 2.

Remark 1. It follows from Theorem 1 that in the case where φ is linear there is no solution of (4) which belongs to $A_{a;b,a'}$ for some $a,b,a' \in \mathbb{R}$ and $a \neq a'$.

We can give a direct argument to prove this fact. Indeed, suppose that there is a solution f of (4) which belongs to $A_{a;b,a'}$. One can easily calculate that then for every $y > 0$ the function $f_y:(0,\infty) \to \mathbb{R}$ given by $f_y(x) = (f(yx) + b(y-1)) \cdot \frac{1}{y}$ for $x > 0$, solves (4) and belongs to $A_{a;b;a'}$. Thus, by Corollary, $f = f_y$ for every $y > 0$ which means that

$$yf(x) = f(yx) + b(y-1)$$

for every $x,y > 0$. Putting $x=1$ in the above equality we obtain

$$f(y) = (f(1) - b)y + b$$

for every $y > 0$. Hence $a = a' = f(1) - b$ contrary to our assumption.

Remark 2. Observe that any solution of (19) which is bounded in a neigh bourhood of 0 and in a neighbourhood of $+\infty$ is bounded in $(0,\infty)$.

The proof of this statement is analogous to the argument we have used in Theorems 1 and 2.

Remark 3. Suppose $g:(0,\infty) \to \mathbb{R}$ is a continuous solution of (19) and $\lim_{x\to 0} g(x) = a$, $\lim_{x\to\infty} g(x) = b$. Then values of g lie (sharply) between a and b.

Indeed, assume without loss of generality that $a < b$. If $a = -\infty$ then $g(x) > a$ for every $x > 0$. Suppose that $a \in \mathbb{R}$ and $g(x) \le a$ for some $x > 0$. It is obvious that g attains its minimum at a point $x_o > 0$ and $g(x_o) \le a$. Let $y > 0$ be any point such that $g(y) = g(x_o)$. Then we infer from (19) that $g(y) = g(v(y)) = g(u(y))$ for otherwise minimality of $g(y)$ would be contradicted. An easy induction shows that in particular $g(u^n(x_o)) = g(x_o)$ for $n \in \mathbb{N}$. Hence taking into account that $\lim_{n\to\infty} u^n(x_o) = +\infty$ we obtain

$$\lim_{x\to\infty} g(x) \le g(x_o) \le a < b = \lim_{x\to\infty} g(x),$$

a contradiction proving that $g(x) > a, x > 0$. Similarly we obtain $g(x) < b, x>0$, which ends the proof.

Remark 4. It remains an open question whether a similar conclusion as in Remark 1 can be obtained in the case of nonlinear φ. The authors conjecture an affirmative answer to this question. However, no answer is known even if we additionally restrict ourselves to continuous solution of (4).

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RÓWNANIE JENSENA NA WYKRESIE

S t r e s z c z e n i e

W pracy rozpatrujemy równanie funkcyjne

$$f(x + \varphi(x)) + f(x - \varphi(x)) = 2f(x), \tag{1}$$

gdzie: $\varphi: (0,\infty) \longrightarrow \mathbb{R}$ jest taką funkcją, że $\varphi(x) \in (0,\infty)$, $x > 0$. Podane są warunki na to, by każde takie rozwiązanie $f: (0,\infty) \longrightarrow \mathbb{R}$ równania (1), że $\lim_{x\to 0} (f(x)-b)/x = \lim_{x\to\infty} f(x)/x$ dla pewnej stałej $b \in \mathbb{R}$ było rozwiązaniem równania Jensena

$$f(x+y) + f(x-y) = 2f(x). \tag{2}$$

Podany jest również przykład pokazujący, że nie wystarczy zakładać dla otrzymania tezy różniczkowalności f w 0, jak to ma miejsce w podobnym zagadnieniu dotyczącym równania Cauchy'ego.

УРАВНЕНИЕ ЕНСЭНА НА ГРАФЕ

Резюме. В настоящей работе рассматривается функциональное уравнение

$$f(x + \varphi(x)) + f(x - \varphi(x)) = 2f(x), \tag{1}$$

Где $\varphi: (0,\infty) \longrightarrow \mathbb{R}$ такая функция, что $\varphi(x) \in (0,\infty), x > 0$. Даны условия для того, чтобы всякое решение $f: (0,\infty) \longrightarrow \mathbb{R}$ уравнения (1) исполняющеё $\lim_{x\to 0} (f(x)-b)/x = \lim_{x\to\infty} f(x)/x$ с некоторой постоянной было также решением уравнения

$$f(x+y) + f(x-y) = 2f(x). \tag{2}$$

Приводится пример указывающий на то, что для получения тезиса не достаточно предположить лишь только диффеpенцируемость f в 0, как это имеет в аналогичной проблеме связанной с уравнением Коши.